CHAPTER 17

Sound Waves

If a (compressible) fluid in equilibrium is perturbed, and the perturbation is sufficiently small, the perturbation will propagate through the fluid as a **sound wave**, which is a mechanical, longitudinal wave.

Let $(\rho_0, P_0, \vec{u}_0)$ be a **uniform, equilibrium solution** of the Euler fluid equations (i.e., ignore viscosity). Also, in what follows we will ignore gravity (i.e., $\nabla\Phi = 0$).

Uniformity implies that $\nabla\rho_0 = \nabla P_0 = \nabla\vec{u}_0 = 0$. In addition, since the only allowed motion is uniform motion of the entire system, we can always use a Galilean coordinate transformation so that $\vec{u}_0 = 0$, which is what we adopt in what follows.

Substitution in the continuity and momentum equations, one obtains that $\partial\rho_0/\partial t = \partial\vec{u}_0/\partial t = 0$, indicative of an equilibrium solution.

Perturbation Analysis: Consider a small perturbation away from the above equilibrium solution:

$$\begin{aligned}
\rho_0 &\rightarrow \rho_0 + \rho_1 \\
P_0 &\rightarrow P_0 + P_1 \\
\vec{u}_0 &\rightarrow \vec{u}_0 + \vec{u}_1 = \vec{u}_1
\end{aligned}$$

where $|\rho_1/\rho_0| \ll 1$, $|P_1/P_0| \ll 1$ and $\vec{u}_1$ is small (compared to the sound speed, to be derived below).

Substitution in the **continuity** and **momentum** equations yields

$$\begin{aligned} \frac{\partial(\rho_0+\rho_1)}{\partial t} + \nabla(\rho_0+\rho_1)\vec{u}_1 &= 0 \\ \frac{\partial \vec{u}_1}{\partial t} + \vec{u}_1 \cdot \nabla \vec{u}_1 &= -\frac{\nabla(P_0+P_1)}{(\rho_0+\rho_1)} \end{aligned}$$

which, using that $\nabla\rho_0 = \nabla P_0 = \nabla \vec{u}_0 = 0$ reduces to

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \vec{u}_1 + \nabla(\rho_1 \vec{u}_1) &= 0 \\ \frac{\partial \vec{u}_1}{\partial t} + \frac{\rho_1}{\rho_0}\frac{\partial \vec{u}_1}{\partial t} + \vec{u}_1 \cdot \nabla \vec{u}_1 + \frac{\rho_1}{\rho_0}\,\vec{u}_1 \cdot \nabla \vec{u}_1 &= -\frac{\nabla P_1}{\rho_0} \end{aligned}$$

The latter follows from first multiplying the momentum equations with $(\rho_0 + \rho_1)/\rho_0$. Next we **linearize** these equations, which means we use that the perturbed values are all small such that terms that contain products of two or more of these quantities are always negligible compared to those that contain only one such quantity. Hence, the above equations reduce to

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \vec{u}_1 &= 0 \\ \frac{\partial \vec{u}_1}{\partial t} + \frac{\nabla P_1}{\rho_0} &= 0 \end{aligned}$$

These equations describe the evolution of perturbations in an **ideal**, **inviscid** and **uniform** fluid. As always, these equations need an additional equation for closure. In what follows we assume a **barotropic equation of state**, $P = P(\rho)$.

Using Taylor series expansion, we then have that

$$P(\rho_0+\rho_1) = P(\rho_0) + \left(\frac{\partial P}{\partial \rho}\right)_0 \rho_1 + \mathcal{O}(\rho_1^2)$$

where we have used $(\partial P/\partial\rho)_0$ as shorthand for the partial derivative of $P(\rho)$ at $\rho = \rho_0$. Using that $P(\rho_0) = P_0$ and $P(\rho_0+\rho_1) = P_0 + P_1$, we find that, when linearized,

$$P_1 = \left(\frac{\partial P}{\partial \rho}\right)_0 \rho_1$$

Note that $P_1 \neq P(\rho_1)$; rather P_1 is the perturbation in pressure associated with the perturbation ρ_1 in the density.

Substitution in the fluid equations of our perturbed quantities yields

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \vec{u}_1 &= 0 \\ \frac{\partial \vec{u}_1}{\partial t} + \left(\frac{\partial P}{\partial \rho}\right)_0 \frac{\nabla \rho_1}{\rho_0} &= 0\end{aligned}$$

Taking the partial time derivative of the above **continuity** equation, and substituting the above **momentum equation**, ultimately yields

$$\frac{\partial^2 \rho_1}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)_0 \nabla^2 \rho_1 = 0$$

which we recognize as a **wave equation**, whose solution is a **plane wave**:

$$\rho_1 \propto e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

with $\vec{k}$ the **wavevector**, $k = |\vec{k}| = 2\pi/\lambda$ the **wavenumber**, λ the **wavelength**, $\omega = 2\pi\nu$ the **angular frequency**, and ν the **frequency**.

To gain some insight, consider the 1D case: $\rho_1 \propto e^{i(kx-\omega t)} \propto e^{ik(x-v_{\rm p}t)}$, where we have defined the **phase velocity** $v_{\rm p} \equiv \omega/k$. This is the velocity with which the wave pattern propagates through space. For our perturbation of a compressible fluid, this phase velocity is called the **sound speed**, $c_{\rm s}$. Substituting the solution $\rho_1 \propto e^{i(kx-\omega t)}$ into the wave equation, we see that

$$c_{\rm s} = \frac{\omega}{k} = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_0}$$

Hence, for a **barotropic fluid**, the sound speed is entirely determined by the equation of state. In particular, for a **polytropic equation of state**, which is a barotropic EoS of the form $P \propto \rho^\Gamma$, with Γ the **polytropic index**, we have that

$$c_{\rm s} = \sqrt{\Gamma \frac{P}{\rho}} = \sqrt{\Gamma \frac{k_{\rm B} T}{\mu\, m_{\rm p}}}$$

Thus, the sound speed increases with temperature, and is higher for a stiffer EoS (i.e., a larger value of Γ). Note also that, for our barotropic fluid, the sound speed is independent of ω. This implies that all waves move equally fast; the shape of a wave packet is preserved as it moves. We say that an ideal (inviscid) fluid with a barotropic EoS is a **non-dispersive medium**.

To gain further insight, let us look once more at the (1D) solution for our perturbation:

$$\rho_1 \propto e^{i(kx-\omega t)} \propto e^{ikx}\, e^{-i\omega t}$$

Recalling **Euler's formula** ($e^{i\theta} = \cos\theta + i\sin\theta$), we see that:

- The e^{ikx} part describes a periodic, spatial oscillation with wavelength $\lambda = 2\pi/k$.
- The $e^{-i\omega t}$ part describes the time evolution:
  - If ω is real, then the solution describes a **sound wave** which propagates through space with a sound speed $c_{\rm s}$.
  - If ω is imaginary then the perturation is either exponentially growing ('unstable') or decaying ('damped') with time.

We will return to this in Chapter 19, when we discuss the **Jeans stability criterion**.