ASTR 320: Problem Set 4

This problem set consists of 6 problems for a total of 64 points.
Due date: Tuesday Apr 25

Problem 1: The Rayleigh-Taylor instability

Following a supernova explosion, a shell of matter with density $\rho_{\rm s}$ is plowing into the ISM with a speed $u_{\rm s} = \mathcal{M}\, c_{\rm s,ISM}$, where $\mathcal{M}$ is the Mach number and $c_{\rm s,ISM}$ is the sound speed of the ISM. The shell has a thickness d and the sound speed of the dense shell material is $c_{\rm s,shell}$.

a) [`4 points`] The shell material experiences a ram pressure $P = \rho_{\rm ISM} u_{\rm s}^2$, which causes it to decelerate. Show that the magnitude of the deceleration is given by

$$a = \frac{\rho_{\rm ISM}\, c_{\rm s,ISM}^2\, \mathcal{M}^2}{\rho_{\rm s}\, d}$$

The shell is subject to Rayleigh-Taylor (RT) instability which obeys the following dispersion relation

$$\omega = \pm ik\sqrt{\frac{a}{k}\, A}$$

where

$$A = \frac{\rho_{\rm s} - \rho_{\rm ISM}}{\rho_{\rm s} + \rho_{\rm ISM}}$$

us called the Atwood number.

b) [`6 points`] Derive an expression for the growth rate of RT perturbations (i.e., the timescale on which the perturbations will grow) with a wavenumber $k = 1/d$. Expresss your answer in terms of the overdensity $\delta = (\rho_{\rm s}/\rho_{\rm ISM}) - 1$ and the sound crossing time of the shall $\tau_{\rm s} \equiv d/c_{\rm s,shell}$. You may assume that at the interface there is pressure equilibrium between the shell and the ISM.

Problem 2: The Hernquist Sphere

A popular model that is often used to describe galaxies is the Hernquist sphere, which is characterized by a density distribution:

$$\rho(r) = \frac{M}{2\pi} \frac{a}{r\,(r+a)^3}$$

The corresponding gravitational potential is given by

$$\Phi(r) = -\frac{G\,M}{(r+a)}$$

Here M is the total mass, and a is a characteristic radius. The corresponding, effective radius, defined as the radius that encloses half of all the light in projection, is $R_{\rm e} \simeq 1.8153\,a$. For this problem you may want to make use of the integrals given at the end of this problem set.

a) [`3 points`] Given an expression for the enclosed mass profile, $M(r)$, in terms of M and a.

b) [`4 points`] The gravitational potential energy is conveniently written in the form

$$W = -\frac{G\,M^2}{r_{\rm g}}$$

where $r_{\rm g}$ is defined as the 'gravitational radius'. Use an alternative expression for W to work out the gravitational radius in units of the scale radius a.

c) [`5 points`] Astronomers have observed a spherical galaxy with an effective radius of $R_{\rm e} = 5\,{\rm kpc}$. Using spectroscopy, they infer that the stars in the galaxy have a line-of-sight velocity dispersion equal to $\sigma = 200\,{\rm km\,s^{-1}}$. Assume that the galaxy is in virial equilibrium, and that it can be adequately described by a Hernquist sphere. Give an estimate for the total mass, M, in solar units.

Problem 3: The Lyα transition

When an electron in the first excited state of the Hydrogren atom falls back to the ground level, it emits a Lyα photon with a wavelength of $\lambda = 121.567$nm. The cross section for a Lyα photon to excite a neutral hydrogen atom (at $T = 10^4$ K) is $\sigma_{\mathrm{Ly}\alpha} = 5.0 \times 10^{-14}\ \mathrm{cm}^2$, while the time scale for spontaneous decay of an electron in the excited Lyα state to the ground state is $\tau_{2,1} = 1.6 \times 10^{-9}$ s. For comparison, at $T = 10^4$ K the collision rate for an electron to de-excite a Hydrogen atom from the Lyα state is $\gamma_{\mathrm{Ly}\alpha} = \langle v \rangle \sigma_{\mathrm{eff}} = 6.8 \times 10^{-9}\ \mathrm{cm}^3\ \mathrm{s}^{-1}$. Here σ_{eff} is the effective cross section, and $\langle v \rangle$ is the typical velocity of a free electron at $T = 10^4$ K.

a) [`6 points`] Consider two ionized clouds of pure hydrogen, each with the same fractional ionization of 99.9 per cent, density of 0.01 cm^{-3} and temperature of $T = 10^4$ K. Cloud 1 has a line of sight thickness of $l = 1$ pc and Cloud 2 has $l = 100$ pc. Compare (by computing) the optical depth of Cloud 1 at a wavelength of $\lambda = 121.567$nm to the optical depth of Cloud 2 at a wavelength of $\lambda = 200$nm. Assume that only scattering processes are important and ignore scattering into the line of sight.

b) [`4 points`] A Lyα photon travels through a pure hydrogen gas of density $n = 10^3\,\mathrm{cm}^{-3}$ and temperature $T = 10^4\,$K. At this temperature, the gas is almost entirely ionized. If the photon excites an upwards Lyα transition in one of the small fraction of particles that are neutral and in the ground state, will the net outcome be scattering or absorption. Explain your answer.

c) [`4 points`] Suppose the pure hydrogen gas described under (b) is in the form of a homogeneous spherical cloud. What is the upper mass of the cloud, in Solar masses, such that it is optically thin for Lyα photons?

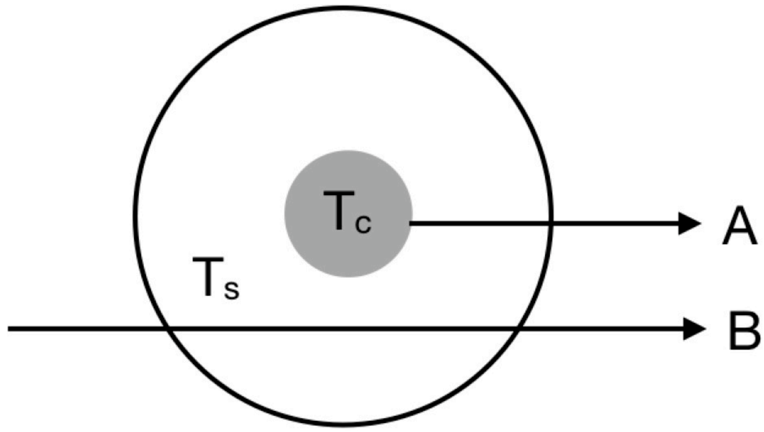

Figure 1:

Problem 4: Radiative Transfer

A spherical, opaque object emits as a blackbody at temperature T_c. Surrounding this central object is a spherical shell of material, thermally emitting at a temperature T_s where $T_s < T_c$. The shell absorbs in a narrow spectral line, i.e., its absoprtion coefficient becomes large at the frequency ν_0 and is negligibly small at other frequencies, such as ν_1. The object is observed at frequencies ν_0 and ν_1 (which is a little bit larger than ν_0) and along two rays, A and B (see Fig. 1). Assume that the Planck function does not vary appreciably from ν_0 to ν_1.

a) [`3 points`] At which frequency, ν_0 or ν_1, will the observed brightness be largest along ray A. Explain you answer and make a small drawing of what the brightness as function of frequency looks like over the range from a bit below ν_0 to a bit larger than ν_1.

b) [`3 points`] Same question as under (a), but now for ray B.

c) [`3 points`] Same question as under (a), but now when $T_s > T_c$.

d) [`3 points`] Same question as under (c), but now for ray B.

Problem 5: More Radiative Transfer [`8 points`]
Consider two gas clouds of identical temperature that are emitting black-body radiation. Cloud 1 is located at a distance of 100 parsec, and has an optical depth of 50 (at frequency ν). Cloud 2, which is located at a distance of 50 parsec, has an optical depth of $\tau_\nu = 0.2$. If both clouds are observed at the same frequency ν, what will the ratio of observed fluxes be?

Problem 6: Hydrogren
a) [`4 points`] The surface of some star is determined to be $T = 26,729$K and to have an electron density of $n_\mathrm{e} = 9.12 \times 10^{11}\ \mathrm{cm}^{-3}$. Assuming that LTE holds, determine the fraction of all hydrogren atoms that are ionized.

b) [`4 points`] Determine the temperature required for the number of hydrogen atoms in the first excited state to be 10 per cent of that in the ground state.

(Potentially) Useful Constants

$$
\begin{aligned}
m_\mathrm{p} &= 1.673 \times 10^{-24}\mathrm{g} \\
m_\mathrm{e} &= 9.109 \times 10^{-28}\mathrm{g} \\
k_\mathrm{B} &= 1.38 \times 10^{-16}\ \mathrm{erg\,K^{-1}} \\
\sigma_\mathrm{T} &= 6.65 \times 10^{-25}\ \mathrm{cm}^2 \\
M_\odot &= 2 \times 10^{33}\mathrm{g} \\
\mathrm{pc} &= 3.086 \times 10^{18}\ \mathrm{cm}
\end{aligned}
$$

Potentially Useful Integrals

$$\int \frac{\mathrm{d}x}{(a+bx)^3} = \frac{-1}{2\,(a+bx)^2}$$

$$\int \frac{x\,\mathrm{d}x}{(a+bx)^3} = \frac{1}{b^2}\left[\frac{-1}{a+bx} + \frac{a}{2(a+bx)^2}\right]$$

$$\int \frac{x\,\mathrm{d}x}{(a+bx)^4} = \frac{1}{b^2}\left[\frac{-1}{2(a+bx)^2} + \frac{a}{3(a+bx)^3}\right]$$