Problem 1: Schwarzschild Criterion

The specific entropy is given by $s = C P/\rho^\gamma$, where $C$ is some constant. Show that the Schwarzschild criterion for convective stability can be written as $ds/dz > 0$.

Problem 2: The Virial Temperature

Virialized dark matter haloes are often defined as having a radius $r_{\text{vir}}$, called the virial radius, that encloses an average density of 200 times the critical density $\rho_{\text{crit}} = 1.36 \times 10^{11} M_\odot/\text{Mpc}^3$. The latter is the density for which the Universe as a whole is ‘flat’ (i.e., has Euclidian geometry). The circular velocity at the virial radius is called the virial velocity and is denoted by $V_{\text{vir}}$. Throughout you may assume that halos are spherically symmetric.

a) Derive expressions for $r_{\text{vir}}$ and $V_{\text{vir}}$ as functions of the halo’s mass $M$, and compute $r_{\text{vir}}$ (in kpc) and $V_{\text{vir}}$ (in km/s) for a halo of mass $M = 10^{12} M_\odot$ (roughly the mass of the Milky Way halo).

b) When gas is accreted by a dark matter halo, it experiences an accretion shock, which converts its infall motion into thermal motion. Derive an expression for the temperature of this shocked gas after it falls into a halo of mass $M$. Assume that the gas comes from infinity where it has zero velocity, and it is accelerated by the gravity of the halo, until it hits the halo’s virial shock at a radius $r_{\text{vir}}$. You may approximate the potential of the halo by a point mass, i.e., $\Phi(r) = -GM(r)/r$. Ignore radiative losses, and express your answer in terms of the virial velocity.

c) Determine the virial temperature for a halo of $M = 10^{12} M_\odot$ in Kelvin. Assume that the gas is made of pure, fully ionized hydrogen.
Problem 3: The Hernquist Sphere

A popular model that is often used to describe galaxies is the Hernquist sphere, which is characterized by a density distribution:

$$\rho(r) = \frac{M}{2\pi} \frac{a}{r(r+a)^3}$$

Here $M$ is the total mass, and $a$ is a characteristic radius. The corresponding, effective radius, defined as the radius that encloses half of all the light in projection, is $R_e \simeq 1.8153a$. For this problem you may want to make use of the integrals given at the end of this problem set.

a) Given an expression for the enclosed mass profile, $M(r)$, in terms of $M$ and $a$.

b) Use the Poisson equation to derive an expression for the gravitational potential, $\Phi(r)$, in terms of $M$ and $a$.

c) The gravitational potential energy is conveniently written in the form

$$W = -\frac{GM^2}{r_g}$$

where $r_g$ is defined as the ‘gravitational radius’. Use an alternative expression for $W$ to work out the gravitational radius in units of the scale radius $a$.

d) Astronomers have observed a spherical galaxy with an effective radius of $R_e = 5$ kpc. Using spectroscopy, they infer that the stars in the galaxy have a line-of-sight velocity dispersion equal to $\sigma = 200$ km s$^{-1}$. Assume that the galaxy is in virial equilibrium, and that it can be adequately described by a Hernquist sphere. Give an estimate for the total mass, $M$, in solar units.

Problem 4: Purely Radial Stellar Oscillations

Consider a spherical, barotropic star for which $P = K\rho^\gamma$. The goal is to derive conditions for $\gamma$ under which the star is stable to radial oscillations.
Suppose the star is uniformly expanded from an initial equilibrium configuration such that the position of a fluid element (or mass shell) changes from \( r_0 \) to \( r_0(1 + \delta) \). Throughout we shall assume that \( \delta \) is small, such that we can use perturbation theory. From the Euler equation (i.e., ignoring viscosity) we can write down the acceleration of a fluid element at a distance \( r \) from the center of the star as

\[
\frac{dv}{dt} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{GM(r)}{r^2}
\]

where \( v \) is the radial component of the velocity (we are considering purely radial motions here) and \( M(r) \) is the mass enclosed within radius \( r \).

**a)** Use Taylor series expansion to show that, to linear order, the density of the perturbed mass shell obeys \( \rho = \rho_0(1 - 3\delta) \).

**b)** Using the same strategy, given a similar expression for \( P \) in terms of \( P_0 \), \( \delta \), and \( \gamma \).

**c)** Substitute the expressions for \( r, \rho \) and \( P \) in the expression for the radial acceleration, keeping only terms up to linear order, and derive for what values of \( \gamma \) the star will be stable to radial oscillations. Note: assume that the initial configuration was one of equilibrium, so that

\[
-\frac{1}{\rho_0} \frac{dP_0}{dr_0} - \frac{GM(r_0)}{r_0^2} = 0
\]

and explain your answer!

(Potentially) Useful Constants
\[ G = 6.674 \times 10^{-8} \, \text{cm}^3 \text{g}^{-1} \text{s}^{-2} \]
\[ = 4.299 \times 10^{-9} \, \text{Mpc} \, M^{-1} \, (\text{km/s})^2 \]
\[ m_{\text{p}} = 1.673 \times 10^{-24} \, \text{g} \]
\[ k_{\text{B}} = 1.38 \times 10^{-16} \, \text{erg K}^{-1} \]
\[ M_{\odot} = 2 \times 10^{33} \, \text{g} \]

**Useful Integrals**

\[
\int \frac{dx}{(a + bx)^3} = \frac{-1}{2(a + bx)^2}
\]

\[
\int \frac{x \, dx}{(a + bx)^3} = \frac{1}{b^2} \left[ \frac{-1}{a + bx} + \frac{a}{2(a + bx)^2} \right]
\]

\[
\int \frac{x \, dx}{(a + bx)^4} = \frac{1}{b^2} \left[ \frac{-1}{2(a + bx)^2} + \frac{a}{3(a + bx)^3} \right]
\]