ASTR 320: Problem Set 3

This problem set consists of 4 problems for a total of 54 points.
Due date: Thursday Mar 9 (day of Midterm Exam)

In this problem set, you will examine a rotating disk without circulation, derive expressions for the Schwarzschild and Jeans stability criteria, and treat stellar pulsations.

Problem 1: Vorticity in a thin disk [`10 points`]

Consider an infinitessimally thin disk of fluid in rotation around the disk's (vertical) symmetry axis. It is given that the flow is symmetric around the same symmetry axis, and that the circulation around any curve C on the disk is zero, as long as the curve does not enclose the symmetry point of the disk, $R = 0$. Derive an expression for the velocity field $\vec{u}(\vec{x})$, i.e., what are the various components of $\vec{u}$ as function of location in the disk? Clearly explain your steps in this derivation.

Problem 2: Schwarzschild Criterion [`6 points`]

The specific entropy is given by $s = CP/\rho^\gamma$, where C is some constant. Starting from

$$\frac{\rho}{\gamma P} \frac{\mathrm{d}P}{\mathrm{d}z} > \frac{\mathrm{d}\rho}{\mathrm{d}z}$$

show that this Schwarzschild criterion for convective stability can be written as $\mathrm{d}s/\mathrm{d}z > 0$.

Problem 3: The Jeans Criterion

In Chapter 13 (Sound Waves) we used perturbation theory to derive the dispersion relation for a perturbed fluid, which we found to be $\omega^2 = k^2 c_\mathrm{s}^2$. In deriving this expression we had ignored gravity. Here we will derive the dispersion relation for a gravitational perturbation, which is done using the same perturbation theory, but this time we are going to include the $\nabla\Phi$ term in the Euler equation. In order to maintain a closed set of equation, we then also have to include the Poisson equation: $\nabla^2\Phi = 4\pi G\rho$.

a) [`5 points`] Derive linearized versions for the perturbed continuity, momentum and Poisson equations. Write down the unperturbed equations, the perturbed versions, and subsequently linearize the latter. As for the sound waves, you may assume that the initial, unperturbed system has $\vec{u}_0 = 0$, $\nabla\rho_0 = 0$, and $\nabla P_0 = 0$. Express the linearized, perturbed momentum equations in terms of the sound speed, $c_{\rm s}$.

b) [`5 points`] Each of the perturbed quantities, ρ_1, u_1, P_1 and Φ_1 can be written as the sum over Fourier modes $\propto \exp[i(\vec{k}\cdot\vec{x} - \omega t)]$. Substitute this solution in each of the three linearized equations. Show that, in the case of the continuity equation, one finds that

$$-\omega\rho_1 + \rho_0\vec{k}\cdot\vec{u}_1 = 0$$

and derive similar expressions for the linearized momentum equations and the linearized Poisson equation.

c) [`5 points`] Combine the expressions derived under **b)** to show that the dispersion relation for this system is given by $\omega^2 = k^2 c_{\rm s}^2 - 4\pi G\rho_0$.

d) [`6 points`] Derive the Jeans length (in parsec) and Jeans mass (in Solar masses) for an ideal fluid with a temperature of $T = 10^4$K, and a density equal to the critical density for closure, which is $\rho_{\rm crit} = 1.4\times 10^{11}\,{\rm M}_\odot/\,{\rm Mpc}^3$. You may assume that the fluid is composed entirely of ionized hydrogen.

Problem 4: Purely Radial Stellar Oscillations

Consider a spherical, barotropic star for which $P = K\rho^\gamma$. The goal is to derive conditions for γ under which the star is stable to radial oscillations. Suppose the star is *uniformly* expanded from an initial equilibrium configuration such that the position of a fluid element (or mass shell) changes from r_0 to $r_0(1+\delta)$. Throughout we shall assume that δ is small, such that we can use perturbation theory. From the Euler equation (i.e., ignoring viscosity) we can write down the acceleration of a fluid element at a distance r from the center of the star as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM(r)}{r^2}$$

where v is the radial component of the velocity (we are considering purely radial motions here) and $M(r)$ is the mass enclosed within radius r.

a) [`6 points`] Use Taylor series expansion to show that, to linear order, the density of the perturbed mass shell obeys $\rho = \rho_0(1 - 3\delta)$.

b) [`5 points`] Using the same strategy, given a similar expression for P in terms of P_0, δ, and γ.

c) [`6 points`] Substitute the expressions for r, ρ and P in the expression for the radial acceleration, keeping only terms up to linear order, and derive for what values of γ the star will be stable to radial oscillations. Note: assume that the initial configuration was one of equilibrium, so that

$$-\frac{1}{\rho_0}\frac{\mathrm{d}P_0}{\mathrm{d}r_0} - \frac{GM(r_0)}{r_0^2} = 0$$

and explain your answer!

(Potentially) Useful Constants

G	$=$	$6.674 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
	$=$	$4.299 \times 10^{-9}\,\mathrm{Mpc}\,M_\odot^{-1}\,(\mathrm{km/s})^2$
m_p	$=$	$1.673 \times^{-24}\,\mathrm{g}$
k_B	$=$	$1.38 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$
$M_\odot$	$=$	$2 \times 10^{33}\mathrm{g}$