CHAPTER 13

Equations of State

Closure: The continuity and momentum (Euler) equations are 4 equations with 6 unknowns (ρ, $\vec{u}$, P, and Φ). With the **Poisson equation**, which relates ρ and Φ, we are still one equation short for closure. This equation can either be an **equation of state** (but only if it is barotropic, i.e., $P = P(\rho)$), or the **energy equation** (see Chapter 14).

Equation of State (EoS): a thermodynamic equation describing the state of matter under a given set of physical conditions. In what follows we will always write our EoS in the form $P = P(\rho, T)$. Other commonly used forms are $P = P(\rho, \varepsilon)$ or $P = P(\rho, S)$.

Ideal Gas: a hypothetical gas that consists of identical point particles (i.e. of zero volume) that undergo perfectly elastic collisions and for which interparticle forces can be neglected.

An ideal gas obeys the **ideal gas law:** $P\,V = N\,k_{\rm B}\,T$.

Here N is the total number of particles, $k_{\rm B}$ is Boltzmann's constant, and V is the volume occupied by the fluid. Using that $\rho = N\,\mu m_{\rm p}/V$, where μ is the **mean molecular weight** in units of the proton mass $m_{\rm p}$, we have that the **EoS for an ideal gas** is given by

$$P = \frac{k_{\rm B}\,T}{\mu\,m_{\rm p}}\,\rho$$

`NOTE:` astrophysical gases are often well described by the ideal gas law. Even for a fully ionized gas, the interparticle forces (Coulomb force) can typically be neglected (i.e., the potential energies involved are typically $< 10\%$ of the kinetic energies). Ideal gas law brakes down for dense, and cool gases, such as those present in gaseous planets.

Maxwell-Boltzmann Distribution: the distribution of particle momenta, $\vec{p} = m\vec{v}$, of an ideal gas follows the Maxwell-Boltzmann distribution.

$$\mathcal{P}(\vec{p})\,\mathrm{d}^3\vec{p} = \left(\frac{1}{2\pi m k_{\mathrm{B}} T}\right)^{3/2} \exp\left(-\frac{p^2}{2 m k_{\mathrm{B}} T}\right)\,\mathrm{d}^3\vec{p}$$

where $p^2 = \vec{p}\cdot\vec{p}$. This distribution follows from maximizing entropy under the following assumptions:

1. all magnitudes of velocity are *a priori* equally likely
2. all directions are equally likely (isotropy)
3. total energy is constrained at a fixed value
4. total number of particles is constrained at a fixed value

Using that $E = p^2/2m$ we thus see that $\mathcal{P}(\vec{p}) \propto e^{-E/k_{\mathrm{B}}T}$.

Pressure: pressure arises from (elastic) collisions of particles. A particle hitting a wall head on with momentum $p = mv$ results in a transfer of momentum to the wall of $2mv$. Using this concept, and assuming isotropy for the particle momenta, it is fairly straightforward to show that

$$P = \zeta\, n\, \langle E\rangle$$

where $\zeta = 2/3$ ($\zeta = 1/3$) in the case of a non-relativistic (relativistic) fluid, and

$$\langle E\rangle = \int_0^\infty E\,\mathcal{P}(E)\,\mathrm{d}E$$

is the average, translational energy of the particles. In the case of our ideal (non-relativistic) fluid,

$$\langle E\rangle = \left\langle \frac{p^2}{2m}\right\rangle = \int_0^\infty \frac{p^2}{2m}\,\mathcal{P}(p)\,\mathrm{d}p = \frac{3}{2} k_{\mathrm{B}} T$$

Hence, we find that the **EoS for an ideal gas** is indeed given by

$$P = \frac{2}{3}\, n \, \langle E \rangle = n \, k_{\rm B} \, T = \frac{k_{\rm B} T}{\mu m_{\rm p}} \, \rho$$

Specific Internal Energy: the internal energy per unit mass for an ideal gas is

$$\varepsilon = \frac{\langle E \rangle}{\mu m_{\rm p}} = \frac{3}{2} \frac{k_{\rm B} T}{\mu m_{\rm p}}$$

Actually, the above derivation is only valid for a true 'ideal gas', in which the particles are point particles. More generally,

$$\varepsilon = \frac{1}{\gamma - 1} \frac{k_{\rm B} T}{\mu m_{\rm p}}$$

where γ is the **adiabatic index**, which for an ideal gas is equal to $\gamma = (q+5)/(q+3)$, with q the internal degrees of freedom of the fluid particles: $q = 0$ for point particles (resulting in $\gamma = 5/3$), while diatomic particles have $q = 2$ (at sufficiently low temperatures, such that they only have rotational, and no vibrational degrees of freedom). The fact that $q = 2$ in that case arises from the fact that a diatomic molecule only has two relevant rotation axes; the third axis is the symmetry axis of the molecule, along which the molecule has negligible (zero in case of point particles) moment of inertia. Consequently, rotation around this symmetry axis carries no energy.

Photon gas: Having discussed the EoS of an ideal gas, we now focus on a gas of photons. Photons have energy $E = h\nu$ and momentum $p = E/c = h\nu/c$, with h the Planck constant.

Black Body: an idealized physical body that absorbs all incident radiation. A black body (BB) in thermal equilibrium emits electro-magnetic radiation called **black body radiation**.

The **spectral number density distribution** of BB photons is given by

$$n_\gamma(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/k_{\rm B}T} - 1}$$

which implies a **spectral energy distribution**

$$u(\nu, T) = n_\gamma(\nu, T)\, h\nu = \frac{8\pi h\nu^3}{c^3}\,\frac{1}{e^{h\nu/k_{\rm B}T} - 1}$$

and thus an **energy density** of

$$u(T) = \int_0^\infty u(\nu, T)\,{\rm d}\nu = \frac{4\sigma_{\rm SB}}{c}\,T^4 \equiv a_{\rm r}\,T^4$$

where

$$\sigma_{\rm SB} = \frac{2\pi^5 k_{\rm B}^4}{15 h^3 c^2}$$

is the **Stefan-Boltzmann constant** and $a_{\rm r} \simeq 7.6 \times 10^{-15} {\rm erg\,cm^{-3}\,K^{-4}}$ is called the **radiation constant**.

Radiation Pressure: when the photons are reflected off a wall, or when they are absorbed and subsequently remitted by that wall, they transfer twice their momentum in the normal direction to that wall. Since photons are relativistic, we have that **the EoS for a photon gas** is given by

$$P = \frac{1}{3}\,n\,\langle E \rangle = \frac{1}{3}\,n_\gamma\,\langle h\nu \rangle = \frac{1}{3}u(T) = \frac{aT^4}{3}$$

where we have used that $u(T) = n_\gamma\,\langle E \rangle$.

Quantum Statistics: according to quantum statistics, a collection of many indistinguishable elementary particles in thermal equilibrium has a momentum distribution given by

$$f(\vec{p})\,{\rm d}^3\vec{p} = \frac{g}{h^3}\left[\exp\left(\frac{E(p) - \mu}{k_{\rm B}T}\right) \pm 1\right]^{-1}{\rm d}^3\vec{p}$$

where the signature $\pm$ takes the positive sign for fermions (which have half-integer spin), in which case the distribution is called the **Fermi-Dirac distribution**, and the negative sign for bosons (particles with zero or integer spin), in which case the distribution is called the **Bose-Einstein distribution**. The factor g is the **spin degeneracy factor**, which expresses the number of spin states the particles can have ($g = 1$ for neutrinos, $g = 2$ for photons and charged leptons, and $g = 6$ for quarks). Finally, μ is called the **chemical potential**, and is a form of potential energy that is related (in a complicated way) to the number density and temperature of the particles.

Classical limit: In the limit where the mean interparticle separation is much larger than the de Broglie wavelength of the particles, so that quantum effects (e.g., Heisenberg's uncertainty principle) can be ignored, the above distribution function of momenta can be accurately approximated by the **Maxwell-Boltzmann distribution**.

Heisenberg's Uncertainty Principle: $\Delta x \; \Delta p_x > h$ (where $h = 6.63 \times 10^{-27} \mathrm{g\,cm^2\,s^{-1}}$ is Planck's constant). One interpretation of this quantum principle is that phase-space is quantized; no particle can be localized in a phase-space element smaller than the fundamental element

$$\Delta x \; \Delta y \; \Delta z \; \Delta p_x \; \Delta p_y \; \Delta p_z = h^3$$

Pauli Exclusion Principle: no more than one fermion of a given spin state can occupy a given phase-space element h^3. Hence, for electrons, which have $g = 2$, the maximum phase-space density is $2/h^3$.

Degeneracy: When compressing and/or cooling a fermionic gas, at some point all possible low momentum states are occupied. Any further compression therefore results in particles occupying high (but the lowest available) momentum states. Since particle momentum is ultimately responsible for pressure, this degeneracy manifests itself as an extremely high pressure, known as **degeneracy pressure**.

Fermi Momentum: Consider a <u>fully degenerate</u> gas of electrons of electron density n_e. It will have fully occupied the part of phase-space with momenta $p \leq p_{\rm F}$. Here $p_{\rm F}$ is the maximum momentum of the particles, and is called the **Fermi momentum**. The energy corresponding to the Fermi momentum is called the **Fermi energy**, $E_{\rm F}$ and is equal to $p_{\rm F}^2/2m$ in the case of a non-relativistic gas, and $p_{\rm F} c$ in the case of a relativistic gas.

Let V_x be the volume occupied in configuration space, and $V_p = \frac{4}{3}\pi p_{\rm F}^3$ the volume occupied in momentum space. If the total number of particles is N, and the gas is fully degenerate, then

$$V_x \, V_p = \frac{N}{2} h^3$$

Using that $n_e = N/V_x$, we find that

$$p_{\rm F} = \left(\frac{3}{8\pi} n_e \right)^{1/3} \, h$$

EoS of Non-Relativistic, Degenerate Gas: Using the information above, it is straightforward to compute the EoS for a fully degenerate gas. Using that for a non-relativistic fluid $E = p^2/2m$ and $P = \frac{2}{3}\, n \, \langle E \rangle$, while degeneracy implies that

$$\langle E \rangle = \frac{1}{N} \int_0^{E_{\rm f}} E \, N(E) \, {\rm d}E = \frac{1}{N} \int_0^{p_{\rm F}} \frac{p^2}{2m} \, \frac{2}{h^3} \, V_x \, 4\pi p^2 \, {\rm d}p = \frac{3}{5} \, \frac{p_{\rm F}^2}{2m}$$

we obtain that

$$P = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \, \frac{h^2}{m^{8/3}} \, \rho^{5/3}$$

EoS of Relativistic, Degenerate Gas: In the case of a relativistic, degenerate gas, we use the same procedure as above. However, this time we have that $P = \frac{1}{3}\, n \, \langle E \rangle$ while $E = p\, c$, which results in

$$P = \frac{1}{8} \left(\frac{3}{\pi} \right)^{1/3} \, \frac{c\, h}{m^{4/3}} \, \rho^{4/3}$$

White Dwarfs and the Chandrasekhar limit: White dwarfs are the end-states of stars with mass low enough that they don't form a neutron star. When the pressure support from nuclear fusion in a star comes to a halt, the core will start to contract until degeneracy pressure kicks in. The star consists of a fully ionized plasme. Assume for simplicity that the plasma consists purely of hydrogen, so that the number density of protons is equal to that of electrons: $n_{\rm p} = n_{\rm e}$. Because of **equipartition**

$$\frac{p_{\rm p}^2}{2m_{\rm p}} = \frac{p_{\rm e}^2}{2m_{\rm e}}$$

Since $m_{\rm p} >> m_{\rm e}$ we have also that $p_{\rm p} >> p_{\rm e}$ (in fact $p_{\rm p}/p_{\rm e} = \sqrt{m_{\rm p}/m_{\rm e}} \simeq 43$). Consequently, when cooling or compressing the core of a star, the electrons will become degenerate well before the protons do. Hence, white dwarfs are held up against collapse by the **degeneracy pressure from electrons**. Since the electrons are typically non-relativistic, the EoS of the white dwarf is: $P \propto \rho^{5/3}$. If the white dwarf becomes more and more massive (i.e., because it is accreting mass from a companion star), the Pauli-exclusion principle causes the Fermi momentum, $p_{\rm F}$, to increase to relativistic values. This **softens** the EoS towards $P \propto \rho^{4/3}$. Such an equation of state is too soft to stabilize the white dwarf against gravitational collapse; the white dwarf collapses until it becomes a **neutron star**, at which stage it is supported against further collapse by the degeneracy pressure from neutrons. This happens when the mass of the white dwarf reaches $M_{\rm lim} \simeq 1.44 M_{\odot}$, the so-called **Chandrasekhar limit**.