CHAPTER 16

Hydrostatic Equilibrium & Stellar Structure

Hydrostatic Equilibrium: A fluid is said to be in hydrostatic equilibrium (HE) when it is at rest. This occurs when external forces such as gravity are balanced by the forces that arise due to pressure gradient.

Starting from the **Navier-Stokes equation** (see Chapter 7)

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \, \vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi + \nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right]$$

we see that $\vec{u} = 0$ implies that

$$\boxed{\nabla P = -\rho \, \nabla \Phi}$$

which is the equation of **hydrostatic equilibrium**.

Stellar Structure: stars are gaseous spheres in hydrostatic equilibrium (except for radial pulsations, which may be considered perturbations away from HE). The structure of stars is therefore largely governed by the above equation. If we assume spherical symmetry (a fairly accurate assumption in the absence of rotation), we have that $\nabla P = \mathrm{d}P/\mathrm{d}r$ and $\nabla \Phi = \mathrm{d}\Phi/\mathrm{d}r = GM(r)/r^2$, where $M(r)$ is the mass enclosed within radius r. The equation of HE therefore can be written as

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G \, M(r) \, \rho(r)}{r^2}$$

In addition, we also have that

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi \rho(r) \, r^2$$

These are two differential equations with three unknowns; P, ρ and M. If the **equation of state** is barotropic, i.e., $P = P(\rho)$, then our set of equations is closed, and the density profile of the star can be solved for (given proper boundary conditions).

However, in general the **equation of state** is of the form $P = P(\rho, T, \{X_i\})$, where $\{X_i\}$ is the set of the abundances of all emements i. The temperature structure of a star and its abundance ratios are governed by **nuclear physics** (which provides the source of energy) and the various **heat transport mechanisms**.

Polytropic Spheres: A barotropic equation of state of the form $P \propto \rho^\Gamma$ is called a polytropic equation of state, and Γ is called the **polytropic index**. Note that $\Gamma = 1$ and $\Gamma = \gamma$ for an isothermal and adiabatic equations of state, respectively. A spherically symmetric, polytropic fluid in HE is called a polytropic sphere.

Lane-Emden equation: Upon substituting the **polytropic EoS** in the equation of **hydrostatic equilibrium** and using the **Poisson equation**, one obtains a single differential equation that completely describes the structure of the polytropic sphere, known as the **Lane-Emden equation**:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n$$

Here $n = 1/(\Gamma - 1)$ is related to the polytropic index (in fact, confusingly, some texts refer to n as the polytropic index),

$$\xi = \left(\frac{4\pi G\rho_\mathrm{c}}{\Phi_0 - \Phi_\mathrm{c}}\right)^{1/2} r$$

is a dimensionless radius,

$$\theta = \left(\frac{\Phi_0 - \Phi(r)}{\Phi_0 - \Phi_\mathrm{c}}\right)$$

with Φ_c and Φ_0 the values of the gravitational potential at the center ($r = 0$) and at the surface of the star (where $\rho = 0$), respectively. The density is related to θ according to $\rho = \rho_\mathrm{c}\theta^n$ with ρ_c the central density.

Solutions to the Lane-Emden equation are called **polytropes of index** n. In general, the Lane-Emden equation has to be solved numerically subject to the boundary conditions $\theta = 1$ and $\mathrm{d}\theta/\mathrm{d}\xi = 0$ at $\xi = 0$. Analytical solutions exist, however, for $n = 0$, 1, and 5. Examples of polytropes are stars that are supported by **degeneracy pressure**. For example, a non-relativistic, degenerate equation of state has $P \propto \rho^{5/3}$ and is therefore describes by a polytrope of index $n = 3/2$. In the relativistic case $P \propto \rho^{4/3}$ which results in a polytrope of index $n = 3$.

Another polytrope that is often encountered in astrophysics is the **isothermal sphere**, which has $P \propto \rho$ and thus $n = \infty$. It has $\rho \propto r^{-2}$ at large radii, which implies an infinite total mass. If one truncates the isothermal sphere at some radius and embeds it in a medium with external pressure (to prevent the sphere from expanding), it is called a **Bonnor-Ebert sphere**, which is a structure that is frequently used to describe molecular clouds.

Heat transport in stars: Typically, ignoring abundance gradients, stars have an equation of state $P = P(\rho, T)$. The equations of stellar structure, equations (2) and (3), are therefore complemented by a third equation

$$\frac{\mathrm{d}T}{\mathrm{d}r} = F(r)$$

Since T is a measure of the internal energy, the rhs of this equation describes the **heat flux**, $F(r)$.

The main heat transport mechanisms in a star are:

- conduction
- convection
- radiation

Note that the fouth heat transport mechanism, advection, is not present in the case of hydrostatic equilibrium, because $\vec{u} = 0$.

Recall that the **thermal conductivity** $\mathcal{K} \propto l\, v_{\rm th}$, where l is the mean free path of the fluid particles, and $v_{\rm th}$ is the thermal (microscopic) velocity (see Chapter 14). Since radiative heat transport in a star is basically the conduction of photons, and since $c \gg v_{\rm e}$ and $l_{\rm photon} \gg l_{\rm e}$ (where 'e' refers to electrons), we have that in stars **radiation is a far more efficient heat transport mechanism than conduction**.

Convection: convection only occurs if the **Schwarzschild Stability Criterion** is violated, which happens when the temperature gradient $\mathrm{d}T/\mathrm{d}r$ becomes too large (i.e., larger than the temperature gradient that would exist if the star was adiabatic; see Chapter 19). If that is the case, convection always dominates over radiation as the most efficient heat transport mechanism. In general, more massive stars are *more radiative* and *less convective*.

Trivia: On average it takes ~ 200.000 years for a photon created at the core of the Sun in nuclear burning to make its way to the Sun's photosphere; from there it only takes ~ 8 minutes to travel to the Earth.

Hydrostatic Mass Estimates: Consider once more the case of an ideal gas with EoS

$$P = \frac{k_{\rm B}T}{\mu m_{\rm p}}\rho$$

We have that

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}r} &= \frac{\partial P}{\partial \rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{\partial P}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{P}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}r} \\
&= \frac{P}{r}\left[\frac{r}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{r}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right] = \frac{P}{r}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right]
\end{aligned}$$

Substitution of this equation in the equation for Hydrostatic equilibrium (HE) yields

$$M(r) = -\frac{k_{\rm B}\,T(r)\,r}{\mu m_{\rm p} G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right]$$

This equation is often used to measure the 'hydrostatic' mass of a galaxy cluster; X-ray measurements can be used to infer $\rho(r)$ and $T(r)$ (after deprojection, which is analytical in the case of spherical symmetry). Substitution of these two radial dependencies in the above equation then yields and estimate for the cluster's mass profile, $M(r)$. Note, though, that this mass estimate is based on three crucial assumptions: (i) sphericity, (ii) hydrostatic equilibrium, and (iii) an ideal-gas EoS. Clusters typically are not spherical, often are turbulent (such that $\vec{u} \neq 0$, violating the assumption of HE), and can have significant contributions from non-thermal pressure due to magnetic fields, cosmic rays and/or turbulence. Including these non-thermal pressure sources the above equation becomes

$$M(r) = -\frac{k_{\rm B}\, T(r)\, r}{\mu m_{\rm p} G} \left[\frac{{\rm d}\ln\rho}{{\rm d}\ln r} + \frac{{\rm d}\ln T}{{\rm d}\ln r} + \frac{P_{\rm nt}}{P_{\rm th}} \frac{{\rm d}\ln P_{\rm nt}}{{\rm d}\ln r}\right]$$

were $P_{\rm nt}$ and $P_{\rm th}$ are the non-thermal and thermal contributions to the total gas pressure. Unfortunately, it is extremely difficult, if not impossible, to properly measure $P_{\rm nt}$, which is therefore often ignored. This may result in systematic biases of the inferred cluster mass.