Problem 1: One-Dimensional Fluid Flow
A fluid is flowing purely in the $x$-direction with a velocity field given by

$$u_x(x) = \begin{cases} u_a & \text{if } x \leq 0 \\ u_a + (u_b - u_a) \frac{x}{L} & \text{if } 0 < x < L \\ u_b & \text{if } x \geq L \end{cases}$$

and with $u_y = u_z = 0$ everywhere. The density of the fluid at all positions with $x < 0$ is given by $\rho_0$, and the system is in a steady state (the density and velocity fields are constant in time).

a) [5 points] What is $\rho(x)$ for $x > 0$?

b) [4 points] Compute the acceleration a fluid element experiences as a function of position $x$ in the flow.

Problem 2: Two-Dimensional Fluid Flow
Consider a two-dimensional fluid with velocity field $\vec{u} = (-ay, bx)$, where $a, b$ are constants and we adopt Cartesian coordinates. Assume that at time $t = 0$ the density everywhere is equal to $\rho_0$.

a) [4 points] Under what condition(s) is the flow incompressible? Under what condition(s) is it curl-free

b) [5 points] Consider the location $A = (2, 2)$. How does the density at $A$ change as a function of time?
Problem 3: Navier-Stokes; from index form to vector form [6 points]

The Navier-Stokes equation in index form, as derived in class, is given by

\[
\rho \frac{du_i}{dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) \right] + \frac{\partial}{\partial x_j} \left( \eta \frac{\partial u_k}{\partial x_j} \right) - \rho \frac{\partial \Phi}{\partial x_i}
\]

Show clearly, step-by-step and using text were needed, that this can be written in vector form as

\[
\rho \frac{d\vec{u}}{dt} = -\nabla P + \mu \nabla^2 \vec{u} + \left( \eta + \frac{1}{3} \mu \right) \nabla (\nabla \cdot \vec{u}) - \rho \nabla \Phi
\]

Problem 4: Momentum Equations in Conservation Law Form [5 points]

The continuity equation in Eulerian vector form is given by

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0
\]

This equation is said to be in ‘conservation law form’, which is a form that is particularly useful for solving numerically.

The Euler equation describing conservation of momentum in Eulerian vector form is

\[
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi
\]

which is not in conservation law form. Show that, for an inviscid fluid, and in the absence of gravity, this equation can be put in the following conservation law form

\[
\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot \Pi = 0
\]

where \( \Pi_{ij} = \rho u_i u_j - \sigma_{ij} \) is the momentum flux density tensor.
Problem 5: The Stress Tensor

Consider a fluid in a 2-dimensional, Cartesian coordinate system \((x_1, x_2)\), with stress tensor

\[
\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}
\]

and let \(\hat{n}\) and \(\hat{t}\) be the unit normal and unit tangent vectors of a surface \(S\), for which \(\hat{n}\) is rotated by angle \(\theta\) with respect to the \(x_1\) axis.

a) [3 points] Express \(\hat{n}\) and \(\hat{t}\) in terms of \(\theta\), i.e., what are \(n_1, n_2, t_1\) and \(t_2\) in \(\hat{n} = (n_1, n_2)\) and \(\hat{t} = (t_1, t_2)\)?

b) [3 points] Show that the normal stress, \(\Sigma_n\), can be written as

\[
\Sigma_n = \sigma_{11} \cos^2 \theta + \sigma_{22} \sin^2 \theta + \sigma_{12} \sin 2\theta
\]

and derive a similar expression for the shear stress, \(\Sigma_t\).

c) [4 points] Consider the case \(\sigma_{11} = \sigma_{12} = \sigma_{21} = 0\). Under what angle \(\theta\) is the shear stress on \(S\) maximal? For what angle \(\theta\) does the shear on \(S\) vanish? What are the normal stresses in both cases?

d) [4 points] Answer the same questions as under c), but now for the case with \(\sigma_{11} = \sigma_{22} = 0\).

e) [3 points] Under what condition are both the normal and shear stress independent of \(\theta\), and what are their values in that case?