CHAPTER 14

The Energy Equation

Heat Transfer: In order to close the fluid equations, we need to add an equation that describes how the internal energy (heat) of a fluid element changes as function of time. There are four fundamental modes of heat transfer:

- **Radiation:** the transfer of energy to and from a fluid element by means of absorption or emission of electro-magnetic radiation.
- **Advection:** the transfer of energy from one location to another as a side effect of physically moving a fluid element containing that energy.
- **Conduction:** the transfer of energy between fluid elements that are in physical contact due to microscopic diffusion (requires temperature gradients).
- **Convection:** the transfer of energy between a fluid element and its environment due to bulk motion plus diffusion (i.e., convection is simply a combination of advection and conduction). Convection occurs whenever the temperature gradient becomes too large (Schwarzschild's stability criterion; see Chapter 19).

Another mode of energy transfer that is relevant for astronomy is the heating due to **cosmic rays**, which are energetic elementary particles (mainly protons) that have been accelerated to relativistic speeds by shocks from supernova etc. In what follows, we will treat cosmic ray heating as a component of radiative heating.

Energy Density: The energy density, E, of a fluid consists of three components: kinetic energy, potential energy, and internal energy:

$$E = \rho\left(\frac{1}{2}u^2 + \Phi + \varepsilon\right)$$

where ε is the **specific internal energy** of the fluid. Note that E as defined here is the energy <u>per unit volume</u>.

Energy equation: The Lagrangian derivate of the energy density is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{E}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\vec{u}\cdot\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \rho\,\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \rho\,\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

which simply follows from applying the chain rule to $E = \rho\left(\frac{1}{2}u^2 + \Phi + \varepsilon\right)$.

We now treat each of these four terms in turn:

1st term: Using the **continuity equation** we have that

$$\frac{E}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -E\,\nabla\cdot\vec{u}$$

2nd term: Using the **(Euler) momentum equation** we have that

$$\rho\,\vec{u}\cdot\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = \vec{u}\cdot\rho\,\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\vec{u}\cdot(\nabla P + \rho\nabla\Phi)$$

3rd term: Using the expression for the **substantial (Lagrangian) derivative** we have that

$$\rho\,\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \rho\,\frac{\partial\Phi}{\partial t} + \rho\,\vec{u}\cdot\nabla\Phi$$

4th term: Using the **first law of thermodynamics**, $\mathrm{d}\varepsilon = \mathrm{d}Q - \mathrm{d}W$, where $\mathrm{d}Q$ is the **specific heat** absorbed and $\mathrm{d}W = P\mathrm{d}(1/\rho)$ is the **specific work** done by the fluid, we have that

$$\rho\,\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \rho\,\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{P}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

Combining all the above, and using that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial t} + \vec{u} \cdot \nabla E$$

we finally obtain the **energy equation for an inviscid fluid**:

$$\boxed{\frac{\partial E}{\partial t} + \nabla \cdot [(E + P)\vec{u}] = -\mathcal{L} + \rho \frac{\partial \Phi}{\partial t}}$$

where we have defined the **net heating rate per unit volume**

$$\mathcal{L} \equiv \rho \frac{\mathrm{d}Q}{\mathrm{d}t} \equiv \mathcal{C} - \mathcal{H}$$

where $\mathcal{C}$ and $\mathcal{H}$ are the volumetric cooling and heating rates, respectively, which express heat transfer due to the emission and/or absorption of radiation (and cosmic rays).

Note that the external (gravitational) potential only enters with a partial time-derivative. Hence, only when the external potential varies with time, does it have an impact on the evolution of the total energy density of fluid elements. If the potential is steady (i.e., $\partial\Phi/\partial t = 0$, then the presence of the gravitational potential can cause the convertion of kinetic energy into potential energy (and vice versa), but it does not change the total energy density. Changes in the energy of individual fluid elements due to a time-variable gravitational potential is called **violent relaxation**, and is the main relaxation mechanisms for collisionless systems.

Using that $E = \rho\left(\frac{1}{2}u^2 + \Phi + \varepsilon\right)$, the energy equation can also be written as:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2} + \varepsilon\right)\right] + \frac{\partial}{\partial x_k}\left[\rho\left(\frac{u^2}{2} + \varepsilon\right) u_k + P\,\delta_{jk}\,u_j\right] = -\mathcal{L} - \rho\,u_k\,\frac{\partial \Phi}{\partial x_k}$$

In deriving the above form of the energy equation we have used that

$$\begin{aligned}
\rho\frac{\partial\Phi}{\partial t} - \frac{\partial\rho\Phi}{\partial t} - \frac{\partial\rho\Phi u_k}{\partial x_k} &= -\Phi\frac{\partial\rho}{\partial t} - \nabla\cdot(\rho\Phi\vec{u}) \\
&= -\Phi\frac{\partial\rho}{\partial t} - \Phi\nabla\cdot\rho\vec{u} - \rho\vec{u}\,\nabla\Phi \\
&= -\Phi\left[\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{u}\right] - \rho\vec{u}\,\nabla\Phi == -\rho\vec{u}\nabla\Phi
\end{aligned}$$

where, in the final step, we have used the **continuity equation**. One of the advantages of this index-form, is that it is easier to incorporate the effects of **viscosity**. By replacing $-P\,\delta_{ij}$ with the **stress tensor** $\sigma_{ij} = -P\,\delta_{ij} + \tau_{ij}$, and adding a term describing **conduction**, we obtain the fully general **energy equation for a viscous fluid**:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2}+\varepsilon\right)\right] = \\
&\quad -\frac{\partial}{\partial x_k}\left[\rho\left(\frac{u^2}{2}+\varepsilon\right)u_k + (P\,\delta_{jk} - \tau_{jk})\,u_j + F_{\mathrm{cond},k}\right] - \mathcal{L} - \rho\,u_k\,\frac{\partial\Phi}{\partial x_k}
\end{aligned}$$

The $\rho\left(\frac{u^2}{2}+\varepsilon\right)u_k$ term on the rhs describes **advection**, the $P\,\delta_{jk}\,u_j$ term describes the **work done**, the $\tau_{jk}\,u_j$ term describes **viscous dissipation** (i.e., the convertion of ordered bulk motion into disordered random motion), $F_{\mathrm{cond},k}$ is the **conduction flux** in the k-direction, $\mathcal{L}$ describes the change in (internal) energy due to the absorption or emission of radiation (or cosmic rays), and the last term on the rhs describes the change of energy due to motion in a gravitational potential.

Conduction: to first order in the ratio of the mean free path l of the particles and the length scale L of the physical system, the **conduction heat flux** can be written as

$$\vec{F}_{\mathrm{cond}} = -\mathcal{K}\,\nabla T$$

where $\mathcal{K}$ is called the **thermal conductivity** and has units of $\mathrm{erg\,s^{-1}\,cm^{-1}\,K^{-1}}$. It is roughly given by $\mathcal{K} \sim \frac{3}{2}\,k_{\mathrm{B}}\,n\,v_{\mathrm{th}}\,l$, where $v_{\mathrm{th}} \propto T^{1/2}$ is the thermal (microscopic) velocity of the particles. Using that the mean free path $l = (n\,\sigma)^{-1}$, with σ the collision cross section, we thus see that $\mathcal{K} \propto T^{1/2}/\sigma$. As expected, conduction increases with temperature (particles move faster) and decreases with increasing cross section (particles move less far).

To see another 'representation' of the conductivity, which links it directly to the microscopic motion of the fluid particles, we now (for the sake of completeness) derive the above energy equation starting from the **master moment equation**

$$\frac{\partial}{\partial t}[n\langle Q\rangle] + \frac{\partial}{\partial x_k}[n\langle v_k Q\rangle] + n\frac{\partial \Phi}{\partial x_k}\left\langle \frac{\partial Q}{\partial v_k}\right\rangle = 0$$

derived in Lecture 9 from the **Boltzmann equation**. For the energy equation, we need to set

$$Q = \frac{1}{2}mv^2 = \frac{m}{2}v_i v_i = \frac{m}{2}(u_i + w_i)(u_i + w_i) = \frac{m}{2}(u^2 + 2u_i w_i + w^2)$$

Hence, we have that $\langle Q\rangle = \frac{1}{2}mu^2 + \frac{1}{2}m\langle w^2\rangle$ where we have used that $\langle u\rangle = u$ and $\langle w\rangle = 0$. Using that $\rho = m\,n$, the first term in the master moment equation thus becomes

$$\frac{\partial}{\partial t}[n\langle Q\rangle] = \frac{\partial}{\partial t}\left[\rho\frac{u^2}{2} + \rho\varepsilon\right]$$

where we have used that the specific internal energy $\varepsilon = \frac{1}{2}\langle w^2\rangle$. For the second term, we use that

$$\begin{aligned}
n\langle v_k Q\rangle &= \frac{\rho}{2}\langle (u_k + w_k)(u^2 + 2u_i w_i + w^2)\rangle \\
&= \frac{\rho}{2}\langle u^2 u_k + 2u_i u_k w_i + w^2 u_k + u^2 w_k + 2u_i w_i w_k + w^2 w_k\rangle \\
&= \frac{\rho}{2}\left[u^2 u_k + u_k\langle w^2\rangle + 2u_i\langle w_i w_k\rangle + \langle w^2 w_k\rangle\right] \\
&= \rho\frac{u^2}{2}u_k + \rho\varepsilon u_k + \rho u_i\langle w_i w_k\rangle + F_{\mathrm{cond},k}
\end{aligned}$$

Here we have defined the **conductivity**

$$F_{\mathrm{cond},k} \equiv \rho\langle w_k \frac{1}{2} w^2\rangle = \langle \rho\varepsilon w_k\rangle$$

This makes it clear that conduction describes how internal energy is dispersed due to the random motion of the fluid particles. Using that $\rho\langle w_i w_k\rangle = -\sigma_{ik} = P\delta_{ik} - \tau_{ik}$, the second term of the master moment equation becomes

$$\frac{\partial}{\partial x_k}\left[n\langle v_k Q\rangle\right] = \frac{\partial}{\partial x_k}\left[\rho\frac{u^2}{2}u_k + \rho\varepsilon u_k + (P\delta_{ik} - \tau_{ik}) + F_{\mathrm{cond},k}\right]$$

Finally, for the third term we use that

$$\frac{\partial Q}{\partial v_k} = \frac{m}{2}\frac{\partial v^2}{\partial v_k} = m v_k$$

To understand the last step, note that in Cartesian coordinates $v^2 = v_x^2 + v_y^2 + v_z^2$. Hence, we have that

$$n\frac{\partial \Phi}{\partial x_k}\left\langle\frac{\partial Q}{\partial v_k}\right\rangle = \rho\frac{\partial \Phi}{\partial x_k}\langle v_k\rangle = \rho\frac{\partial \Phi}{\partial x_k}u_k$$

Combining the three terms in the master moment equation, we finally obtain the following **energy equation**:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2} + \varepsilon\right)\right] = \\ -\frac{\partial}{\partial x_k}\left[\rho\left(\frac{u^2}{2} + \varepsilon\right)u_k + (P\,\delta_{jk} - \tau_{jk})\,u_j + F_{\mathrm{cond},k}\right] - \rho\,u_k\,\frac{\partial \Phi}{\partial x_k}$$

which is exactly the same as that derived above, except for the $-\mathcal{L}$ term, which is absent from the derivation based on the Boltzmann equation, since the later does not include the effects of radiation.

The final task of this lecture on the energy equation is to derive an equation that describes the evolution of the **internal energy**. This is obtained by subtracting u_i times the **Navier-Stokes equation** in conservative, Eulerian form from the energy equation derived above.
The Navier-Stokes equation in Eulerian index form is

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} = \frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k} - \frac{\partial \Phi}{\partial x_i}$$

Using the **continuity equation**, this can be rewritten in the so-called **conservation form** as

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_k}\left[\rho u_i u_k - \sigma_{ik}\right] = -\rho \frac{\partial \Phi}{\partial x_i}$$

Next we multiply this equation with u_i. Using that

$$\begin{aligned}
u_i \frac{\partial \rho u_i}{\partial t} &= \frac{\partial \rho u^2}{\partial t} - \rho u_i \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] + \frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] - \rho u_i \frac{\partial u_i}{\partial t} \\
&= \frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] + \frac{\rho}{2}\frac{\partial u^2}{\partial t} + \frac{u^2}{2}\frac{\partial \rho}{\partial t} - \rho u_i \frac{\partial u_i}{\partial t} \\
&= \frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] + \frac{u^2}{2}\frac{\partial \rho}{\partial t}
\end{aligned}$$

where we have used that $\partial u^2/\partial t = 2 u_i \partial u_i/\partial t$. Similarly, we have that

$$\begin{aligned}
u_i \frac{\partial}{\partial x_k} &= \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] + \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] - \rho u_i u_k \frac{\partial u_i}{\partial x_k} \\
&= \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] + \frac{\rho}{2} u_k \frac{\partial u^2}{\partial x_k} + \frac{u^2}{2}\frac{\partial \rho u_k}{\partial x_k} - \rho u_i u_k \frac{\partial u_i}{\partial x_k} \\
&= \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] + \frac{u^2}{2}\frac{\partial \rho u_k}{\partial x_k}
\end{aligned}$$

Combining the above two terms, and using the **continuity equation** to expose of the two terms containing the factor $u^2/2$, the Navier-Stokes equation in conservation form multiplied by u_i becomes

$$\frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] + \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] = u_i \frac{\partial \sigma_{ik}}{\partial x_k} - \rho u_i \frac{\partial \Phi}{\partial x_i}$$

Subtracting this from the energy equation ultimately yields the **internal energy equation** in Eulerian index form:

$$\boxed{\frac{\partial}{\partial t}(\rho \varepsilon) + \frac{\partial}{\partial x_k}(\rho \varepsilon u_k) = -P \frac{\partial u_k}{\partial x_k} + \mathcal{V} - \frac{\partial F_{\mathrm{cond},k}}{\partial x_k} - \mathcal{L}}$$

where

$$\mathcal{V} \equiv \pi_{ik} \frac{\partial u_i}{\partial x_k}$$

is the **rate of viscous dissipation** which describes the rate at which the work done against viscous forces is irreversibly converted into internal energy. In words, the internal energy equation states that the internal energy at some fixed location in space changes due to **advection**, (described by the $\partial \rho \varepsilon u_k / \partial x_k$ term), due to the **work done** (described by the $P(\partial u_k / \partial x_k)$ term), due to **radiation** (described by the $-\mathcal{L}$ term), due to **conduction** (described by the $\partial F_{\mathrm{cond},k} / \partial x_k$ term) and due to **viscous dissipation** (described by the $\mathcal{V}$ term). The latter term describes the rate at which heat is added to the internal energy budget via viscous conversion of ordered energy in differential fluid motions to disordered energy in random particle motions. Finally, we mention that the Lagrangian vector form of the **internal energy equation** is given by

$$\boxed{\rho \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -P\, \nabla \cdot \vec{u} - \nabla \cdot \vec{F}_{\mathrm{cond}} - \mathcal{L} + \mathcal{V}}$$

Note that in this Lagrangian form, there is no term describing **advection**; after all, we are moving with the fluid.