Problem 1: Traffic Flow
Consider a steady traffic flow on one side of a divided highway. The density of cars per unit length is given by \( \rho \) and has units of cars per kilometer. The velocities of cars (in the direction of the highway) is given by \( u \), and has units of kilometers per hour. Let \( x \) describe distance along the road. For \( x < 0 \) the speed limit is \( u_a \) and assume that the cars are driving at this speed limit (the highway is obviously not located in CT). At \( x = 0 \) there is a sign letting the drivers know that there is a change in speed limit. They begin to decelerate until they reach the new speed limit, \( u_b \) at \( x = L \). At \( x > L \) the cars continue with constant speed \( u_b \).

Assume that the car speed is described by

\[
  u(x) = \begin{cases} 
    u_a & \text{if } x \leq 0 \\
    u_a + (u_b - u_a)\frac{x}{L} & \text{if } 0 < x < L \\
    u_b & \text{if } x \geq L
  \end{cases}
\]

a) [5 points] What is \( \rho(x) \) if the density of cars at \( x = 0 \) is equal to \( \rho_a \)?
b) [5 points] What is the position of a car as a function of time that at \( t = 0 \) is located at \( x = 0 \)?
c) [3 points] Using the Lagrangian derivate, compute the acceleration a car experiences as a function of position \( x \) in the flow.

Problem 2: Streamlines
A streamline is defined as a curve that is instantaneously tangent to the velocity vector of a flow. Streamlines show the direction a massless fluid element will travel at any point in time (see Chapter 2 of Lecture Notes).
Consider a two-dimensional fluid with velocity field \( \vec{u} = (-ay, bx) \), where \( a, b \) are constants and we adopt Cartesian coordinates. Assume that at time \( t = 0 \) the density everywhere is equal to \( \rho_0 \).

a) [4 points] Draw streamlines for the following two cases: (i) \( a = b > 0 \), and (ii) \( a = -b > 0 \).

b) [2 points] Under what condition(s) is the flow incompressible? Under what condition(s) is it curl-free?

c) [3 points] Consider the location \( A = (2, 2) \). How does the density at \( A \) change as a function of time in cases (i) and (ii) above?

d) [3 points] Now consider a two-dimensional fluid with velocity field \( \vec{u} = (2x, y) \), with \( a > 0 \) and \( b > 0 \). Compute the density at \( A \) at time \( t = 2 \) in units of the initial density \( \rho_0 \) at \( t = 0 \).

Problem 3: Navier-Stokes; from index form to vector form [6 points]
The Navier-Stokes equation in index form, as derived in class, is given by

\[
\rho \frac{du_i}{dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \eta \frac{\partial u_i}{\partial x_k} \right] - \rho \frac{\partial \Phi}{\partial x_i} \]

Show clearly, step-by-step and using text were needed, that this can be written in vector form as

\[
\rho \frac{d\vec{u}}{dt} = -\nabla P + \mu \nabla^2 \vec{u} + \left( \eta + \frac{1}{3} \mu \right) \nabla (\nabla \cdot \vec{u}) - \rho \nabla \Phi
\]

Problem 4: The Stress Tensor
Consider a fluid in a 2-dimensional, Cartesian coordinate system \((x_1, x_2)\), with stress tensor

\[
\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}
\]

and let \( \hat{n} \) and \( \hat{t} \) be the unit normal and unit tangent vectors of a surface \( S \), for which \( \hat{n} \) is rotated by angle \( \theta \) with respect to the \( x_1 \) axis.

a) [3 points] Express \( \hat{n} \) and \( \hat{t} \) in terms of \( \theta \), i.e., what are \( n_1, n_2, t_1 \) and \( t_2 \) in \( \hat{n} = (n_1, n_2) \) and \( \hat{t} = (t_1, t_2) \)?
b) [3 points] Show that the normal stress, $\Sigma_n$, can be written as

$$\Sigma_n = \sigma_{11} \cos^2 \theta + \sigma_{22} \sin^2 \theta + \sigma_{12} \sin 2\theta$$

and derive a similar expression for the shear stress, $\Sigma_t$.

c) [4 points] Consider the case $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$. Under what angle $\theta$ is the shear stress on $S$ maximal? For what angle $\theta$ does the shear on $S$ vanish? What are the normal stresses in both cases?

d) [4 points] Answer the same questions as under c), but now for the case with $\sigma_{11} = \sigma_{22} = 0$.

e) [3 points] Under what condition are both the normal and shear stress independent of $\theta$, and what are their values in that case?