ASTR 320: Solutions to Problem Set 5

Problem 1: Schwarzschild Criterion

The specific entropy is given by $s = CP/\rho^\gamma$, where C is some constant. Show that the Schwarzschild criterion for convective stability can be written as $\mathrm{d}s/\mathrm{d}z > 0$.

SOLUTION: The Schwarzschild criterion for convective stability is given by

$$\frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}z} > \frac{\mathrm{d}\rho}{\mathrm{d}z}$$

Using that $s = CP/\rho^\gamma$, we have that

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}z} &= C\,\frac{1}{\rho^\gamma}\,\frac{\mathrm{d}P}{\mathrm{d}z} - \gamma\, C\,\frac{P}{\rho^{\gamma+1}}\,\frac{\mathrm{d}\rho}{\mathrm{d}z} \\ &= \frac{s}{P}\,\frac{\mathrm{d}P}{\mathrm{d}z} - \gamma\,\frac{s}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}z} \qquad (1)\end{aligned}$$

We can use this to write the above stability criterion as:

$$\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}z} > 0$$

and since the specific entropy is always positive, we thus see that the Schwarzschild criterion can be written as $\mathrm{d}s/\mathrm{d}z > 0$ (i.e., entropy has to decrease in the direction of increasing gravitational force for the system to be stable against convection).

Problem 2: The Virial Temperature

Virialized dark matter haloes are often defined as having a radius $r_{\rm vir}$, called the virial radius, that encloses an average density of 200 times the critical density $\rho_{\rm crit} = 1.36 \times 10^{11} M_\odot/\mathrm{Mpc}^3$. The latter is the density for which the Universe as a whole is 'flat' (i.e., has Euclidian geometry). The circular velocity at the virial radius is called the virial velocity and is denoted by $V_{\rm vir}$. Throughout you may assume that halos are spherically symmetric.

a) Derive expressions for $r_{\rm vir}$ and $V_{\rm vir}$ as functions of the halo's mass M, and compute $r_{\rm vir}$ (in kpc) and $V_{\rm vir}$ (in km/s) for a halo of mass $M = 10^{12} M_\odot$ (roughly the mass of the Milky Way halo).

SOLUTION: The average density inside the virial radius obeys

$$\frac{3\,M}{4\pi r_{\rm vir}^3} = 200\rho_{\rm crit}$$

Hence, the virial radius can be written as

$$r_{\rm vir} = \left(\frac{3M}{800\pi\rho_{\rm crit}}\right)^{1/3}$$

For convenience, we express the mass in units of $M_{12} = M/(10^{12} M_\odot)$, which gives

$$r_{\rm vir} = 0.206\,{\rm Mpc}\,M_{12}^{1/3}$$

where we have used that $\rho_{\rm crit} = 1.36 \times 10^{11} M_\odot/{\rm Mpc}^3$ as given.
Next we derive the virial velocity. The circular velocity for a spherical system is given by $V_{\rm circ} = \sqrt{GM(r)/r}$. Hence, we have that

$$V_{\rm vir} = \left(\frac{GM}{r_{\rm vir}}\right)^{1/2}$$

Substituting the above expression for the virial radius, and using that $G = 4.299 \times 10^{-9}\,{\rm Mpc}\,M_\odot^{-1}\,({\rm km/s})^2$, we find that

$$V_{\rm vir} = 144.5{\rm km/s}\,M_{12}^{1/3}$$

Thus, the virial radius and velocity of a halo of $M = 10^{12} M_\odot$ are 206kpc and 144.5km/s, respectively.

b) When gas is accreted by a dark matter halo, it experiences an accretion shock, which converts its infall motion into thermal motion. Derive an expression for the temperature of this shocked gas after it falls into a halo of mass M. Assume that the gas comes from infinity where it has zero velocity, and it is accelerated by the gravity of the halo, until it hits the halo's virial shock at a radius $r_{\rm vir}$. You may approximate the potential of the halo by a point mass, i.e., $\Phi(r) = -GM(r)/r$. Ignore radiative losses, and express your answer in terms of the virial velocity.

SOLUTION: Let $M_{\rm gas}$ be the mass that passes the accretion shock with an infall velocity $v_{\rm in}$. The total kinetic energy of this infalling gas, at infall, is then $E_{\rm kin} = \frac{1}{2} M_{\rm gas} v_{\rm in}^2$. This entire energy is converted to thermal energy, without radiative losses. The thermal energy of the gas is $E_{\rm thermal} = \frac{3}{2} N k_{\rm B} T$, where $N = M_{\rm gas}/(\mu m_{\rm p})$ is the number of gas particles. Equating $E_{\rm thermal}$ to $E_{\rm kin}$ then gives us

$$T = T_{\rm vir} = \frac{\mu m_{\rm p}}{3 k_{\rm B}} \, v_{\rm in}^2$$

Since the gas starts out from infinity with zero velocity, and is accelerated by the halo, the velocity it has when it crosses the virial shock is exaclty equal to the escape velocity of the gas from that same radius. Hence

$$v_{\rm in} = v_{\rm esc}(r = r_{\rm vir}) = \sqrt{2|\Phi(r_{\rm vir})|}$$

Using that $\Phi(r) = -GM(r)/r$, and that the mass enclosed by the virial radius is equal to the halo mass itself, we obtain

$$v_{\rm in} = \sqrt{2|\frac{GM}{r_{\rm vir}}|} = \sqrt{2} V_{\rm vir}$$

Hence, we find that

$$T_{\rm vir} = \frac{2}{3} \frac{\mu m_{\rm p}}{k_{\rm B}} \, V_{\rm vir}^2$$

c) Determine the virial temperature for a halo of $M = 10^{12} M_\odot$ in Kelvin. Assume that the gas is made of pure, fully ionized hydrogen.

SOLUTION: Using the above expressions for $T_{\rm vir}$ and $V_{\rm vir}$ we see that

$$T_{\rm vir}(M = 10^{12} M_\odot) = \frac{2}{3} \frac{\mu m_{\rm p}}{k_{\rm B}} \, (144.5 {\rm km/s})^2$$

Since the gas is fully ionized hydrogen, we have that $\mu = 0.5$ (i.e., one proton mass per two particles (the proton plus the corresponding electron). Using the values for $m_{\rm p}$ and $k_{\rm B}$ given, and using that $1\text{erg} = 1\,\text{g}\,\text{cm}^2/\text{s}^{-2} = 10^{10}\text{g}\,\text{km}^2/\text{s}^{-2}$, we find that $T_{\rm vir} \simeq 2.5 \times 10^6 K$.

Problem 3: The Hernquist Sphere

A popular model that is often used to describe galaxies is the Hernquist sphere, which is characterized by a density distribution:

$$\rho(r) = \frac{M}{2\pi} \frac{a}{r\,(r+a)^3}$$

Here M is the total mass, and a is a characteristic radius. The corresponding, effective radius, defined as the radius that encloses half of all the light in projection, is $R_{\rm e} \simeq 1.8153\,a$. For this problem you may want to make use of the integrals given at the end of this problem set.

a) Given an expression for the enclosed mass profile, $M(r)$, in terms of M and a.

SOLUTION: The enclosed mass is given by

$$M(r) = 4\pi \int_0^r \rho(r')\, r'^2\, {\rm d}r'$$

Substituting the expression for the density profile then yields

$$M(r) = 2M \int_0^r \frac{a r'^2}{r'(r'+a)^3}\,{\rm d}r'$$

Substituting $x = (r/a)$ reduces this to

$$M(r) = 2M \int_0^{r/a} \frac{x\,{\rm d}x}{(1+x)^3} = 2M \left[\frac{-1}{1+x} + \frac{1}{2(1+x)^2} \right]_0^{r/a}$$

which, after some simple algebra, yields

$$M(r) = M \frac{r^2}{(r+a)^2}$$

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b) Use the Poisson equation to derive an expression for the gravitational potential, $\Phi(r)$, in terms of M and a.

SOLUTION: The Poisson equation for a spherically symmetric system can be written in the form

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r')\, r'^2\, \mathrm{d}r' + \int_r^\infty \rho(r')\, r'\, \mathrm{d}r' \right]$$

The first terms can be written in terms of the enclosed mass, $M(r)$. Using the result under (a) above this yields

$$\begin{aligned}
\Phi(r) &= -GM \frac{r}{(r+a)^2} + 4\pi G \frac{M}{2\pi} \int_r^\infty \frac{r'\mathrm{d}r'}{(r'/a)(r'+a)^3} && (2)\\
&= -GM \frac{r}{(r+a)^2} + \frac{2GM}{a} \int_{r/a}^\infty \frac{\mathrm{d}x}{(x+1)^3} && (3)\\
&= -GM \frac{r}{(r+a)^2} + \frac{2GM}{a} \left[\frac{-1}{2(1+x)^2} \right]_{r/a}^\infty && (4)\\
&= -GM \frac{r}{(r+a)^2} + \frac{GMa}{(r+a)^2} = -\frac{GM}{(r+a)} && (5)
\end{aligned}$$

where once again we made use of the substitution $x = (r/a)$.

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c) The gravitational potential energy is conveniently written in the form

$$W = -\frac{G\,M^2}{r_{\mathrm{g}}}$$

where r_{g} is defined as the 'gravitational radius'. Use an alternative expression for W to work out the gravitational radius in units of the scale radius a.

SOLUTION: An alternative expression for the gravitational potential energy is

$$W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, \mathrm{d}^3\vec{x} = 2\pi \int_0^\infty \rho(r)\, \Phi(r)\, r^2\, \mathrm{d}r$$

where we have made use of the fact that the system is spherically symmetric. Upon substituting the above expressions for $\rho(r)$ and $\Phi(r)$, we get

$$\begin{aligned} W &= -GM^2 \int_0^\infty \frac{a}{r(r+a)^3}\, r^2\, \mathrm{d}r && (6) \\ &= -\frac{GM^2}{a} \int_0^\infty \frac{x\,\mathrm{d}x}{(1+x)^4} && (7) \\ &= -\frac{GM^2}{a} \left[\frac{-1}{2(1+x)^2} + \frac{1}{3(1+x)^3}\right]_0^\infty && (8) \\ &= -\frac{GM^2}{a} \left[\frac{1}{2} - \frac{1}{3}\right] = -\frac{GM^2}{6a} && (9) \end{aligned}$$

We thus see that the gravitational radius $r_{\rm g} = 6a$.

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d) Astronomers have observed a spherical galaxy with an effective radius of $R_{\rm e} = 5\,{\rm kpc}$. Using spectroscopy, they infer that the stars in the galaxy have a line-of-sight velocity dispersion equal to $\sigma = 200\,{\rm km\,s^{-1}}$. Assume that the galaxy is in virial equilibrium, and that it can be adequately described by a Hernquist sphere. Give an estimate for the total mass, M, in solar units.

SOLUTION: Virial equilibrium implies that $2K + W = 0$. We have seen above that

$$W = -\frac{GM^2}{r_{\rm g}} = -\frac{GM^2}{6a}$$

The kinetic energy of the galaxy is simply $K = \frac{1}{2} M \sigma_{\rm 3D}^2$, where $\sigma_{\rm 3D}$ is the *average* velocity disperion of the stars in the galaxy. Observations have revealed that the line-of-sight velocity dispersion of the system is $\sigma_{\rm los} = 200\,{\rm km\,s^{-1}}$. Since this is measured along the line-of-sight, it is a measure of the average 1D velocity dispersion. Using that the system has spherical symmetry, we have that $\sigma_{\rm 3D}^2 = 3\sigma_{\rm los}^2$, and thus $K = \frac{3}{2} M \sigma_{\rm los}^2$. Substituting our expressions for K and W in the virial equation, we infer that

$$M = \frac{3\,\sigma_{\rm los}^2\, r_{\rm g}}{G} = \frac{9.916\,\sigma_{\rm los}^2\, R_{\rm e}}{G}$$

where we have used that $r_{\rm g} = 6a = 6(R_{\rm e}/1.8153)$. Substituting the observed values for $\sigma_{\rm los}$ and $R_{\rm e}$, then yields that $M \simeq 4.6 \times 10^{11}\,{\rm M_\odot}$.

Problem 4: Purely Radial Stellar Oscillations

Consider a spherical, barotropic star for which $P = K\rho^{\gamma}$. The goal is to derive conditions for γ under which the star is stable to radial oscillations. Suppose the star is *uniformly* expanded from an initial equilibrium configuration such that the position of a fluid element (or mass shell) changes from r_0 to $r_0(1+\delta)$. Throughout we shall assume that δ is small, such that we can use perturbation theory. From the Euler equation (i.e., ignoring viscosity) we can write down the acceleration of a fluid element at a distance r from the center of the star as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM(r)}{r^2}$$

where v is the radial component of the velocity (we are considering purely radial motions here) and $M(r)$ is the mass enclosed within radius r.

a) Use Taylor series expansion to show that, to linear order, the density of the perturbed mass shell obeys $\rho = \rho_0(1 - 3\delta)$.

SOLUTION: Consider a mass shell defined by radii r and $r + \mathrm{d}r$. The density of the unperturbed mass shell is $\rho_{\rm sh,0} = M_{\rm sh}/V_{\rm sh}$, with $M_{\rm sh,0}$ and $V_{\rm sh,0}$ the mass and volume of the shell, respectively. The latter is given by

$$V_{\rm sh} = \frac{4\pi}{3}\left[(r + \mathrm{d}r)^3 - r^3\right] \simeq 4\pi r^2 \mathrm{d}r$$

where we have used that $\mathrm{d}r \ll r$. Hence, we have that

$$\rho_{\rm sh,0} = \frac{M_{\rm sh,0}}{4\pi r^2 \mathrm{d}r}$$

The density of the perturbed shell is

$$\rho_{\rm sh}(\delta) = \frac{M_{\rm sh,0}}{4\pi r^2(1+\delta)^2 \mathrm{d}r(1+\delta)}$$

where we have used that the mass of the shell is conserved, while both r and $\mathrm{d}r$ increase by a factor $(1+\delta)$. Combining the above expressions we have that

$$\rho_{\rm sh}(\delta) = \frac{\rho_{\rm sh,0}}{(1+\delta)^3}$$

Using a Taylor-series expansion of $\rho_{\rm sh}(\delta)$ around $\delta = 0$, we obtain

$$\rho_{\rm sh}(\delta) = \rho_{\rm sh,0} + \left(\frac{{\rm d}\rho_{\rm sh}}{{\rm d}\delta}\right)_{\delta=0} \delta + \frac{1}{2}\left(\frac{{\rm d}^2\rho_{\rm sh}}{{\rm d}\delta^2}\right)_{\delta=0} \delta^2 + ...$$

Using only the first two terms, which is what 'to linear order' means, and using that

$$\frac{{\rm d}\rho_{\rm sh}}{{\rm d}\delta} = -3\frac{\rho_{\rm sh,0}}{(1+\delta)^4}$$

we obtain that $\rho_{\rm sh}(\delta) \simeq \rho_{\rm sh,0} - 3\rho_{\rm sh,0}\delta = \rho_{\rm sh,0}(1-3\delta)$.

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b) Using the same strategy, given a similar expression for P in terms of P_0, δ, and γ.

SOLUTION: We have that $P_0 = K\rho_0^\gamma$. Hence, for the pressure in the perturbed shell we have that

$$P(\delta) = K\rho^\gamma = K[\rho_0(1-3\delta)]^\gamma = P_0(1-3\delta)^\gamma$$

Using once again the Taylor series expansion, we can write this as

$$P(\delta) = P_0 + \left(\frac{{\rm d}P}{{\rm d}\delta}\right)_{\delta=0} \delta + \frac{1}{2}\left(\frac{{\rm d}^2 P}{{\rm d}\delta^2}\right)_{\delta=0} \delta^2 + ...$$

which to linear order reduces to

$$P(\delta) = P_0 - 3\gamma P_0 \left[(1-3\delta)^{\gamma-1}\right]_{\delta=0} \delta = P_0(1-3\gamma\delta)$$

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c) Substitute the expressions for r, ρ and P in the expression for the radial acceleration, keeping only terms up to linear order, and derive for what values of γ the star will be stable to radial oscillations. Note: assume that the initial configuration was one of equilibrium, so that

$$-\frac{1}{\rho_0}\frac{{\rm d}P_0}{{\rm d}r_0} - \frac{GM(r_0)}{r_0^2} = 0$$

and <u>explain your answer</u>!

SOLUTION: First we summarize what we have learned above:

$$r = r_0(1+\delta) \qquad \rho = \rho_0(1-3\delta) \qquad P = P_0(1-3\gamma\delta)$$

Hence we have that

$$\frac{1}{\rho} = \frac{1}{\rho_0}(1+3\delta)$$

where we have used the well-known result that $1/(1-\epsilon) = 1+\epsilon$ whenever ϵ is small. Similarly, we have that

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}P_0}{\mathrm{d}r_0}\frac{\mathrm{d}P}{\mathrm{d}P_0}\frac{\mathrm{d}r_0}{\mathrm{d}r} = \frac{\mathrm{d}P_0}{\mathrm{d}r_0}(1-3\gamma\delta)\,(1-\delta) \tag{10}$$

$$= \frac{\mathrm{d}P_0}{\mathrm{d}r_0}\,(1-\delta-3\gamma\delta) \tag{11}$$

where we have ignored terms of order δ^2. Combining the above results yields

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{1}{\rho_0}\frac{\mathrm{d}P_0}{\mathrm{d}r_0}[(1+3\delta)(1-\delta-3\gamma\delta)] = \frac{1}{\rho_0}\frac{\mathrm{d}P_0}{\mathrm{d}r_0}(1+2\delta-3\gamma\delta)$$

Along similar lines, it is easy to show that

$$\frac{GM(r)}{r^2} = \frac{GM(r_0)}{r_0^2(1+\delta)^2} = \frac{GM(r_0)}{r_0^2}(1-2\delta)$$

Substituting all these results in the expression for the radial acceleration, and using that the initial configuration was in equilibrium, we find that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{GM(r_0)}{r_0^2}(1+2\delta-3\gamma\delta) - \frac{GM(r_0)}{r_0^2}(1-2\delta) = \frac{GM(r_0)}{r_0^2}(4\delta-3\gamma\delta)$$

Suppose we consider a uniform expansion of the star so that δ is positive. If the radial acceleration given by the above expression is positive, then the expansion rate will be accelerated, leading to an instability. Hence, stability requires that $\mathrm{d}v/\mathrm{d}t < 0$ and thus that $4\delta - 3\gamma\delta < 0$. The latter implies that $\gamma > 4/3$.