CHAPTER 12

Reynold's Number & Turbulence

Non-linearity: The Navier-Stokes equation is non-linear. This non-linearity arises from the convective (material) derivative term

$$\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2 - \vec{u} \times \vec{w}$$

which describes the "inertial acceleration" and is ultimately responsible for the origin of the **chaotic character** of many flows and of **turbulence**. Because of this non-linearity, we cannot say whether a solution to the Navier-Stokes equation with nice and smooth initial conditions will remain nice and smooth for all time (at least not in 3D).

Laminar flow: occurs when a fluid flows in parallel layers, without lateral mixing (no cross currents perpendicular to the direction of flow). It is characterized by high momentum diffusion and low momentum convection.

Turbulent flow: is characterized by chaotic and stochastic property changes. This includes low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time.

The Reynold's number: In order to gauge the importance of viscosity for a fluid, it is useful to compare the ratio of the inertial acceleration ($\vec{u} \cdot \nabla \vec{u}$) to the viscous acceleration ($\nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u})\right]$). This ratio is called the Reynold's number, $\mathcal{R}$, and can be expressed in terms of the typical velocity scale $U \sim |\vec{u}|$ and length scale $L \sim 1/\nabla$ of the flow, as

$$\mathcal{R} = \left| \frac{\vec{u} \cdot \nabla \vec{u}}{\nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u})\right]} \right| \sim \frac{U^2/L}{\nu U/L^2} = \frac{U\,L}{\nu}$$

If $\mathcal{R} \gg 1$ then viscosity can be ignored (and one can use the Euler equations to describe the flow). However, if $\mathcal{R} \ll 1$ then viscosity is important.

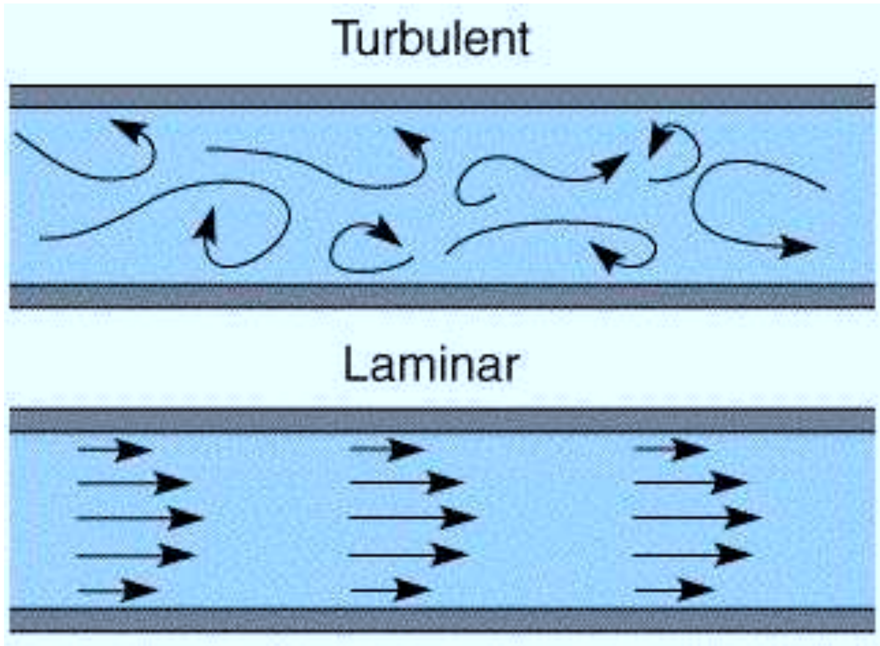

Figure 6: Illustration of laminar *vs.* turbulent flow.

Similarity: Flows with the same Reynold's number are similar. This is evident from rewriting the Navier-Stokes equation in terms of the following dimensionless variables

$$\tilde{u} = \frac{\vec{u}}{U} \qquad \tilde{x} = \frac{\vec{x}}{L} \qquad \tilde{t} = t\,\frac{U}{L} \qquad \tilde{p} = \frac{P}{\rho\, U^2} \qquad \tilde{\Phi} = \frac{\Phi}{U^2} \qquad \tilde{\nabla} = L\,\nabla$$

This yields (after multiplying the Navier-Stokes equation with L/U^2):

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \tilde{\nabla}\tilde{u} + \tilde{\nabla}\tilde{p} + \tilde{\nabla}\tilde{\Phi} = \frac{1}{\mathcal{R}}\tilde{\nabla}^2\tilde{u}$$

which shows that the form of the solution depends only on $\mathcal{R}$.

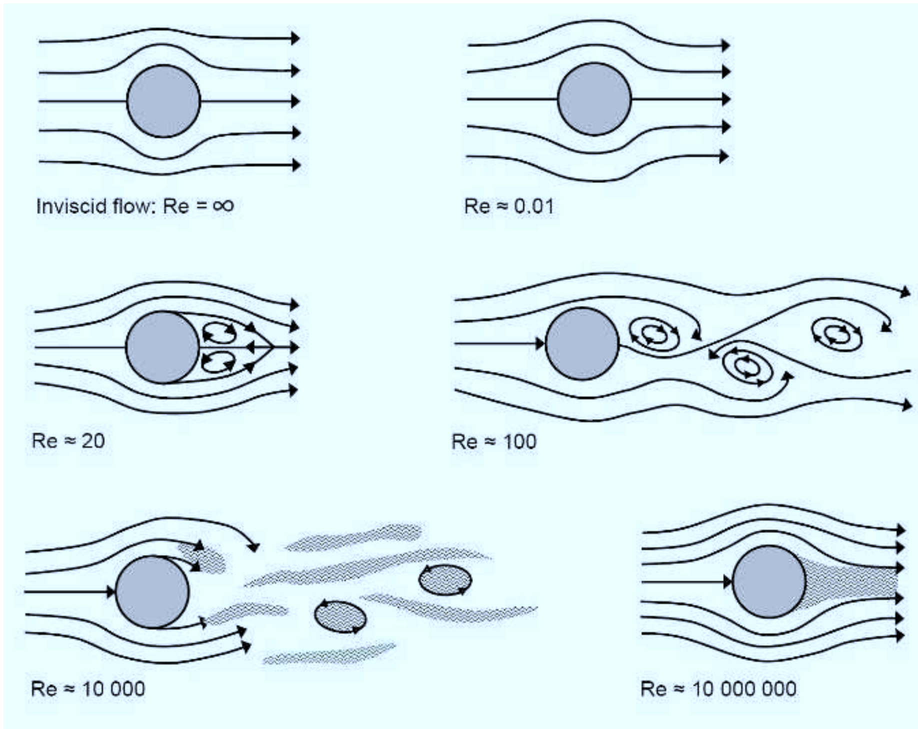

Figure 7: Illustration of flows at different Reynolds number.

As a specific example, consider fluid flow past a cylinder of diameter L:

- $\mathcal{R} \ll 1$: "creeping flow". In this regime the flow is viscously dominated and (nearly) symmetric upstream and downstream. The inertial acceleration $(\vec{u} \cdot \nabla\vec{u})$ can be neglected, and the flow is (nearly) time-reversible.
- $\mathcal{R} \sim 1$: Slight asymmetry develops
- $10 \leq \mathcal{R} \leq 41$: Separation occurs, resulting in two counter-rotating votices in the wake of the cylinder. The flow is still steady and laminar, though.
- $41 \leq \mathcal{R} \leq 10^3$: "von Kármán vortex street"; <u>unsteady laminar flow</u> with counter-rotating vortices shed periodically from the cylinder. Even at this stage the flow is still 'predictable'.
- $\mathcal{R} > 10^3$: vortices are unstable, resulting in a <u>turbulent wake</u> behind the cylinder that is 'unpredictable'.

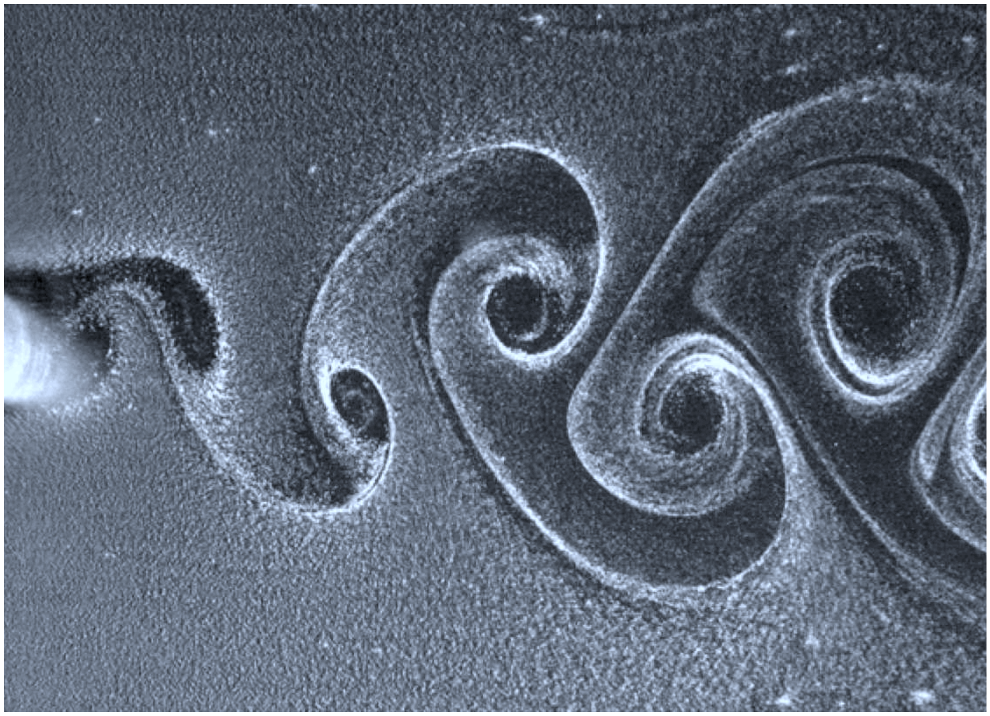

Figure 8: The image shows the von Kármán Vortex street behind a 6.35 mm diameter circular cylinder in water at Reynolds number of 168. The visualization was done using hydrogen bubble technique. `Credit:` Sanjay Kumar & George Laughlin, Department of Engineering, The University of Texas at Brownsville

The following movie shows a $\mathcal{R} = 250$ flow past a cylinder. Initially one can witness separation, and the creation of two counter-rotating vortices, which then suddenly become 'unstable', resulting in the von Kármán vortex street:
`http://www.youtube.com/watch?v=IDeGDFZSYo8`

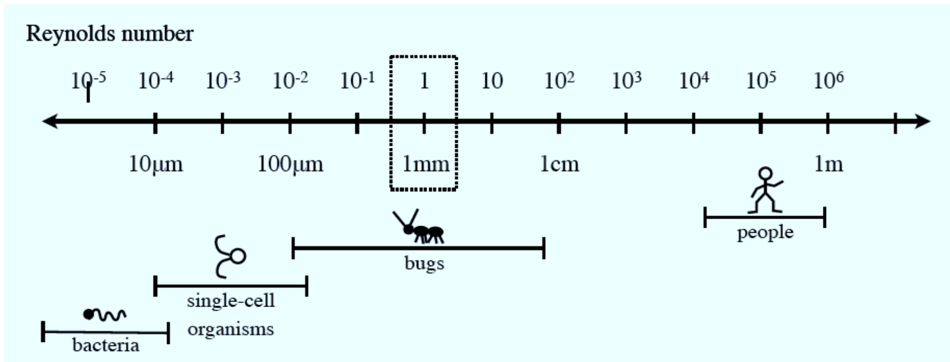

Figure 9: Typical Reynolds numbers for various biological organisms. Reynolds numbers are estimated using the length scales indicated, the rule-of-thumb in the text, and material properties of water.

Locomotion at Low-Reynolds number: Low Reynolds number corresponds to high kinetic visocisity *for a given U and L.* In this regime of 'creeping flow' the flow past an object is (nearly) time-reversible. Imagine trying to move (swim) in a highly viscous fluid (take honey as an example). If you try to do so by executing time-symmetric movements, you will not move. Instead, you need to think of a symmetry-breaking solution. Nature has found many solutions for this problem. If we make the simplifying "rule-of-thumb" assumption that an animal of size L meters moves roughly at a speed of $U = L$ meters per second (yes, this is very, very rough, but an ant *does* move close to 1 mm/s, and a human at roughly 1 m/s), then we have that $\mathcal{R} = UL/\nu \simeq L^2/\nu$. Hence, with respect to a fixed substance (say wayter, for which $\nu \sim 10^{-2}\text{cm}^2/s$), smaller organisms move at lower Reynolds number (effectively in a fluid of higher viscosity). Scaling down from a human to bacteria and single-cell organisms, the motion of the latter in water has $\mathcal{R} \sim 10^{-5} - 10^{-2}$. Understanding the locomotion of these organisms is a fascinating sub-branch of bio-physics.

Boundary Layers: Even when $\mathcal{R} \gg 1$, viscosity always remains important in thin boundary layers adjacent to any solid surface. This boundary layer must exist in order to satisfy the no-slip boundary condition. If the Reynolds number exceeds a critical value, the boundary layer becomes turbulent. Turbulent layes and their associated turbulent wakes exert a much bigger drag on moving bodies than their laminar counterparts.

Turbulence: Turbulence is still considered as one of the last "unsolved problems of classical physics" [Richard Feynman]. Indeed, it is an extremely difficult subject. Salmon (1998) nicely sums up the challenge of defining turbulence:

> Every aspect of turbulence is controversial. Even the definition of fluid turbulence is a subject of disagreement. However, nearly everyone would agree with some elements of the following description:
>
> - Turbulence requires the presence of vorticity; irrotational flow is smooth and steady to the extent that the boundary conditions permit.
> - Turbulent flow has a complex structure, involving a broad range of space and time scales.
> - Turbulent flow fields exhibit a high degree of apparent randomness and disorder. However, close inspection often reveals the presence of embedded cohererent flow structures
> - Turbulent flows have a high rate of viscous energy dissipation.
> - Advected tracers are rapidly mixed by turbulent flows.
>
> However, one further property of turbulence seems to be more fundamental that all of these because it largely explains why turbulence demands a statistical treatment...turbulence is chaotic.

The following is a brief, qualitative description of turbulence

Turbulence kicks in at sufficiently high Reyolds number (typically $\mathcal{R} > 10^3 - 10^4$). Turbulent flow is characterized by irregular and seemingly random motion. Large vortices (called **eddies**) are created. These contain a large amount of kinetic energy. Due to **vortex stretching** these eddies are stretched thin until they 'brake up' in smaller eddies. This results in a **cascade** in which the turbulent energy is transported from large scales to small scales. This cascade is largely **inviscid**, conserving the total turbulent energy. However, once the length scale of the eddies becomes comparable to the mean free path of the particles, the energy is dissipated; the kinetic energy associated with the eddies is transformed into internal energy. The scale at which this happens is called the **Kolmogorov length scale**.

Molecular clouds: an example of turbulence in astrophysics are molecular clouds. These are gas clouds of masses $10^5 - 10^6 M_\odot$, densities $n_H \sim 100 - 500\,\mathrm{cm}^{-3}$, and temperatures $T \sim 10$K. They consist mainly of molecular hydrogen and are the main sites of **star formation**. Observations show that their velocity linewidths are $\sim 6 - 10$km/s, which is much higher than their sound speed ($c_s \sim 0.2$km/s). Hence, they are supported against (gravitational) collapse by **supersonic turbulence**. On small scales, however, the turbulent motions compress the gas to high enough densities that stars can form. A numerical simulation of a molecular cloud with supersonic turbulence is available here:

`http://www.youtube.com/watch?v=3z9ZKAkbMhY`