ASTR 320: Solutions to Problem Set 1

Problem 1: Vector Field Gymnastics
Let $\vec{A}(\vec{x}) = (x^2 + y, 2x^3 + y^3 z, x + 3z^2)$.

a) Compute $\nabla \cdot \vec{A}$.

SOLUTION:

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2x + 3y^2 z + 6z$$

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b) Compute $\nabla \times \vec{A}$.

SOLUTION:

$$\begin{aligned}
\nabla \times \vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \vec{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \vec{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \vec{e}_z \\
&= (0 - y^3)\, \vec{e}_x + (0 - 1)\, \vec{e}_y + (6x^2 - 1)\, \vec{e}_z \\
&= (-y^3, -1, 6x^2 - 1)
\end{aligned}$$

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c) Compute $\nabla^2 \vec{A}$.

SOLUTION:

$$\begin{aligned}
\nabla^2 \vec{A} &= (\nabla \cdot \nabla)\vec{A} \\
&= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right) \vec{e}_x \\
&+ \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right) \vec{e}_y
\end{aligned}$$

$$+ \quad \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \vec{e}_z$$
$$= \quad (2+0+0)\,\vec{e}_x + (12x+6yz+0)\,\vec{e}_y + (0+0+6)\,\vec{e}_z$$
$$= \quad (2, 12x+6yz, 6)$$

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d) Compute $\nabla \times (\nabla \times \vec{A})$

SOLUTION:

$\nabla\times(\nabla\times\vec{A}) = \nabla\times(-y^3, -1, 6x^2-1) = (0-0)\,\vec{e}_x + (0-12x)\,\vec{e}_y + (0+3y^2)\,\vec{e}_z = (0, -12x, 3y^2)$

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e) Compute $\nabla(\nabla \cdot \vec{A})$ and verify that $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

SOLUTION:

$$\begin{aligned}
\nabla(\nabla \cdot \vec{A}) &= \nabla(2x + 3y^2 z + 6z) \\
&= \left(\frac{\partial(2x+3y^2z+6z)}{\partial x}, \frac{\partial(2x+3y^2z+6z)}{\partial y}, \frac{\partial(2x+3y^2z+6z)}{\partial z}\right) \\
&= (2, 6yz, 3y^2+6)
\end{aligned}$$

Thus, we see that $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ as required.

Problem 2: A Simple Scalar Field
Consider the scalar field $\rho(\vec{x}) = \rho(x, y, z) = x^2 + 2xy - z$, and the spherical coordinate system (r, θ, ϕ).
a) What is $\partial\rho/\partial r$ at $(x, y, z) = (2, -2, 1)$?

SOLUTION: Using the chain rule, you have that

$$\frac{\partial\rho}{\partial r} = \frac{\partial\rho}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\rho}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial\rho}{\partial z}\frac{\partial z}{\partial r}$$

Using the transformation rules between (x, y, z) and (r, θ, ϕ), we have that

$$\begin{aligned}
\frac{\partial x}{\partial r} &= \sin\theta\,\cos\phi = \frac{x}{r} \\
\frac{\partial y}{\partial r} &= \sin\theta\,\sin\phi = \frac{y}{r} \\
\frac{\partial z}{\partial r} &= \cos\theta = \frac{z}{r}
\end{aligned}$$

Substituting in the above chain rule, and using the partial derivatives of ρ, we obtain that

$$\frac{\partial\rho}{\partial r} = (2x + 2y)\,\frac{x}{r} + 2x\,\frac{y}{r} - \frac{z}{r} = \frac{2x^2 + 4xy - z}{\sqrt{x^2 + y^2 + z^2}}$$

Now we can substitute $(x, y, z) = (2, -2, 1)$, which yields that $\partial\rho/\partial r = -9/\sqrt{9} = -3$.
NOTE: it is NOT correct to write $\partial x/\partial r = 1/(\partial r/\partial x) = 1/(x/r) = r/x$ With partial derivatives this doesn't work (i.e., $\partial x/\partial r$ means 'keeping θ and ϕ fixed', while $\partial r/\partial x$ means 'keeping y and z fixed').

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b) What is $\partial\rho/\partial r$ at the origin?

SOLUTION: It is easiest to first express the above expression for $\partial\rho/\partial r$ in spherical coordinates, which yields

$$\begin{aligned}\frac{\partial\rho}{\partial r} &= \frac{2r^2\sin^2\theta\cos^2\phi + 4r^2\sin^2\theta\cos\phi\sin\phi - r\cos\theta}{r} \\ &= 2r\sin^2\theta\left[\cos^2\phi + \sin 2\phi\right] - \cos\theta\end{aligned}$$

At the origin, $r = 0$, we thus have that $\partial\rho/\partial r = -\cos\theta$. Hence, the partial derivative of ρ with respect to radius depends on θ. Since the latter is not defined, we can't specify $\partial\rho/\partial r$, and without any further information, it is basically undefined.

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c) What is the derivative of ρ at $(x, y, z) = (2, -2, 1)$ in the direction of $\vec{w} = (3, 1, -2)$?

SOLUTION: The derivative of ρ in the direction of a vector $\vec{w}$ is given by

$$D_w\,\rho = \nabla\rho \cdot \frac{\vec{w}}{|\vec{w}|}$$

Using that $|\vec{w}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$ and that $\nabla\rho = (\partial\rho/\partial x, \partial\rho/\partial y, \partial\rho/\partial z)$, we have that $D_w = (2x + 2y, 2x, -1) \cdot (3/\sqrt{14}, 1/\sqrt{14}, -2/\sqrt{14}) = 6/\sqrt{14}$.

Problem 3: Conservative Force Fields

Consider a force field $\vec{F}(\vec{x}) = (F_x, F_y, F_z) = (axy, x^2 + z^3, byz^2 - 4z^3)$, with a and b two constants and $\vec{x} = (x, y, z)$ Cartesian coordinates.

a) For what a and b is $\vec{F}(\vec{x})$ conservative?

SOLUTION: A conservative vector field is curl-free. Hence, we need to determine for which (a, b) we have that $\nabla \times \vec{F} = 0$. Using that

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \\
&= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z} \\
&= (bz^2 - 3z^2)\,\hat{x} + (0 - 0)\,\hat{y} + (2x - ax)\,\hat{z}
\end{aligned}$$

Demanding that each component is equal to zero, we see that $(a, b) = (2, 3)$.

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b) Derive the corresponding scalar potential field, $\Phi(\vec{x})$.

SOLUTION: The scalar potential field is defined by $\vec{F} = \nabla\Phi$, or, in index form $F_i = \partial\Phi/\partial x_i$. Starting with the x-component, and making our way forward, we obtain that

$$\begin{aligned}
F_x &= \frac{\partial \Phi}{\partial x} \Rightarrow 2xy = \frac{\partial \Phi}{\partial x} \Rightarrow \Phi(\vec{x}) = x^2 y + g(y, z) \\
F_y &= \frac{\partial \Phi}{\partial y} = x^2 + \frac{\partial g}{\partial y} \Rightarrow x^2 + z^3 = x^2 + \frac{\partial g}{\partial y} \Rightarrow \Phi(\vec{x}) = x^2 y + z^3 y + f(z) \\
F_z &= \frac{\partial \Phi}{\partial z} = 3z^2 y + \frac{\partial f}{\partial z} \Rightarrow 3yz^2 - 4z^3 = 3z^2 y + \frac{\partial f}{\partial z}
\end{aligned}$$

$$\Rightarrow \Phi(\vec{x}) = x^2 y + yz^3 - z^4 + C$$

where C is an integration constant.

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c) Let F_r, F_θ and F_ϕ be the components of $\vec{F}$ in the spherical coordinate system. Write down expressions of F_r, F_θ and F_ϕ in terms of F_x, F_y and F_z. Show your derivation.

SOLUTION: To transform $\vec{F}$ from the Cartesian basis, $\mathcal{C} = (\vec{e}_x, \vec{e}_y, \vec{e}_z)$ to the basis $\mathcal{B} = (\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ we use that

$$[\vec{F}]_{\mathcal{B}} = \mathbf{T}^{-1} [\vec{F}]_{\mathcal{C}}$$

where $\mathbf{T}$ is the "transformation of basis matrix", whose column vectors are the unit direction vectors of the $\mathcal{B}$-basis. Using that the inverse of $\mathbf{T}$ is equal to its transpose (i.e., $\mathbf{T}$ is an orthogonal matrix), we have that

$$\mathbf{T}^{-1} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

which implies that

$$\begin{aligned} F_r &= F_x \sin\theta\cos\phi + F_y \sin\theta\sin\phi + F_z \cos\theta \\ F_\theta &= F_x \cos\theta\cos\phi + F_y \cos\theta\sin\phi - F_z \sin\theta \\ F_\phi &= -F_x \sin\phi + F_y \cos\phi \end{aligned}$$

Problem 4: Solenoidal Vector Fields
Consider the 2D solenoidal vector field $\vec{F} = -y\,\hat{x} + x\,\hat{y}$ and the two points $x_0 = (1, 2)$ and $x_1 = (3, 4)$. Consider two different paths from x_0 to x_1:

Path 1: $(1, 2) \to (1, 4) \to (3, 4)$
Path 2: $(1, 2) \to (1, 0) \to (3, 0) \to (3, 4)$

a) Compute $\int_{x_0}^{x_1} \vec{F} \cdot \mathrm{d}\vec{l}$ along both paths 1 and 2.

SOLUTION: Note that $\int \vec{F} \cdot \mathrm{d}\vec{l} = \int (F_x\,\mathrm{d}x + F_y\,\mathrm{d}y) = \int F_x\,\mathrm{d}x + \int F_y\,\mathrm{d}y = -\int y\,\mathrm{d}x + \int x\,\mathrm{d}y$. Hence, when integrating along Path 1, we get

$$\int \vec{F} \cdot \mathrm{d}\vec{l} = \int_2^4 1\,\mathrm{d}y - \int_1^3 4\,\mathrm{d}x = 2 - 8 = -6$$

Similarly, for Path 2 we get

$$\int \vec{F} \cdot \mathrm{d}\vec{l} = \int_2^0 1\,\mathrm{d}y - \int_1^3 0\,\mathrm{d}x + \int_0^4 3\,\mathrm{d}y = -2 - 0 + 12 = 10$$

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b) Paths 1 and 2 combined make up a closed curve c. What is $\oint_c \vec{F} \cdot \mathrm{d}\vec{l}$?

SOLUTION: By convention, the closed path has to be traced in the counter-clockwise direction. Hence, we obtain that

$$\oint_c \vec{F} \cdot \mathrm{d}\vec{l} = \int_{\text{path2}} \vec{F} \cdot \mathrm{d}\vec{l} - \int_{\text{path1}} \vec{F} \cdot \mathrm{d}\vec{l} = 10 - (-6) = 16$$

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c) Show that Green's Theorem holds by computing $\int\int_A \nabla \times \vec{F}\,\mathrm{d}A$ where A is the region enclosed by c.

SOLUTION: Green's Theorem, which holds in two-dimensions, states that

$$\oint_c \vec{F} \cdot \mathrm{d}\vec{l} = \int\int_A \nabla \times \vec{F} \cdot \hat{n}\,\mathrm{d}A$$

where A is the area bounded by c, whose normal direction vector is $\hat{n}$. Using that

$$\nabla \times \vec{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \vec{e}_z = (1+1)\,\vec{e}_z = 2\,\vec{e}_z\,,$$

that $\hat{n} = \vec{e}_z$, and that $\vec{e}_z \cdot \vec{e}_z = 1$, we obtain that

$$\int\int_A \nabla \times \vec{F} \cdot \hat{n}\, \mathrm{d}A = 2 \int\int_A \mathrm{d}A = 2A = 2 \int_1^3 \mathrm{d}x \int_0^4 \mathrm{d}y = 16$$

which demonstrates that Green's Theorem indeed holds.