ASTR 320: Problem Set 2

This problem set consists of 5 problems for a total of 46 points.
Due date: tuesday February 18

In this problem set, you will experiment with flows, practice the conversion between index-form and vector-form, and develop a feeling for the stress tensor.

Problem 1: One-Dimensional Fluid Flow

A fluid is flowing purely in the x-direction with a velocity field given by

$$u_x(x) = \begin{cases} u_a & \text{if } x \leq 0 \\ u_a + (u_b - u_a)\frac{x}{L} & \text{if } 0 < x < L \\ u_b & \text{if } x \geq L \end{cases}$$

and with $u_y = u_z = 0$ everywhere. The density of the fluid at all positions with $x < 0$ is given by ρ_0, and the system is in a steady state.

a) [`5 points`] What is $\rho(x)$ for $x > 0$?

b) [`4 points`] Compute the acceleration a fluid element experiences as a function of position x in the flow. Compute this using a Lagrangian derivative.

Problem 2: Two-Dimensional Fluid Flow

Consider a two-dimensional fluid with velocity field $\vec{u} = (-ay, bx)$, where a, b are constants and we adopt Cartesian coordinates. Assume that at time $t = 0$ the density everywhere is equal to ρ_0.

a) [`4 points`] Under what condition(s) is the flow incompressible? Under what condition(s) is it curl-free

b) [`5 points`] Consider the location $A = (2, 2)$. How does the density at A change as a function of time?

Problem 3: Navier-Stokes; from index form to vector form [`6 points`]
The Navier-Stoker equation in index form, as derived in class, is given by

$$\rho\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\right] + \frac{\partial}{\partial x_i}\left(\eta\,\frac{\partial u_k}{\partial x_k}\right) - \rho\frac{\partial \Phi}{\partial x_i}$$

Show clearly, step-by-step and using text were needed, that this can be written in vector form as

$$\rho\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \mu\nabla^2\vec{u} + \left(\eta + \frac{1}{3}\mu\right)\,\nabla(\nabla\cdot\vec{u}) - \rho\nabla\Phi$$

Problem 4: Momentum Equations in Conservation Law Form [`5 points`]
The continuity equation in Eulerian vector form is given by

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

This equation is said to be in 'conservation law form', which is a form that is particularly useful for solving numerically.
The Euler equation describing conservation of momentum in Eulerian vector form is

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{u} = -\frac{\nabla P}{\rho} - \nabla\Phi$$

which is not in conservation law form. Show that, for an inviscid fluid, and in the absence of gravity, this equation can be put in the following conservation law form

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla\cdot\mathbf{\Pi} = 0$$

where $\Pi_{ij} = \rho\, u_i\, u_j - \sigma_{ij}$ is the momentum flux density tensor.

Problem 5: The Stress Tensor

Consider a fluid in a 2-dimensional, Cartesian coordinate system (x_1, x_2), with stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

and let $\hat{n}$ and $\hat{t}$ be the unit normal and unit tangent vectors of a surface S, for which $\hat{n}$ is rotated by angle θ with respect to the x_1 axis.

a) [`3 points`] Express $\hat{n}$ and $\hat{t}$ in terms of θ, i.e., what are n_1, n_2, t_1 and t_2 in $\hat{n} = (n_1, n_2)$ and $\hat{t} = (t_1, t_2)$?

b) [`3 points`] Show that the normal stress, $\Sigma_{\rm n}$, can be written as

$$\Sigma_{\rm n} = \sigma_{11}\,\cos^2\theta + \sigma_{22}\,\sin^2\theta + \sigma_{12}\,\sin 2\theta$$

and derive a similar expression for the shear stress, $\Sigma_{\rm t}$.

c) [`4 points`] Consider the case $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$. Under what angle θ is the shear stress on S maximal? For what angle θ does the shear on S vanish? What are the normal stresses in both cases?

d) [`4 points`] Answer the same questions as under **c)**, but now for the case with $\sigma_{11} = \sigma_{22} = 0$.

e) [`3 points`] Under what condition are both the normal and shear stress independent of θ, and what are their values in that case?