CHAPTER 15

Gravity: Poisson Equation & Virial Theorem

Gravity in Astrophysical Fluids: Many of the fluids encountered in astrophysics are self-gravitating, which means that the gravitational force due to the fluid itself exceeds the gravitational force from the external mass distribution. Arguably the most important example of self-gravitating, astrophysical fluids are stars. But Cold Dark Matter halos are also examples of self-gravitating fluids (albeit collisionless). The interstellar medium (ISM) can and cannot be self-gravitating, depending on the conditions. The intracluster medium (ICM) is generally not self-gravitating; rather the gravitating potential is dominated by the dark matter.

Gravitational Potential: Gravity is a conservative force, which means that it can be written as the gradient of a scalar field. Newton's gravitational potential, $\Phi(\vec{x})$, is defined such that the gravitational force per unit mass

$$\vec{F}_{\rm g} = -\nabla\Phi$$

Note that the absolute normalization of Φ has no physical relevance; only the gradients of Φ matter.

Consider a density distribution $\rho(\vec{x})$. What is the gravitational force $\vec{F}_{\rm g}$ acting on a particle of mass m at location $\vec{x}$? We can sum the small constributions $\delta\vec{F}_{\rm g}$ from different regions $\vec{x}\,' \pm {\rm d}^3\vec{x}\,'$, given by

$$\delta\vec{F}_{\rm g}(\vec{x}) = G\frac{m\,\delta m(\vec{x}\,')}{|\vec{x}\,'-\vec{x}|^2}\,\frac{\vec{x}\,'-\vec{x}}{|\vec{x}\,'-\vec{x}|} = Gm\,\frac{\vec{x}\,'-\vec{x}}{|\vec{x}\,'-\vec{x}|^3}\rho(\vec{x}\,'){\rm d}^3\vec{x}\,'$$

Adding up all the small contributions yields $\vec{F}_{\rm g}(\vec{x}) = \int\delta\vec{F}_{\rm g}(\vec{x}) \equiv m\,\vec{g}(\vec{x})$, where

$$\vec{g}(\vec{x}) = G\int{\rm d}^3\vec{x}\,'\frac{\vec{x}\,'-\vec{x}}{|\vec{x}\,'-\vec{x}|^3}\rho(\vec{x}\,')$$

is the **gravitational field** (i.e., the force per unit mass). Using that

$$\frac{\vec{x}\,' - \vec{x}}{|\vec{x}\,' - \vec{x}|^3} = \nabla_x \left(\frac{1}{|\vec{x}\,' - \vec{x}|} \right)$$

we can rewrite $g(\vec{x})$ as

$$\vec{g}(\vec{x}) = G \int \mathrm{d}^3\vec{x}\,' \nabla_x \left(\frac{1}{|\vec{x}\,' - \vec{x}|} \right) \rho(\vec{x}\,') = \nabla_x \int \mathrm{d}^3\vec{x}\,' \frac{G\rho(\vec{x}\,')}{|\vec{x}\,' - \vec{x}|} \equiv -\nabla_x \Phi$$

where in the last step we have defined the gravitational potential

$$\Phi(\vec{x}) = -G \int \mathrm{d}^3\vec{x}\,' \frac{\rho(\vec{x}\,')}{|\vec{x}\,' - \vec{x}|}$$

Poisson Equation: It can be shown that the gravitational potential obeys the **Poisson equation**:

$$\nabla^2 \Phi = 4\pi \, G \, \rho$$

For a derivation, see Section 3.2 of `Astrophysical Fluid Dynamics` by Clarke & Carswell, or Section 2.1 of `Galactic Dynamics` by Binney & Tremaine.

In general, it is extremely complicated to solve the **Poisson equation** for $\Phi(\vec{x})$ given $\rho(\vec{x})$ [see Chapter 2 of `Galactic Dynamics` by Binney & Tremaine for a detailed discussion]. However, under certain symmetries, solutions to the Poisson equation are fairly straightforward. In particular, under **spherical symmetry** the general solution to the **Poisson equation** is

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r') \, r'^2 \, \mathrm{d}r' + \int_r^\infty \rho(r') \, r' \, \mathrm{d}r' \right]$$

Note that the potential at r depends on the mass distribution outside of r. However, if we now compute the gravitational force per unit mass

$$\vec{F}_\mathrm{g}(r) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} \, \hat{e}_r = -\frac{G \, M(r)}{r^2} \, \hat{e}_r$$

where

$$M(r) \equiv 4\pi \int_0^r \rho(r') \, r'^2 \, \mathrm{d}r$$

is the **enclosed mass** within r. This shows that the gravitational force does *not* depend on the mass distribution outside of r.

Newton's first theorem: a body that is inside a spherical shell of matter experiences no net gravitational force from that shell. The equivalent in general relativity is called **Birkhoff's theorem**.

This is easily understood from the fact that the solid angles that extent from a point inside a sphere to opposing directions have areas on the sphere that scale as r^2 (where r is the distance from the point to the sphere), while the gravitational force per unit mass scales as r^{-2}. Hence, the gravitational forces from the two opposing areas exactly cancel.

Circular velocity: the velocity of a particle or fluid element on a circular orbit. For a spherical mass distribution

$$V_{\rm circ}(r) = \sqrt{r\,\frac{{\rm d}\Phi}{{\rm d}r}} = \sqrt{\frac{G\,M(r)}{r}}$$

In the case of an axisymmetric mass distribution, the cicular velocity in the **equatorial plane** ($z=0$, where z is one of the three cylindrical coordinates (R,ϕ,z)) is given by

$$V_{\rm circ}(R) = \sqrt{R\,\frac{{\rm d}\Phi}{{\rm d}R}} \neq \sqrt{\frac{G\,M(R)}{R}}$$

Escape velocity: the velocity needed for a particle or fluid element to escape to infinity. Since $E = v^2/2 + \Phi(\vec{x})$, and escape requires $E > 0$, the escape velocity is

$$V_{\rm esc}(\vec{x}) = \sqrt{2\,|\Phi(\vec{x})|}$$

independent of the symmetry (or lack thereof) of the mass distribution.

Since gas cannot be on self-intersecting orbits, gas in disk galaxies generally orbits on circular orbits. The measured rotation velocities therefore reflect the circular velocities, which can be used to infer the enclosed mass as function of radius. This method is generaly used to infer the presence of **dark matter haloes** surrounding disk galaxies.

Consider a gravitational system consisting of N particles (e.g., stars, fluid elements). The **total energy** of the system is $E = K + W$, where

$$\textbf{Total Kinetic Energy:} \quad K = \sum_{i=1}^{N} \tfrac{1}{2}\, m_i\, v_i^2$$

$$\textbf{Total Potential Energy:} \quad W = -\tfrac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \tfrac{G\, m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

The latter follows from the fact that gravitational binding energy between a pair of masses is proportional to the product of their masses, and inversely proportional to their separation. The factor 1/2 corrects for double counting the number of pairs.

Potential Energy in Continuum Limit: To infer an expression for the gravitational potential energy in the continuum limit, it is useful to rewrite the above expression as

$$W = \frac{1}{2} \sum_{i=1}^{N} m_i\, \Phi_i$$

where

$$\Phi_i = -\sum_{j=1}^{N} \frac{G\, m_j}{r_{ij}}$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$. In the continuum limit this simply becomes

$$W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, \mathrm{d}^3\vec{x}$$

One can show (see e.g., `Galactic Dynamics`) that this is equal to the trace of the **Chandrasekhar Potential Energy Tensor**

$$W_{ij} \equiv -\int \rho(\vec{x})\, x_i\, \frac{\partial \Phi}{\partial x_j}\, \mathrm{d}^3\vec{x}$$

In particular,

$$W = \mathrm{Tr}(W_{ij}) = \sum_{i=1}^{3} W_{ii} = -\int \rho(\vec{x})\, \vec{x} \cdot \nabla\Phi \, \mathrm{d}^3\vec{x}$$

which is another, equally valid, expression for the gravitational potential energy in the continuum limit.

Virial Theorem: A stationary, gravitational system obeys

$$\boxed{2K + W = 0}$$

Actually, the correct virial equation is $2K + W + \Sigma = 0$, *where* Σ *is the surface pressure. In many, but certainly not all, applications in astrophysics this term can be ignored. Many textbooks don't even mention the surface pressure term.*

Combining the **virial equation** with the expression for the total energy, $E = K + W$, we see that for a system that obeys the virial theorem

$$\boxed{E = -K = W/2}$$

Example: Consider a cluster consisting of N galaxies. If the cluster is in virial equilibrium then

$$2\sum_{i=1}^{N}\frac{1}{2}m\,v_i^2-\frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\frac{G\,m_i m_j}{r_{ij}}=0$$

If we assume, for simplicity, that all galaxies have equal mass then we can rewrite this as

$$N\,m\,\frac{1}{N}\sum_{i=1}^{N}v_i^2-\frac{G\,(Nm)^2}{2}\,\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\neq i}\frac{1}{r_{ij}}=0$$

Using that $M=N\,m$ and $N(N-1)\simeq N^2$ for large N, this yields

$$M=\frac{2\,\langle v^2\rangle}{G\,\langle 1/r\rangle}$$

with

$$\langle 1/r\rangle=\frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j\neq i}\frac{1}{r_{ij}}$$

It is useful to define the **gravitational radius** $r_{\rm g}$ such that

$$W=-\frac{G\,M^2}{r_{\rm g}}$$

Using the relations above, it is clear that $r_{\rm g}=2/\langle 1/r\rangle$. We can now rewrite the above equation for M in the form

$$M=\frac{r_{\rm g}\langle v^2\rangle}{G}$$

Hence, one can infer the mass of our cluster of galaxies from its velocity dispersion and its gravitation radius. In general, though, neither of these is observable, and one uses instead

$$M=\alpha\frac{R_{\rm eff}\langle v_{\rm los}^2\rangle}{G}$$

where $v_{\rm los}$ is the line-of-sight velocity, $R_{\rm eff}$ is some measure for the 'effective' radius of the system in question, and α is a parameter of order unity that depends on the radial distribution of the galaxies. Note that, under the assumption of isotropy, $\langle v_{\rm los}^2 \rangle = \langle v^2 \rangle/3$ and one can also infer the mean reciprocal pair separation from the projected pair separations; in other words under the assumption of isotropy one can infer α, and thus use the above equation to compute the total, gravitational mass of the cluster. This method was applied by **Fritz Zwicky** in 1933, who inferred that the total dynamical mass in the Coma cluster is much larger than the sum of the masses of its galaxies. This was the first observational evidence for **dark matter**, although it took the astronomical community until the late 70's to generally accept this notion.

For a self-gravitating fluid

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i \, v_i^2 = \frac{1}{2} N \, m \, \langle v^2 \rangle = \frac{3}{2} N \, k_{\rm B} \, T$$

where the last step follows from what we have learned in Chapter 13 about ideal gases of monoatomic particles. In fact, we can use the above equation for *any* fluid (including a collisionless one), if we interpret T as an effective temperature that measures the rms velocity of the constituent particles. If the system is in virial equilibrium, then

$$E = -K = -\frac{3}{2} N \, k_{\rm B} \, T$$

which, as we show next, has some important implications...

Heat Capacity: the amount of heat required to increase the temperature by one degree Kelvin (or Celsius). For a self-gravitating fluid this is

$$C \equiv \frac{{\rm d}E}{{\rm d}T} = -\frac{3}{2} N \, k_{\rm B}$$

which is negative! This implies that **by loosing energy, a gravitational system gets hotter**!! This is a very counter-intuitive result, that often leads to confusion and wrong expectations. Below we give two examples of implications of the **negative heat capacity** of gravitating systems,

Example 1: Drag on satellites Consider a satellite orbiting Earth. When it experiences friction against the (outer) atmosphere, it looses energy. This causes the system to become more strongly bound, and the orbital radius to shrink. Consequently, the energy loss results in the gravitational potential energy, W, becoming more negative. In order for the satellite to re-establish virial equilibrium ($2K+W=0$), its kinetic energy needs to <u>increase</u>. Hence, contrary to common intuition, *friction causes the satellite to speed up*, as it moves to a lower orbit (where the circular velocity is higher).

Example 2: Stellar Evolution A star is a gaseous, self-gravitating sphere that radiates energy from its surface at a luminosity L. Unless this energy is replenished (i.e., via some energy production mechanism in the star's interior), the star will react by shrinking (i.e., the energy loss implies an increase in binding energy, and thus a potential energy that becomes more negative). In order for the star to remain in virial equilibrium its kinetic energy, which is proportional to temperature, has to increase; *the star's energy loss results in an increase of its temperature.*

In the Sun, hydrogren burning produces energy that replenishes the energy loss from the surface. As a consequence, the system is in equilibrium, and will not contract. However, once the Sun has used up all its hydrogren, it will start to contract and heat up, because of the **negative heat capacity**. This continues until the temperature in the core becomes sufficiently high that helium can start to fuse into heavier elements, and the Sun settles in a new equilibrium.

Example 3: Core Collapse *a system with negative heat capacity in contact with a heat bath is thermodynamically unstable.* Consider a self-gravitating fluid of 'temperature' T_1, which is in contact with a heat bath of temperature T_2. Suppose the system is in thermal equilibrium, so that $T_1 = T_2$. If, due to some small disturbance, a small amount of heat is tranferred from the system to the heat bath, the negative heat capacity implies that this results in $T_1 > T_2$. Since heat always flows from hot to cold, more heat will now flow from the system to the heat bath, further increasing the temperature difference, and T_1 will continue to rise without limit. This run-away instability is called the **gravothermal catastrophe**. An example of this instability is the **core collapse of globular clusters:** Suppose the formation of a gravitational system results in the system having a declining velocity dispersion profile, $\sigma^2(r)$ (i.e., σ decreases with increasing radius). This implies that the central region is (dynamically) hotter than the outskirts. **IF** heat can flow from the center to those outskirts, the gravothermal catastrophe kicks in, and σ in the central regions will grow without limits. Since $\sigma^2 = GM(r)/r$, the central mass therefore gets compressed into a smaller and smaller region, while the outer regions expand. This is called **core collapse**. Note that this does NOT lead to the formation of a supermassive black hole, because regions at smaller r always shrink faster than regions at somewhat larger r. In dark matter haloes, and elliptical galaxies, the velocity dispersion profile is often declining with radius. However, in those systems the two-body relaxation time is soo long that there is basically no heat flow (which requires two-body interactions). However, globular clusters, which consist of $N \sim 10^4$ stars, and have a crossing time of only $t_{\rm cross} \sim 5 \times 10^6$yr, have a two-body relaxation time of only $\sim 5 \times 10^8$yr. Hence, heat flow in globulars is not negligible, and they can (and do) undergo core collapse. The collapse does not proceed indefinitely, because of binaries (see `Galactic Dynamics` by Binney & Tremaine for more details).