Problem 1: Bondi Accretion

Consider a homogeneous, ideal fluid of density $\rho_\infty$ and with sound speed $c_\infty$. If a spherical body of mass $M$ (i.e., a star or a black hole) is placed in this medium (at $\vec{r} = 0$), the gravity due to $M$ will induce a flow towards it. Consequently, the mass $M$ will start to accrete matter from its ambient medium. In what follows we assume that the flow that develops is steady and isothermal (i.e., it is assumed that the heating associated with the compression of the fluid is radiated away, such that the temperature of the accreting fluid remains constant).

a) [4 points] Show that the pressure potential $h = \int \frac{dP}{\rho}$ for the isothermal accretion flow is equal to $c_\infty^2 \ln(\rho/\rho_\infty)$, where $\rho$ is the density in the flow at the radius where $h$ is evaluated.

b) [4 points] Use the continuity equation to show that $r^2 \rho(r) u(r)$ is independent of the distance $r$ from $M$, and show that this implies that

$$\frac{d \ln \rho}{dr} = -\frac{2}{r} \frac{d \ln u}{dr}$$

Here $u(r) = |\vec{u}(\vec{r})|$ is the velocity of the accretion flow.

The above implies that the mass accretion rate $\dot{M} = 4\pi r^2 \rho u$ is also constant with radius (and, under the assumption of an infinite supply of ambient medium) and time.

c) [5 points] Starting from the momentum equation for the accretion flow in Eulerian form, show that

$$(u^2 - c_s^2) \frac{d \ln u}{dr} = \frac{2c_s^2}{r} \left[ 1 - \frac{GM}{2c_s^2 r} \right]$$

**Hint:** Use the results derived under b).
d) [3 points] The radius at which the fluid flow transits from being subsonic to being supersonic is called the sonic radius, \( r_s \). Derive an expression for \( r_s \) in terms of \( M \) and \( c_s \).

e) [4 points] Use the fact that in a steady, barotropic flow Bernoulli’s constant is conserved along particle paths to show that

\[
\rho_s \equiv \rho(r_s) = \rho_\infty e^{1.5}
\]

Hint: use that \( \lim_{r \to \infty} u = 0 \).

f) [3 points] Write down an expression for the accretion rate onto \( M \) as a function of \( \rho_\infty \), \( c_\infty \) and \( M \).

g) [6 points] The center of the Milky Way harbors a black hole of mass \( M \simeq 10^6 M_\odot \). Assuming the (infinite reservoir of) ambient gas has a number density \( n = 1 \text{ cm}^{-3} \), a mean molecular weight \( \mu = 1 \) (i.e., the mean particle mass is equal to the mass of a proton), and a temperature \( T = 10^7 \text{K} \), how long does it take for the black hole to double its mass due to steady, isothermal Bondi accretion of gas from its ambient medium? Show your derivation in detail and express your answer in units of the Hubble time \( t_H \approx 10^{10} \text{ year} \), which is roughly the age of the Universe.

Problem 2: The Hernquist Sphere

Hint: see chapter 18 of the lecture notes.

A popular model that is often used to describe galaxies is the Hernquist sphere, which is characterized by a density distribution:

\[
\rho(r) = \frac{M}{2\pi r (r + a)^3} \frac{a}{(r + a)^3}
\]

The corresponding gravitational potential is given by

\[
\Phi(r) = -\frac{G M}{(r + a)}
\]
Here $M$ is the total mass, and $a$ is a characteristic radius. The corresponding, effective radius, defined as the radius that encloses half of all the light in projection, is $R_e \simeq 1.8153 \, a$. For this problem you may want to make use of the integrals given at the end of this problem set.

**a)** [3 points] Given an expression for the enclosed mass profile, $M(r)$, in terms of $M$ and $a$.

**b)** [4 points] The gravitational potential energy is conveniently written in the form

$$ W = -\frac{G \, M^2}{r_g} $$

where $r_g$ is defined as the ‘gravitational radius’ (see Chapter 18 of the lecture notes). Use an alternative expression for $W$ to work out the gravitational radius in units of the scale radius $a$.

**c)** [5 points] Astronomers have observed a spherical galaxy with an effective radius of $R_e = 5$ kpc. Using spectroscopy, they infer that the stars in the galaxy have a line-of-sight velocity dispersion equal to $\sigma = 200 \, \text{km/s}^{-1}$. Assume that the galaxy is in virial equilibrium, and that it can be adequately described by a Hernquist sphere. Give an estimate for the total mass, $M$, in solar units.

**Problem 3: The Virial Temperature**

Virialized dark matter haloes are often defined as having a radius $r_{\text{vir}}$, called the virial radius, that encloses an average density of 200 times the critical density $\rho_{\text{crit}} = 1.36 \times 10^{11} M_\odot/\text{Mpc}^3$. The latter is the density for which the Universe as a whole is ‘flat’ (i.e., has Euclidian geometry). The circular velocity at the virial radius is called the virial velocity and is denoted by $V_{\text{vir}}$. Throughout you may assume that halos are spherically symmetric.

**a)** [4 points] Derive expressions for $r_{\text{vir}}$ and $V_{\text{vir}}$ as functions of the halo’s mass $M$, and compute $r_{\text{vir}}$ (in kpc) and $V_{\text{vir}}$ (in km/s) for a halo of mass $M = 10^{12} M_\odot$ (roughly the mass of the Milky Way halo).
b) [5 points] When gas is accreted by a dark matter halo, it experiences an accretion shock, which converts its infall motion into thermal motion. Derive an expression for the temperature of this shocked gas after it falls into a halo of mass $M$. Assume that the gas comes from infinity where it has zero velocity, and it is accelerated by the gravity of the halo, until it hits the halo’s virial shock at a radius $r_{\text{vir}}$. You may approximate the potential of the halo by a point mass, i.e., $\Phi(r) = -GM(r)/r$. Ignore radiative losses, and express your answer in terms of the virial velocity.

c) [3 points] Determine the virial temperature for a halo of $M = 10^{12} M_\odot$ in Kelvin. Assume that the gas is made of pure, fully ionized hydrogen.

**Problem 4: The Rayleigh-Taylor instability**

Following a supernova explosion, a shell of matter with density $\rho_s$ is plowing into the ISM with a speed $u_s = \mathcal{M} c_{s,\text{ISM}}$, where $\mathcal{M}$ is the Mach number and $c_{s,\text{ISM}}$ is the sound speed of the ISM. The shell has a thickness $d$ and the sound speed of the dense shell material is $c_{s,\text{shell}}$.

a) [4 points] The shell material experiences a ram pressure $P = \rho_{\text{ISM}} u_s^2$, which causes it to decelerate. Show that the magnitude of the deceleration is given by

$$a = \frac{\rho_{\text{ISM}} c_{s,\text{ISM}}^2 \mathcal{M}^2}{\rho_s d}$$

**HINT:** use that pressure is force per unit area.

The shell is subject to Rayleigh-Taylor (RT) instability which obeys the following dispersion relation

$$\omega = \pm i k \sqrt{\frac{a}{k}} A$$

where

$$A = \frac{\rho_s - \rho_{\text{ISM}}}{\rho_s + \rho_{\text{ISM}}}$$

called the Atwood number.
b) [6 points] Derive an expression for the growth rate of RT perturbations (i.e., the timescale on which the perturbations will grow) with a wavenumber $k = 1/d$. Express your answer in terms of the overdensity $\delta = (\rho_s/\rho_{\text{ISM}}) - 1$ and the sound crossing time of the shell $\tau_s \equiv d/c_{s,\text{shell}}$. You may assume that at the interface there is pressure equilibrium between the shell and the ISM.

Problem 5: The Isothermal Sphere

HINT: see chapter 16 of the lecture notes.

An isothermal sphere is a sphere with a density profile $\rho(r) \propto r^{-2}$ and a velocity dispersion profile $\sigma^2(r) = \sigma^2$ that is independent of radius.

a) [4 points] A galaxy is observed to have an effective radius $R_e = 1\, \text{kpc}$ and a line-of-sight velocity dispersion of $100\, \text{km\,s}^{-1}$. Use the spherical Jeans equation to compute the mass enclosed within a sphere of radius $1\, \text{kpc}$ under the assumption that the galaxy is an isotropic isothermal sphere in equilibrium.

b) [3 points] What is the enclosed mass if you were to assume that the sphere is not isotropic, but instead has an anisotropy parameter $\beta = -1$?
Potentially Useful Constants

\[
\begin{align*}
G &= 6.674 \times 10^{-8} \text{ cm}^3 \text{g}^{-1} \text{s}^{-2} \\
&= 4.299 \times 10^{-9} \text{ Mpc} \, M_\odot^{-1} (\text{km} / \text{s})^2 \\
m_p &= 1.673 \times 10^{-24} \text{ g} \\
m_e &= 9.109 \times 10^{-28} \text{ g} \\
e &= -4.803 \times 10^{-10} \text{ esu} \\
k_B &= 1.38 \times 10^{-16} \text{ erg K}^{-1} \\
M_\odot &= 2 \times 10^{33} \text{ g}
\end{align*}
\]

Potentially Useful Integrals

\[
\int \frac{dx}{(a + bx)^3} = \frac{-1}{2(a + bx)^2}
\]

\[
\int \frac{x \, dx}{(a + bx)^3} = \frac{1}{b^2} \left[\frac{-1}{a + bx} + \frac{a}{2(a + bx)^2}\right]
\]

\[
\int \frac{x \, dx}{(a + bx)^4} = \frac{1}{b^2} \left[\frac{-1}{2(a + bx)^2} + \frac{a}{3(a + bx)^3}\right]
\]