CHAPTER 10

Vorticity & Circulation

Vorticity: The vorticity of a flow is defined as the curl of the velocity field:

$$\boxed{\textbf{vorticity}: \qquad \vec{w} = \nabla \times \vec{u}}$$

It is a microscopic measure of rotation (vector) at a given point in the fluid, which can be envisioned by placing a paddle wheel into the flow. If it spins about its axis at a rate Ω, then $w = |\vec{w}| = 2\Omega$.

Circulation: The circulation around a closed contour C is defined as the line integral of the velocity along that contour:

$$\boxed{\textbf{circulation}: \qquad \Gamma_C = \oint_C \vec{u} \cdot \mathrm{d}\vec{l} = \int_S \vec{w} \cdot \mathrm{d}\vec{S}}$$

where S is an *arbitrary* surface bounded by C. The circulation is a macroscopic measure of rotation (scalar) for a finite area of the fluid.

Irrotational fluid: An irrotational fluid is defined as being curl-free; hence, $\vec{w} = 0$ and therefore $\Gamma_C = 0$ for any C.

Vortex line: a line that points in the direction of the vortex vector. Hence is vortex line is to $\vec{w}$ what a streamlines is to $\vec{u}$. Note that a vortex line associated with a fluid line is always perpendicular to the streamline associated with that fluid element.

Vortex tube: a bundle of vortex lines. The circularity of a curve C is proportional to the number of vortex lines that thread the enclosed area.

In an inviscid fluid the vortex lines/tubes move **with** the fluid: a vortex line anchored to some fluid element remains anchored to that fluid element.

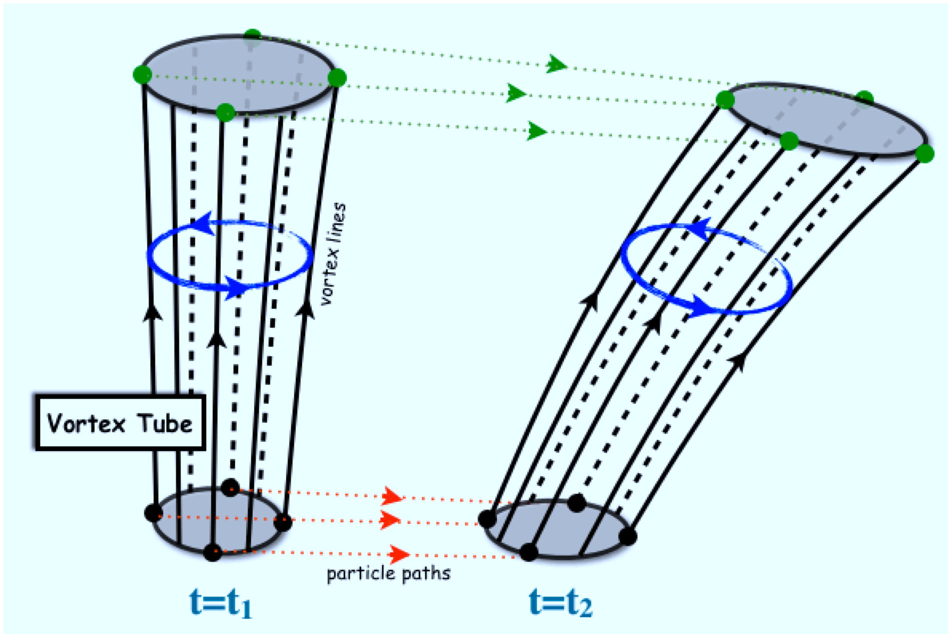

Figure 3: Evolution of a vortex tube. Solid dots correspond to fluid elements. Due to the shear in the velocity field, the vortex tube is stretched and tilted. However, as long as the fluid is inviscid and barotropic, incompressible or isobaric, Kelvin's circularity theorem assures that the circularity is conserved with time. In addition, since vorticity is divergence-free, the circularity along different cross sections of the same vortex-tube is the same.

Vorticity equation: The Navier-Stokes momentum equations, in the absence of bulk viscosity, in Eulerian vector form, are given by

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\,\vec{u} = -\frac{\nabla P}{\rho} - \nabla\Phi + \nu\left[\nabla^2\vec{u} + \frac{1}{3}\nabla(\nabla \cdot \vec{u})\right]$$

Using the vector identity $(\vec{u} \cdot \nabla)\,\vec{u} = \frac{1}{2}\nabla u^2 + (\nabla \times \vec{u}) \times \vec{u} = \nabla(u^2/2) - \vec{u} \times \vec{w}$ allows us to rewrite this as

$$\frac{\partial \vec{u}}{\partial t} - \vec{u} \times \vec{w} = -\frac{\nabla P}{\rho} - \nabla\Phi - \frac{1}{2}\nabla u^2 + \nu\left[\nabla^2\vec{u} + \frac{1}{3}\nabla(\nabla \cdot \vec{u})\right]$$

If we now take the curl on both sides of this equation, and we use that $\mathrm{curl}(\mathrm{grad}\, S) = 0$ for any scalar field S, and that $\nabla \times (\nabla^2\vec{A}) = \nabla^2(\nabla \times \vec{A})$, we obtain the **vorticity equation**:

$$\boxed{\frac{\partial \vec{w}}{\partial t} = \nabla \times (\vec{u} \times \vec{w}) - \nabla \left(\frac{\nabla P}{\rho} \right) + \nu \nabla^2 \vec{w}}$$

To write this in Lagrangian form, we first use that $\nabla \times (S\,\vec{A}) = \nabla S \times \vec{A} + S\,(\nabla \times \vec{A})$ [see Chapter 1] to write

$$\nabla \times (\frac{1}{\rho}\,\nabla P) = \nabla(\frac{1}{\rho}) \times \nabla P + \frac{1}{\rho}(\nabla \times \nabla P) = \frac{\rho \nabla(1) - 1 \nabla \rho}{\rho^2} \times \nabla P = \frac{\nabla P \times \nabla \rho}{\rho^2}$$

where we have used, once more, that curl(grad S) $= 0$. Next, using the vector identities from Chapter 1, we write

$$\nabla \times (\vec{w} \times \vec{u}) = \vec{w}(\nabla \cdot \vec{u}) - (\vec{w} \cdot \nabla)\vec{u} - \vec{u}(\nabla \cdot \vec{w}) + (\vec{u} \cdot \nabla)\vec{w}$$

The third term vanishes because $\nabla \cdot \vec{w} = \nabla \cdot (\nabla \times \vec{u}) = 0$. Hence, using that $\partial \vec{w}/\partial t - (\vec{u} \cdot \nabla)\vec{w} = \mathrm{d}\vec{w}/\mathrm{d}t$ we finally can write the **vorticity equation in Lagrangian form:**

$$\boxed{\frac{\mathrm{d}\vec{w}}{\mathrm{d}t} = (\vec{w} \cdot \nabla)\vec{u} - \vec{w}(\nabla \cdot \vec{u}) + \frac{\nabla \rho \times \nabla P}{\rho^2} + \nu \nabla^2 \vec{w}}$$

This equation describes how the vorticity of a fluid element evolves with time. We now describe the various terms of the *lhs* of this equation in turn:

- $(\vec{w} \cdot \nabla)\vec{u}$: This term represents the stretching and tilting of vortex tubes due to velocity gradients. To see this, we pick $\vec{w}$ to be pointing in the z-direction. Then

$$(\vec{w} \cdot \nabla)\vec{u} = w_z \frac{\partial \vec{u}}{\partial z} = w_z \frac{\partial u_x}{\partial z}\vec{e}_x + w_z \frac{\partial u_y}{\partial z}\vec{e}_y + w_z \frac{\partial u_z}{\partial z}\vec{e}_z +$$

  The first two terms on the *rhs* describe the tilting of the vortex tube, while the third term describes the stretching.

- $\vec{w}(\nabla \cdot \vec{u})$: This term describes stretching of vortex tubes due to flow compressibility. This term is zero for an incompressible fluid ($\nabla \cdot \vec{u} = 0$). Note that, again under the assumption that the vorticity is pointing in the z-direction,

$$\vec{w}(\nabla \cdot \vec{u}) = w_z \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right] \vec{e}_z$$

- $(\nabla\rho \times \nabla P)/\rho^2$: This is the baroclinic term. It describes the production of vorticity due to a misalignment between pressure and density gradients. This term is zero for a barotropic EoS: if $P = P(\rho)$ the pressure and density gradiens are parallel so that $\nabla P \times \nabla \rho = 0$. Obviously, this baroclinic also vanishes for an incompressible fluid ($\nabla\rho = 0$) or for an isobaric fluid ($\nabla P = 0$). The baroclinic term is responsible, for example, for creating vorticity in pyroclastic flows (see Figure 3).

- $\nu\nabla^2\vec{w}$: This term describes the diffusion pf vorticity due to viscosity, and is obvisouly zero for an inviscid fluid ($\nu = 0$). Typically, viscosity generates/creates vorticity at a bounding surface: due to the *no-slip* boundary condition shear arises giving rise to vorticity, which is subsequently diffused into the fluid by the viscosity. In the interior of a fluid, no new vorticity is generated; rather, viscosity diffuses and dissipates vorticity.

- $\nabla \times \vec{F}$: There is a fifth term that can create vorticity, which however does not appear in the vorticity equation above. The reason is that we assumed that the only external force is gravity, which is a conservative force and can therefore be written as the gradient of a (gravitational) potential. More generally, though, there may be non-conservative, external body forces present, which would give rise to a $\nabla \times \vec{F}$ term in the rhs of the vorticity equation. An example of a non-conservative force creating vorticity is the Coriolis force, which is responsible for creating hurricanes.

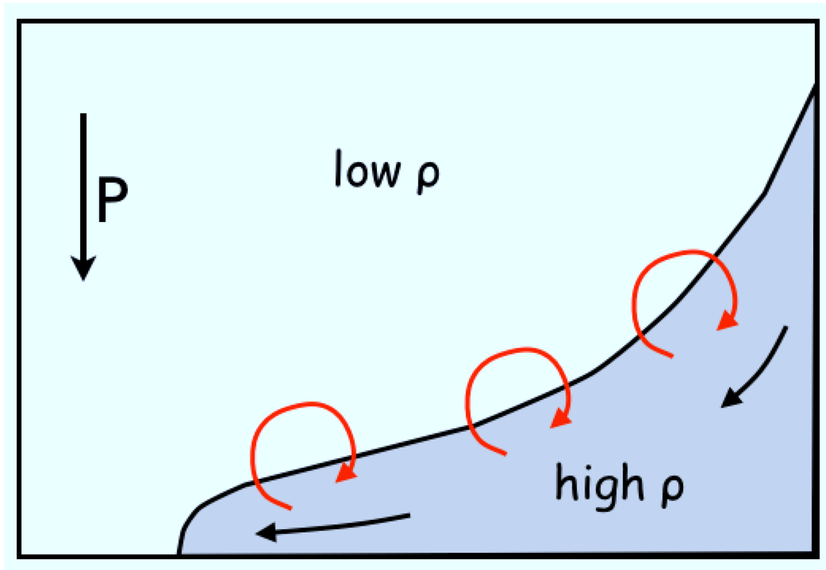

Figure 4: The baroclinic creation of vorticity in a pyroclastic flow. High density fluid flows down a mountain and shoves itself under lower-density material, thus creating non-zero baroclinicity.

Using the definition of **circulation**, one can shown that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_S \left[\frac{\partial \vec{w}}{\partial t} + \nabla \times (\vec{w} \times \vec{u})\right] \cdot \mathrm{d}\vec{S}$$

Using the **vorticity equation**, this can be rewritten as

$$\boxed{\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_S \left[\frac{\nabla \rho \times \nabla P}{\rho^2} + \nu \nabla^2 \vec{w} + \nabla \times \vec{F}\right] \cdot \mathrm{d}\vec{S}}$$

where, for completeness, we have added in the contribution of an external force $\vec{F}$ (which vanishes if $\vec{F}$ is conservative). Using Stokes' Curl Theorem we can also write this equation in a line-integral form as

$$\boxed{\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\oint \frac{\nabla P}{\rho} \cdot \mathrm{d}\vec{l} + \nu \oint \nabla^2 \vec{u} \cdot \mathrm{d}\vec{l} + \oint \vec{F} \cdot \mathrm{d}\vec{l}}$$

which is the form that is more often used.

NOTE: By comparing the equations expressing $\mathrm{d}\vec{w}/\mathrm{d}t$ and $\mathrm{d}\Gamma/\mathrm{d}t$ it is clear that the stretching a tilting terms present in the equation describing $\mathrm{d}\vec{w}/\mathrm{d}t$, are absent in the equation describing $\mathrm{d}\Gamma/\mathrm{d}t$. This implies that stretching and tilting changes the vorticity, but keeps the circularity invariant. This is basically the first theorem of Helmholtz described below.

Kelvin's Circulation Theorem: The number of vortex lines that thread any element of area that moves with the fluid remains unchanged in time for an inviscid, barotropic (or incompressible) fluid, in the absence of non-conservative forces.

The proof of **Kelvin's Circulation Theorem** is immediately evident from the above equation, which shows that $\mathrm{d}\Gamma/\mathrm{d}t = 0$ if the fluid is both inviscid ($\nu = 0$) and either barotropic ($P = P(\rho) \Rightarrow \nabla\rho \times \nabla P = 0$) or incompressible ($\nabla\rho = 0$), and there are no non-conservative forces ($\vec{F} = 0$).

We end this chapter on vorticity and circulation with the three theorems of Helmholtz, which hold in the absence of non-conservative forces (i.e., $\vec{F} = 0$).

Helmholtz Theorem 1: The strength of a vortex tube, which is defined as the circularity of the circumference of any cross section of the tube, is constant along its length. This theorem holds for any fluid, and simply derives from the fact that the vorticity field is divergence-free: $\nabla \cdot \vec{w} = \nabla \cdot (\nabla \times \vec{u}) = 0$. To see this, use Gauss' divergence theorem to write that

$$\int_V \nabla \cdot \vec{w}\,\mathrm{d}V = \int_S \vec{w} \cdot \mathrm{d}^2S = 0$$

Here V is the volume of a subsection of the vortex tube, and S is its bounding surface. Since the vorticity is, by definition, perpendicular to S along the sides of the tube, the only non-vanishing components to the surface integral come from the areas at the top and bottom of the vortex tube; i.e.

$$\int_S \vec{w} \cdot \mathrm{d}^2\vec{S} = \int_{A_1} \vec{w} \cdot (-\hat{n})\,\mathrm{d}A + \int_{A_2} \vec{w} \cdot \hat{n}\,\mathrm{d}A = 0$$

where A_1 and A_2 the areas of the cross sections of that bound the volume V of the vortex tube. Using Stokes' curl theorem, we have that

$$\int_A \vec{w} \cdot \hat{n} \, \mathrm{d}A = \oint_C \vec{u} \cdot \mathrm{d}\vec{l}$$

Hence we have that $\Gamma_{C_1} = \Gamma_{C_2}$ where C_1 and C_2 are the curves bounding A_1 and A_2, respectively.

Helmholtz Theorem 2: A vortex line cannot end in a fluid. Vortex lines and tubes must appear as closed loops, extend to infinity, or start/end at solid boundaries.

Helmholtz Theorem 3: A barotropic (or incompressible), inviscid fluid that is initially irrotational will remain irrotational. Hence, such a fluid does not and cannot create vorticity (except across curved shocks, see Chapter 11).
The proof of Helmholtz' third theorem is straightforward. According to Kelvin's circulation theorem, a barotropic (or incompressible), inviscid fluid has $\mathrm{d}\Gamma/\mathrm{d}t = 0$ everywhere. Using that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_S \left[\frac{\partial \vec{w}}{\partial t} + \nabla \times (\vec{w} \times \vec{u}) \right] \cdot \mathrm{d}^2\vec{S} = 0$$

Since this has to hold for any S, we have that $\partial \vec{w}/\partial t = \nabla \times (\vec{u} \times \vec{w})$. Hence, if $\vec{w} = 0$ initially, the vorticity remains zero for ever.

Potential Flow: An implication of Helmholtz' third theorem is that if an <u>inviscid</u> fluid is <u>incompressible</u> ($\nabla \cdot \vec{u} = 0$) and <u>irrotational</u> ($\nabla \times \vec{u} = 0$), then it will remain irrotational. Such a flow is called **Potential Flow**, and obeys **Laplace's equation**:

$$\nabla^2 \Phi_u = o$$

where Φ_u is called the **velocity potential**, defined according to $\vec{u} = \nabla \Phi_u$. Although there is no such thing as an inviscid liquid, viscosity typically only manifests itself in thin <u>boundary layers</u> (where Kelvin's circulation theorem doesn't apply). Outside of the boundary layer, flow is often accurately described by potential flow.

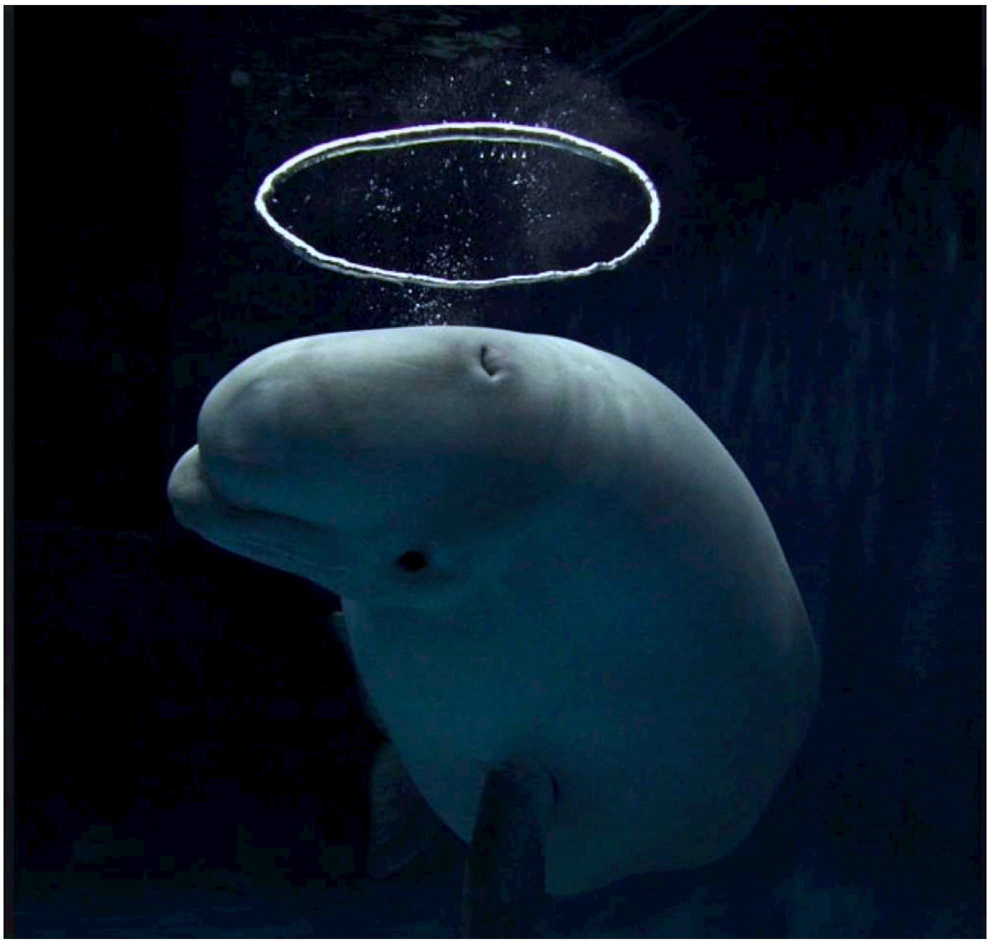

Figure 5: A beluga whale demonstrating Kelvin's circulation theorem and Helmholtz' second theorem by producing a closed vortex tube under water, made out of air.