Problem 1: The Equation of State of a Photon Gas

Consider a photon gas enclosed in a box. The energy per unit volume that is flowing inside the box into a unit solid angle is $U(T)/4\pi$. This follows from the fact that the energy density of the photon gas is $U(T) = a_r T^4$, with $a_r$ the radiation constant, and that a full sphere covers $4\pi$ steradians of solid angle. The momentum flux flowing in a beam of solid angle $d\Omega$ is therefore equal to $U(T)d\Omega/(4\pi c)$, which follows from the fact that each photon carries a momentum $p = E/c$.

a) [5 points] Now consider a beam of photons hitting a surface area $A$ under an angle $\theta$ with respect to the area’s normal. Give an expression for the amount of momentum that this beam transfers to the surface, per unit time, per unit area. For simplicity, you may assume that the surface acts like a perfect mirror.

b) [5 points] The radiation pressure on our surface $A$ is the momentum transfer per unit area, per unit time, and follows from integrating the result obtained under (a) over all possible beams. Show that the resulting radiation pressure is equal to $U(T)/3$. HINT: integrate over the solid angle of half the hemisphere that is pointing from the surface towards the inside of the box.
Problem 2: The Equation of State of Degenerate Gases
Consider a fully degenerate, non-relativistic gas. The particles in the gas have a spin degeneracy of \( g = 2 \), and a number density \( n \). Since the gas is non-relativistic, its pressure is given by

\[
P = \frac{2}{3} n \langle E \rangle
\]

where

\[
\langle E \rangle = \frac{1}{N} \int_{0}^{\infty} E N(E) dE
\]

with \( N(E) \) the number of particles with energy \( E \), and \( N \) the total number of particles. Since \( E = \frac{p^2}{2m} \), with \( p \) the particles momentum, and since the gas is fully degenerate, we have that

\[
N(E)dE = N(\vec{p})d^3\vec{p} = \begin{cases} \frac{2}{h^3} V_x d^3\vec{p} & \text{if } 0 \leq |\vec{p}| \leq p_F \\ 0 & \text{otherwise} \end{cases}
\]

with \( h \) Planck’s constant, \( p_F \) the Fermi momentum, and \( V_x \) the volume of the gas in configuration space.

a) [4 points] Show that \( \langle E \rangle = \frac{3}{5} \frac{p_F^2}{2m} \).

b) [5 points] Show that the pressure is given by \( P = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2}{m} n^{5/3} \).

c) [5 points] Using the same method as above, show that in the ultra-relativistic case \( \langle E \rangle = \frac{3}{4} p_F c \) and \( P = \frac{1}{8} \left( \frac{3}{\pi} \right)^{1/3} c h n^{4/3} \).
Problem 3: Ideal Gas
Consider a gas consisting of an equal mixture of free electrons and free protons (fully ionized). The gas particles follow a Maxwell-Boltzmann distribution of velocities, such that their average kinetic energy per particle is given by

\[ E_K = \frac{3}{2} k_B T \]

The particles feel each others Coulomb force, resulting in an average potential energy per particle equal to

\[ E_P = \frac{e^2}{\langle r \rangle} \]

Here \( e \) is the electrical charge (in statcoulomb) of an electron and \( \langle r \rangle \) is the mean particle separation. We are only allowed to approximate this gas as ‘ideal’ as long as this average potential energy per particle is small compared to the average kinetic energy per particle.

a) [3 points] Compute the average ratio \( E_K/E_P \) for the warm ISM which has a temperature of \( T \sim 10^4 \) K and an average number density of \( n \sim 1 \) cm\(^{-3}\).

b) [3 points] Same as under (a), but now for the photosphere of the Sun, which has an average temperature of \( \sim 5500 \) K and a density \( \rho \sim 3 \times 10^{-4} \) kg m\(^{-3}\).

c) [3 points] Same as under (a), but now for the core of the Sun, which has an average temperature of \( \sim 1.6 \times 10^7 \) K and a density roughly equal to 160 times the density of water.

d) [3 points] Which of the cases under (a), (b) and (c) can be described as a ideal gas.