ASTR 320: Problem Set 0

This problem set consists of 4 problems and will NOT be graded
It is intended as a voluntary test of required math skills

This material is covered in Appendices A-C of the Lecture Notes

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Problem 1: Vector Field Gymnastics
Let $\vec{A}(\vec{x}) = (x^2 + y, 2x^3 + y^3 z, x + 3z^2)$.

a) Compute $\nabla \cdot \vec{A}$.

b) Compute $\nabla \times \vec{A}$.

c) Compute $\nabla^2 \vec{A}$.

d) Compute $\nabla \times (\nabla \times \vec{A})$

e) Compute $\nabla (\nabla \cdot \vec{A})$ and verify that $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Problem 2: A Simple Scalar Field
Consider the scalar field $\rho(\vec{x}) = \rho(x, y, z) = x^2 + 2xy - z$, and the spherical coordinate system (r, θ, ϕ).

a) What is $\partial\rho/\partial r$ at $(x, y, z) = (2, -2, 1)$?

b) What is $\partial\rho/\partial r$ at the origin?

c) What is the derivative of ρ at $(x, y, z) = (2, -2, 1)$ in the direction of $\vec{w} = (3, 1, -2)$?

Problem 3: Conservative Force Fields
Consider a force field $\vec{F}(\vec{x}) = (F_x, F_y, F_z) = (axy, x^2 + z^3, byz^2 - 4z^3)$, with a and b two constants and $\vec{x} = (x, y, z)$ Cartesian coordinates.

a) For what a and b is $\vec{F}(\vec{x})$ conservative?

b) Derive the corresponding scalar potential field, $\Phi(\vec{x})$.

c) Let F_r, F_θ and F_ϕ be the components of $\vec{F}$ in the spherical coordinate system. Write down expressions of F_r, F_θ and F_ϕ in terms of F_x, F_y and F_z. Show your derivation.
`HINT:` Use the transformation of basis matrix (see App. C of Lecture Notes)

Problem 4: Solenoidal Vector Fields

Consider the 2D solenoidal vector field $\vec{F} = -y\,\hat{x} + x\,\hat{y}$ and the two points $x_0 = (1, 2)$ and $x_1 = (3, 4)$. Consider two different paths from x_0 to x_1:

Path 1: $(1, 2) \rightarrow (1, 4) \rightarrow (3, 4)$

Path 2: $(1, 2) \rightarrow (1, 0) \rightarrow (3, 0) \rightarrow (3, 4)$

a) Compute $\int_{x_0}^{x_1} \vec{F} \cdot \mathrm{d}\vec{l}$ along both paths 1 and 2.

b) Paths 1 and 2 combined make up a closed curve c. What is $\oint_c \vec{F} \cdot \mathrm{d}\vec{l}$?

c) Show that Green's Theorem holds by computing $\int\int_A \nabla \times \vec{F}\,\mathrm{d}A$ where A is the region enclosed by c.