ASTR 320: Problem Set 6

This FINAL problem set consists of 3 problems.
Due date: Tuesday Apr 23

Problem 1: Bondi Accretion

Consider a homogeneous, ideal fluid of density ρ_∞ and with sound speed c_∞. If a spherical body of mass M (i.e., a star or a black hole) is placed in this medium (at $\vec{r} = 0$), the gravity due to M will induce a flow towards it. Consequently, the mass M will start to accrete matter from its ambient medium. In what follows we assume that the flow that develops is steady and isothermal (i.e., it is assumed that the heating associated with the compression of the fluid is radiated away, such that the temperature of the accreting fluid remains constant).

a) Show that the pressure potential $h = \int \mathrm{d}P/\rho$ for the isothermal accretion flow is equal to $c_\infty^2 \ln(\rho/\rho_\infty)$, where ρ is the density in the flow at the radius where h is evaluated.

b) Use the continuity equation to show that $r^2\,\rho(r)\,u(r)$ is independent of the distance r from M, and show that this implies that

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} = -\frac{2}{r} - \frac{\mathrm{d}\ln u}{\mathrm{d}r}$$

Here $u(r) = |\vec{u}(\vec{r})|$ is the velocity of the accretion flow.

The above implies that the mass accretion rate $\dot{M} = 4\pi r^2 \rho u$ is also constant with radius (and, under the assumption of an infinite supply of ambient medium) and time.

c) Starting from the momentum equation for the accretion flow in Eulerian form, show that

$$(u^2 - c_\mathrm{s}^2)\frac{\mathrm{d}\ln u}{\mathrm{d}r} = \frac{2c_\mathrm{s}^2}{r}\left[1 - \frac{GM}{2c_\mathrm{s}^2 r}\right]$$

`Hint:` Use the results derived under **b)**.

d) The radius at which the fluid flow transits from being subsonic to being supersonic is called the sonic radius, r_s. Derive an expression for r_s is terms of M and c_s.

e) Use the fact that in a steady, barotropic flow Bernoulli's constant is conserved along particle paths to show that

$$\rho_{\rm s} \equiv \rho(r_{\rm s}) = \rho_\infty {\rm e}^{1.5}$$

`Hint:` use that $\lim_{r\to\infty} u = 0$.

f) Write down an expression for the accretion rate onto M as a function of ρ_∞, c_∞ and M.

g) The center of the Milky Way harbors a black hole of mass $M \simeq 10^6 M_\odot$. Assuming the (infinite reservoir of) ambient gas has a number density $n = 1\,{\rm cm}^{-3}$, a mean molecular weight $\mu = 1$ (i.e., the mean particle mass is equal to the mass of a proton), and a temperature $T = 10^7$K, how long does it take for the black hole to double its mass due to steady, isothermal Bondi accretion of gas from its ambient medium? Show your derivation in detail and express your answer in units of the Hubble time $t_{\rm H} \simeq 10^{10}$ year, which is roughly the age of the Universe.

Problem 2: Hydrogren

a) The surface of some star is determined to be $T = 26,729$K and to have an electron density of $n_{\rm e} = 9.12 \times 10^{11}\ {\rm cm}^{-3}$. Assuming that LTE holds, determine the fraction of all hydrogren atoms that are ionized.

b) Determine the temperature required for the number of hydrogen atoms in the first excited state to be 10 per cent of that in the ground state.

Problem 3: Cylclotron & Synchrotron emission

Consider an interstellar medium (ISM) with a magnetic field strength of $B = 3\mu$G.

a) Determine the radius of gyration (in km), period and gyrofrequency of a typical electron within the warm ($T \sim 10^4$K) component of the ISM. You may assume that the electrons move perpendicular to the magnetic field lines.

b) Determine the same as under (a), but now for an ultra-relativistic electron with a Lorentz boost of $\gamma = 10^4$.

c) Assuming that the relativistic electron under (b) has a pitch angle of $\phi = 45^{\rm o}$, determine the opening angle of the emission cone (in degreees), the critical frequency, and the spacing of the harmonics for this particle.

Useful Constants

G	$=$	$6.674 \times 10^{-8}\,{\rm cm}^3\,{\rm g}^{-1}\,{\rm s}^{-2}$
	$=$	$4.299 \times 10^{-9}\,{\rm Mpc}\,M_\odot^{-1}\,({\rm km/s})^2$
$m_{\rm p}$	$=$	$1.673\,\times^{-24}\,{\rm g}$
$m_{\rm e}$	$=$	$9.109 \times 10^{-28}{\rm g}$
e	$=$	$-4.803 \times 10^{-10}{\rm esu}$
$k_{\rm B}$	$=$	$1.38 \times 10^{-16}\,{\rm erg\,K}^{-1}$
$M_\odot$	$=$	$2 \times 10^{33}{\rm g}$