ASTR 320: Solutions to Problem Set 4

Problem 1: The Equation of State of a Photon Gas
Consider a photon gas enclosed in a box. The energy per unit volume that is flowing inside the box into a unit solid angle is $U(T)/4\pi$. This follows from the fact that the energy density of the photon gas is $U(T) = a_r\, T^4$, with a_r the radiation constant, and that a full sphere covers 4π sterradian of solid angle. The momentum flux flowing in a beam of solid angle $\mathrm{d}\Omega$ is therefore equal to $U(T)\mathrm{d}\Omega/(4\pi c)$, which follows from the fact that each photon carries a momentum $p = E/c$.

a) Now consider a beam of photons hitting a surface area A under an angle θ with respect to the area's normal. Give an expression for the amount of momentum that this beam transfers to the surface, per unit time, per unit area. For simplicity, you may assume that the surface acts like a perfect mirror.

SOLUTION: The momentum being carried to surface A in time Δt by photons coming from solid angle $\mathrm{d}\Omega$ is given by the momentum flux in that beam multiplied by the volume of the parallelipiped shown in Fig.1. This volume is given by $V = c\,\Delta t\, A\, \cos\theta$, which is easily inferred by inspection. Hence, the momentum being carried to surface A in time Δt by photons coming from solid angle $\mathrm{d}\Omega$ is

$$\Delta p = \frac{U(T)}{c}\,\frac{\mathrm{d}\Omega}{4\pi}\, c\,\Delta t\, A\, \cos\theta$$

The momentum *transferred* to the surface is twice the normal component:

$$\Delta p_{\mathrm{trans}} = 2\,\Delta p\, \cos\theta$$

Hence, we have that the amount of momentum that this beam transfers to the surface, per unit time, per unit area is given by

$$\frac{\Delta p_{\mathrm{trans}}}{A\,\Delta t} = U(T)\,\frac{\mathrm{d}\Omega}{2\pi}\, \cos^2\theta$$

b) The radiation pressure on our surface A is the momentum transfer per unit area, per unit time, and follows from integrating the result obtained under (a) over all possible beams. Show that the resulting radiation pressure is equal to $U(T)/3$. `HINT:` integrate over the solid angle of half the hemisphere that is pointing from the surface towards the inside of the box.

SOLUTION: To compute the radiation pressure, we use that the contribution from a beam coming from solid angle $\mathrm{d}\Omega$ is given by $\mathrm{d}P_{\rm rad} = U(T)\,(\mathrm{d}\Omega/2\pi)\,\cos^2\theta$, where θ is the angle between the normal of the surface and the direction of propogation of the beam. Using that $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$, we have that integration over all beams yields

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\begin{aligned}
P_{\rm rad} &= \int \mathrm{d}P_{\rm rad} = \frac{U(T)}{2\pi} \int_{\rm halfsphere} U(T)\,\cos^2\theta\,\mathrm{d}\Omega \\
&= \frac{U(T)}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} \sin\theta\,\cos^2\theta\,\mathrm{d}\theta \\
&= -U(T) \int_1^0 \cos^2\theta\,\mathrm{d}\cos\theta \\
&= U(T) \int_0^1 x^2 \mathrm{d}x = \frac{1}{3} U(T) \qquad (1)
\end{aligned}
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Problem 2: The Equation of State of Degenerate Gases

Consider a fully degenerate, non-relativistic gas. The particles in the gas have a spin degeneracy of $g = 2$, and a number density n. Since the gas is non-relativistic, its pressure is given by

$$P = \frac{2}{3}\, n\, \langle E \rangle$$

where

$$\langle E \rangle = \frac{1}{N} \int_0^\infty E\, N(E)\, \mathrm{d}E$$

with $N(E)$ the number of particles with energy E, and N the total number of particles. Since $E = p^2/2m$, with p the particles momentum, and since the gas is fully degenerate, we have that

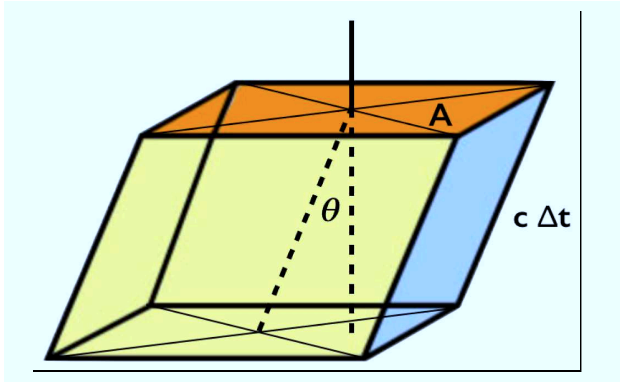

Figure 1: The parallelipiped that contains the photons that will deposit their momentum to surface area A in a time Δt. The volume of this parallelipiped is $c\Delta t\, A\, \cos\theta$.

$$N(E)\mathrm{d}E = N(\vec{p})\mathrm{d}^3\vec{p} = \begin{cases} \frac{2}{h^3}\, V_{\mathrm{x}}\, \mathrm{d}^3\vec{p} & \text{if } 0 \le |\vec{p}| \le p_{\mathrm{F}} \\ 0 & \text{otherwise} \end{cases}$$

with h Planck's constant, p_{F} the Fermi momentum, and V_{x} the volume of the gas in configuration space.

a) Show that $\langle E\rangle = \frac{3}{5}\frac{p_{\mathrm{F}}^2}{2m}$.

SOLUTION: We simply use that

$$\begin{aligned} \langle E\rangle &= \frac{1}{N}\int_0^\infty E\, N(E)\, \mathrm{d}E \\ &= \frac{1}{N}\int_0^\infty \frac{p^2}{2m}\, N(\vec{p})\, \mathrm{d}^3\vec{p} \\ &= \frac{1}{N}\int_0^{p_{\mathrm{F}}} \frac{p^2}{2m}\, \frac{2}{h^3}\, V_{\mathrm{x}}\, 4\pi p^2 \mathrm{d}p \\ &= \frac{1}{nm}\frac{4\pi}{h^3}\frac{1}{5}p_{\mathrm{F}}^5 \end{aligned} \tag{2}$$

Here we have used that $n = N/V_{\mathrm{x}}$ and that $\mathrm{d}^3\vec{p} = 4\pi p^2 \mathrm{d}p$, which follows from isotropy. Next we use that

$$p_{\mathrm{F}} = \left(\frac{3}{8\pi}n\right)^{1/3} h$$

to rewrite this as

$$\langle E\rangle = \frac{1}{nm}\frac{4\pi}{h^3}\frac{1}{5}p_{\mathrm{F}}^2\frac{3}{8\pi}nh^3 = \frac{3}{5}\frac{p_{\mathrm{F}}^2}{2m}$$

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b) Show that the pressure is given by $P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m}\,n^{5/3}$.

SOLUTION: Here we simply use that $P = \frac{2}{3}n\langle E\rangle$. Substitution of the above solution for the average energy per particle yields

$$\begin{aligned} P &= \frac{1}{5}\frac{n}{m}p_{\mathrm{F}}^2 \\ &= \frac{1}{5}\frac{n}{m}\left(\frac{3}{8\pi}n\right)^{2/3} h^2 \\ &= \frac{1}{20}n^{5/3}\frac{h^2}{m}\left(\frac{3}{\pi}\right)^{2/3} \end{aligned} \tag{3}$$

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c) Using the same method as above, show that in the ultra-relativistic case $\langle E\rangle = \frac{3}{4}\,p_{\mathrm{F}}\,c$ and $P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} c\,h\,n^{4/3}$.

SOLUTION: When going from non-relativistic to ultra-relativistic, the main changes are that now the energy per particle is $E = pc$, while the pressure is $P = \frac{1}{3}n\langle E\rangle$.
We start by computing the average energy per particle:

$$\begin{aligned} \langle E\rangle &= \frac{1}{N}\int_0^\infty E\,N(E)\,\mathrm{d}E \\ &= \frac{1}{N}\int_0^\infty pc\,N(\vec{p})\,\mathrm{d}^3\vec{p} \\ &= \frac{1}{N}\int_0^{p_{\mathrm{F}}} pc\,\frac{2}{h^3}\,V_{\mathrm{x}}\,4\pi p^2\mathrm{d}p \\ &= \frac{c}{n}\frac{8\pi}{h^3}\frac{1}{4}p_{\mathrm{F}}^4 \end{aligned} \tag{4}$$

Next we once again use that

$$p_{\rm F} = \left(\frac{3}{8\pi}n\right)^{1/3} h$$

to rewrite this as

$$\langle E\rangle = \frac{c}{n}\frac{8\pi}{h^3}\frac{1}{4}p_{\rm F}\frac{3}{8\pi}nh^3 = \frac{3}{4}cp_{\rm F}$$

We can now use this to compute the pressure:

$$\begin{aligned} P &= \frac{1}{3}n\langle E\rangle = \frac{1}{4}ncp_{\rm F} \\ &= \frac{1}{4}nc\left(\frac{3}{8\pi}n\right)^{1/3} h \\ &= \frac{1}{8}n^{4/3}ch\left(\frac{3}{\pi}\right)^{1/3} \end{aligned} \tag{5}$$

Problem 3: Primordial Matter

About three minutes after the Big Bang, the Universe undergoes Big-Bang Nucleosynthesis (BBN), in which protons and neutrons combine to create helium (and a tiny amount of lithium). At the end of BBN, the primordial gas consists, to good approximation of only hydrogen and helium (all other elements have negligible abundances). The mass fractions of hydrogen and helium are 75 and 25 percent, respectively. Both hydrogen and helium are fully ionized; i.e., there are no atoms, only free electrons, nuclei of hydrogen (single protons), and nuclei of helium (two protons plus two neutrons). Primordial gas has an equation of state that is accurately represented by that of an ideal gas, $P = k_{\rm B}\,T\,\rho/\mu\,m_{\rm p}$.

a) What is the mean particle mass, μ, in units of the proton mass $m_{\rm p}$, for such a primordial, fully ionized gas? You may assume that the masses of protons and neutrons are equal to each other, and that the mass of an electron is negligible compared to that of a proton or neutron. Also, note that a nucleus is considered a single particle.

SOLUTION: The mean mass per particle, for a fully ionized gas consisting of hydrogen and helium is given by

$$\mu m_{\rm p} = \frac{n_{\rm H} m_{\rm H} + n_{\rm He} m_{\rm He} + n_e m_e}{n_{\rm H} + n_{\rm He} + n_e}$$

Using that electrons have negligble mass, while protons and neutrons have equal mass, we have that $m_e = 0$ and $m_{\rm He} = 4m_{\rm p}$ (helium nuclei consist of two protons and two neutrons), we can rewrite this as

$$\mu m_{\rm p} = \frac{n_{\rm H} m_{\rm p} + 4 n_{\rm He} m_{\rm p}}{n_{\rm H} + n_{\rm He} + n_e}$$

Next we realize that under complete ionization each hydrogen atom yields one electron, while each helium atom yields two electrons. Hence, $n_e = n_{\rm H} + 2n_{\rm He}$, and we obtain that

$$\mu = \frac{n_{\rm H} + 4n_{\rm He}}{2n_{\rm H} + 3n_{\rm He}}$$

The mass fractions of hydrogen and helium are 3/4 and 1/4, respectively, which implies that

$$\frac{n_{\rm H} m_{\rm H}}{n_{\rm H} m_{\rm H} + n_{\rm He} m_{\rm He} + n_e m_e} = \frac{3}{4}$$

and

$$\frac{n_{\rm He} m_{\rm He}}{n_{\rm H} m_{\rm H} + n_{\rm He} m_{\rm He} + n_e m_e} = \frac{1}{4}$$

Taking the ratio of these two expressions, we have that

$$\frac{n_{\rm H} m_{\rm H}}{n_{\rm He} m_{\rm He}} = \frac{3/4}{1/4} = 3$$

Using that $m_{\rm He} = 4m_{\rm H}$, we thus infer that $n_{\rm H}/n_{\rm He} = 12$. Substituting that $n_{\rm He} = n_{\rm H}/12$ in the above expression for the mean mass per particle we finally find that

$$\mu = \frac{n_{\rm H} + \frac{4}{12} n_{\rm H}}{2n_{\rm H} + \frac{3}{12} n_{\rm H}} = \frac{16/12}{27/12} = \frac{16}{27}$$

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b) What are the number densities of hydrogen and helium nuclei relative to that of electrons?

SOLUTION: Using the above, we have that $n_e = n_{\rm H} + 2n_{\rm He} = n_{\rm H} + \frac{2}{12}n_{\rm H} = \frac{14}{12}n_{\rm H}$. Hence, we have that $n_{\rm H} = 6/7n_{\rm e}$.
For helium, we have that

$$\frac{n_e}{n_{\rm He}} = \frac{n_e}{n_{\rm H}}\frac{n_{\rm H}}{n_{\rm He}} = \frac{14}{12}\frac{12}{1} = 14$$

Hence we have that $n_{\rm He} = 1/14n_{\rm e}$

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c) By how much would the pressure of the primordial gas decrease if it were to become atomic (i.e., electrons combine with hydrogen and helium nuclei to produce neutral hydrogen and helium atoms), while the temperature and density of the gas remain fixed?

SOLUTION: Let $P_{\rm atomic}$ and $P_{\rm ionized}$ be the pressures when the gas is atomic and ionized, respectively. When the gas goes from being ionized to being atomic, its mass density doesn't change (no mass is created or destroyed per unit volume). It is also given that the temperature doesn't change (this is not really physical, as the temperature plays an important role in determining whether the gas will be ionized or not, but since it is a given, we will work under that assumption). Using that $P = k_{\rm B}\,T\,\rho/\mu\,m_{\rm p}$, we have that $P_{\rm atomic}/P_{\rm ionized} = \mu_{\rm ionized}/\mu_{\rm atomic}$. Under (a) we have seen that $\mu_{\rm ionized} = 16/27$. All we need to do, therefore, is to calculate $\mu_{\rm atomic}$. Following the same procedure as under (a):

$$\mu m_{\rm p} = \frac{n_{\rm H}m_{\rm H} + n_{\rm He}m_{\rm He}}{n_{\rm H} + n_{\rm He}} = \frac{n_{\rm H}m_{\rm p} + 4n_{\rm He}m_{\rm p}}{n_{\rm H} + n_{\rm He}}$$

Using that, as before, $n_{\rm He} = n_{\rm H}/12$ (this is the same, whether the gas is ionized or atomic), we thus infer that

$$\mu = \frac{n_{\rm H} + \frac{4}{12}n_{\rm H}}{n_{\rm H} + \frac{1}{12}n_{\rm H}} = \frac{16}{13}$$

Thus, when the primordial gas transits from being ionized to being atomic (under fixed density and temperature), the pressure changes by a factor $P_{\rm atomic}/P_{\rm ionized} = \mu_{\rm ionized}/\mu_{\rm atomic} = 13/27$.
