Problem 1: Vector Field Gymnastics
Let \( \vec{A}(x) = (x^2 + y, 2x^3 + y^3z, x + 3z^2) \).

a) Compute \( \nabla \cdot \vec{A} \).

**SOLUTION:**
\[
\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2x + 3y^2z + 6z
\]

b) Compute \( \nabla \times \vec{A} \).

**SOLUTION:**
\[
\nabla \times \vec{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) \hat{e}_x + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) \hat{e}_y + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \hat{e}_z
\]
\[
= (0 - y^3) \hat{e}_x + (0 - 1) \hat{e}_y + (6x^2 - 1) \hat{e}_z
\]
\[
= (-y^3, -1, 6x^2 - 1)
\]

c) Compute \( \nabla^2 \vec{A} \).

**SOLUTION:**
\[
\nabla^2 \vec{A} = (\nabla \cdot \nabla) \vec{A}
\]
\[
= \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \right) \hat{e}_x
\]
\[
+ \left( \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \right) \hat{e}_y
\]
\[
\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \cdot \vec{e}_z \\
= (2 + 0 + 0) \vec{e}_x + (12x + 6yz + 0) \vec{e}_y + (0 + 0 + 6) \vec{e}_z \\
= (2, 12x + 6yz, 6)
\]

d) Compute \( \nabla \times (\nabla \times \vec{A}) \)

**SOLUTION:**

\[
\nabla \times (\nabla \times \vec{A}) = \nabla \times (-y^3, -1, 6x^2 - 1) = (0 - 0) \vec{e}_x + (0 - 12x) \vec{e}_y + (0 + 3y^2) \vec{e}_z = (0, -12x, 3y^2)
\]

e) Compute \( \nabla (\nabla \cdot \vec{A}) \) and verify that \( \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \)

**SOLUTION:**

\[
\nabla (\nabla \cdot \vec{A}) = \nabla (2x + 3y^2z + 6z) \\
= \left( \frac{\partial (2x + 3y^2z + 6z)}{\partial x}, \frac{\partial (2x + 3y^2z + 6z)}{\partial y}, \frac{\partial (2x + 3y^2z + 6z)}{\partial z} \right) \\
= (2, 6yz, 3y^2 + 6)
\]

Thus, we see that \( \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \) as required.
Problem 2: A Simple Scalar Field
Consider the scalar field \( \rho(\vec{x}) = \rho(x, y, z) = x^2 + 2xy - z \), and the spherical coordinate system \((r, \theta, \phi)\).

a) What is \( \partial \rho / \partial r \) at \((x, y, z) = (2, -2, 1)\)?

**SOLUTION:** Using the chain rule, you have that

\[
\frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial \rho}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial \rho}{\partial z} \frac{\partial z}{\partial r}
\]

Using the transformation rules between \((x, y, z)\) and \((r, \theta, \phi)\), we have that

\[
\begin{align*}
\frac{\partial x}{\partial r} &= \sin \theta \cos \phi = \frac{x}{r} \\
\frac{\partial y}{\partial r} &= \sin \theta \sin \phi = \frac{y}{r} \\
\frac{\partial z}{\partial r} &= \cos \theta = \frac{z}{r}
\end{align*}
\]

Substituting in the above chain rule, and using the partial derivatives of \( \rho \), we obtain that

\[
\frac{\partial \rho}{\partial r} = (2x + 2y) \frac{x}{r} + 2x \frac{y}{r} - \frac{z}{r} = \frac{2x^2 + 4xy - z}{\sqrt{x^2 + y^2 + z^2}}
\]

Now we can substitute \((x, y, z) = (2, -2, 1)\), which yields that \( \partial \rho / \partial r = -9 / \sqrt{9} = -3 \).

**NOTE:** It is NOT correct to write \( \partial x / \partial r = 1 / (\partial r / \partial x) = 1 / (x / r) = r / x \).

With partial derivatives this doesn’t work (i.e., \( \partial x / \partial r \) means ‘keeping \( \theta \) and \( \phi \) fixed’, while \( \partial r / \partial x \) means ‘keeping \( y \) and \( z \) fixed’).
b) What is $\partial \rho / \partial r$ at the origin?

**SOLUTION:** It is easiest to first express the above expression for $\partial \rho / \partial r$ in spherical coordinates, which yields

$$
\frac{\partial \rho}{\partial r} = \frac{2r^2 \sin^2 \theta \cos^2 \phi + 4r^2 \sin^2 \theta \cos \phi \sin \phi - r \cos \theta}{r} = 2r \sin^2 \theta \left[ \cos^2 \phi + \sin 2\phi \right] - \cos \theta
$$

At the origin, $r = 0$, we thus have that $\partial \rho / \partial r = -\cos \theta$. Hence, the partial derivative of $\rho$ with respect to radius depends on $\theta$. Since the latter is not defined, we can’t specify $\partial \rho / \partial r$, and without any further information, it is basically undefined.

c) What is the derivative of $\rho$ at $(x, y, z) = (2, -2, 1)$ in the direction of $\vec{w} = (3, 1, -2)$?

**SOLUTION:** The derivative of $\rho$ in the direction of a vector $\vec{w}$ is given by

$$D_w \rho = \nabla \rho \cdot \frac{\vec{w}}{|\vec{w}|}$$

Using that $|\vec{w}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$ and that $\nabla \rho = (\partial \rho / \partial x, \partial \rho / \partial y, \partial \rho / \partial z)$, we have that $D_w = (2x + 2y, 2x, -1) \cdot (3/\sqrt{14}, 1/\sqrt{14}, -2/\sqrt{14}) = 6/\sqrt{14}$. 


Problem 3: Conservative Force Fields

Consider a force field \( \vec{F}(\vec{x}) = (F_x, F_y, F_z) = (axy, x^2 + z^3, byz^2 - 4z^3) \), with \( a \) and \( b \) two constants and \( \vec{x} = (x, y, z) \) Cartesian coordinates.

a) For what \( a \) and \( b \) is \( \vec{F}(\vec{x}) \) conservative?

**SOLUTION:** A conservative vector field is curl-free. Hence, we need to determine for which \((a, b)\) we have that \(\nabla \times \vec{F} = 0\). Using that

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
F_x & F_y & F_z \\
\end{vmatrix}
\]

\[
= \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}
\]

\[
= (bz^2 - 3z^2) \hat{x} + (0 - 0) \hat{y} + (2x - ax) \hat{z}
\]

Demanding that each component is equal to zero, we see that \((a, b) = (2, 3)\).

b) Derive the corresponding scalar potential field, \( \Phi(\vec{x}) \).

**SOLUTION:** The scalar potential field is defined by \( \vec{F} = \nabla \Phi \), or, in index form \( F_i = \partial \Phi / \partial x_i \). Starting with the \( x \)-component, and making our way forward, we obtain that

\[
F_x = \frac{\partial \Phi}{\partial x} \Rightarrow 2xy = \frac{\partial \Phi}{\partial x} \Rightarrow \Phi(\vec{x}) = x^2y + g(y, z)
\]

\[
F_y = \frac{\partial \Phi}{\partial y} = x^2 + \frac{\partial g}{\partial y} \Rightarrow x^2 + z^3 = x^2 + \frac{\partial g}{\partial y} \Rightarrow \Phi(\vec{x}) = x^2y + z^3y + f(z)
\]

\[
F_z = \frac{\partial \Phi}{\partial z} = 3z^2y + \frac{\partial f}{\partial z} \Rightarrow 3yz^2 - 4z^3 = 3z^2y + \frac{\partial f}{\partial z} \Rightarrow \Phi(\vec{x}) = x^2y + yz^3 - z^4 + C
\]

where \( C \) is an integration constant.
c) Let \( F_r, F_\theta \) and \( F_\phi \) be the components of \( \vec{F} \) in the spherical coordinate system. Write down expressions of \( F_r, F_\theta \) and \( F_\phi \) in terms of \( F_x, F_y \) and \( F_z \). Show your derivation.

**SOLUTION:** To transform \( \vec{F} \) from the Cartesian basis, \( \mathcal{C} = (\vec{e}_x, \vec{e}_y, \vec{e}_z) \) to the basis \( \mathcal{B} = (\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi) \) we use that

\[
[\vec{F}]_\mathcal{B} = T^{-1} [\vec{F}]_\mathcal{C}
\]

where \( T \) is the “transformation of basis matrix”, whose column vectors are the unit direction vectors of the \( \mathcal{B} \)-basis. Using that the inverse of \( T \) is equal to its transpose (i.e., \( T \) is an orthogonal matrix), we have that

\[
T^{-1} = \begin{pmatrix}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{pmatrix}
\]

which implies that

\[
F_r = F_x \sin \theta \cos \phi + F_y \sin \theta \sin \phi + F_z \cos \theta \\
F_\theta = F_x \cos \theta \cos \phi + F_y \cos \theta \sin \phi - F_z \sin \theta \\
F_\phi = -F_x \sin \phi + F_y \cos \phi
\]
**Problem 4: Solenoidal Vector Fields**

Consider the 2D solenoidal vector field $\vec{F} = -y \hat{x} + x \hat{y}$ and the two points $x_0 = (1, 2)$ and $x_1 = (3, 4)$. Consider two different paths from $x_0$ to $x_1$:

Path 1: $(1, 2) \rightarrow (1, 4) \rightarrow (3, 4)$
Path 2: $(1, 2) \rightarrow (1, 0) \rightarrow (3, 0) \rightarrow (3, 4)$

a) Compute $\oint_{x_0 x_1} \vec{F} \cdot d\vec{l}$ along both paths 1 and 2.

**SOLUTION:** Note that

\[ \oint \vec{F} \cdot d\vec{l} = \int (F_x \, dx + F_y \, dy) = \int F_x \, dx + \int F_y \, dy = -\int y \, dx + \int x \, dy. \]

Hence, when integrating along Path 1, we get

\[ \int \vec{F} \cdot d\vec{l} = \int_2^4 1 \, dy - \int_1^3 4 \, dx = 2 - 8 = -6 \]

Similarly, for Path 2 we get

\[ \int \vec{F} \cdot d\vec{l} = \int_2^0 1 \, dy - \int_0^3 0 \, dx + \int_0^4 3 \, dy = -2 - 0 + 12 = 10 \]

b) Paths 1 and 2 combined make up a closed curve $c$. What is $\oint_c \vec{F} \cdot d\vec{l}$?

**SOLUTION:** By convention, the closed path has to be traced in the counterclockwise direction. Hence, we obtain that

\[ \oint_c \vec{F} \cdot d\vec{l} = \int_{\text{path2}} \vec{F} \cdot d\vec{l} - \int_{\text{path1}} \vec{F} \cdot d\vec{l} = 10 - (-6) = 16 \]

c) Show that Green’s Theorem holds by computing $\iint_A \nabla \times \vec{F} \, dA$ where $A$ is the region enclosed by $c$.

**SOLUTION:** Green’s Theorem, which holds in two-dimensions, states that

\[ \oint_c \vec{F} \cdot d\vec{l} = \int \int_A \nabla \times \vec{F} \cdot \hat{n} \, dA \]

where $A$ is the area bounded by $c$, whose normal direction vector is $\hat{n}$. Using that
\[ \nabla \times \vec{F} = \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \vec{e}_z = (1 + 1) \vec{e}_z = 2 \vec{e}_z, \]

that \( \hat{n} = \vec{e}_z \), and that \( \vec{e}_z \cdot \vec{e}_z = 1 \), we obtain that

\[
\int \int_A \nabla \times \vec{F} \cdot \hat{n} \, dA = 2 \int \int_A \, dA = 2A = 2 \int_1^3 \, dx \int_0^4 \, dy = 16
\]

which demonstrates that Green’s Theorem indeed holds.