Physical Processes in Astronomy

a concise treatment of fluid dynamics, collisionless dynamics, plasma physics, and radiative processes in astrophysics

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These lecture notes are constantly being updated, extended and improved

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EXTRA-CURRICULAR MATERIAL

The following chapters are NOT part of the course material on which the students will be examined. They are solely provided here for those students interested in learning some extra material.

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The material covered and presented in these lecture notes has relied heavily on a number of excellent textbooks listed below.

- **The Physics of Fluids and Plasmas**  
  by A. Choudhuri (ISBN-0-521-55543)

- **Modern Fluid Dynamics for Physics and Astrophysics**  

- **Principles of Astrophysical Fluid Dynamics**  

- **The Physics of Astrophysics–I. Radiation**  
  by F. Shu (ISBN-0-935702-64-4)

- **The Physics of Astrophysics–II. Gas Dynamics**  
  by F. Shu (ISBN-0-935702-65-2)

- **Astrophysics: Decoding the Cosmos**  

- **Astrophysics Processes: The physics of Astronomical Phenomena**  
  by H. Bradt (ISBN-978-0-521-84656-1)

- **Theoretical Astrophysics**  
  by M. Bartelmann (ISBN-978-3-527-41004-0)

- **Radiative Processes in Astrophysics**  

- **Galactic Dynamics**  

- **Galaxy Formation & Evolution**  
  by H. Mo, F. van den Bosch & S. White (ISBN-978-0-521-85793-2)
CHAPTER 1

Introduction to Fluids

What is a fluid?
A fluid is a substance that can flow, has no fixed shape, and offers little resistance to an external stress

- In a fluid the constituent particles (atoms, ions, molecules, stars) can ‘freely’ move past one another.
- Fluids take on the shape of their container.
- A fluid changes its shape at a steady rate when acted upon by a stress force.

Different types of Fluids:
We distinguish neutral fluids and charged fluids, depending on whether the constituent particles are electrically neutral or charged. Neutral fluids can be either gaseous or liquid. Liquids are (to good approximation) incompressible, which means that a liquid of given mass occupies a given volume. Gases can be either compressible or incompressible: When the gas is in a container, you can easily compress it with a piston, but if I move my hand (sub-sonically) through the air, the gas adjust itself to the perturbation in an incompressible fashion (it moves out of the way at the speed of sound). The small compression at my hand propagates forward at the speed of sound (sound wave) and disperses the gas particles out of the way. In astrophysics we rarely encounter containers, and subsonic gas flow is often treated (to good approximation) as being incompressible. Charged fluids are called plasmas, and can be either gaseous or liquid. In almost all cases, certainly in astrophysics, plasmas are gaseous and we therefore won’t consider liquid plasmas in what follows.

Both neutral fluids and plasmas can be either collisional or collisionless, depending on whether the mean-free-path of the particles is small or large, respectively, compared to the size of the system. Most of your ‘daily intuition’ comes from air and water, and therefore applies to collisional, neutral fluids. In astrophysics, though, plasmas are far more common than neutral fluids (both by volume and by mass),
and we also frequently encounter collisionless fluids in the form of galaxies and dark matter halos.

Figure 1 shows examples of trajectories in three different fluids. Panel (a) depicts the trajectory in a **collisional, neutral fluid**, where particles move along straight lines in between (short-range) collisions. In neutral fluids, the typical cross section for interactions is the size of the particles (i.e., the Bohr radius for atoms), which is very small.

Panel (b) depicts the typical trajectory of a charged particle in a **plasma**, regulated by long-range electromagnetic interactions. A charged particle feels the cumulative electrostatic force of all other charged particles. A plasma is typically quasi-neutral, in that it contains (roughly) as many positive as negative charges. A charged particle therefore mainly undergoes **collective interactions**. Only when a charged particle comes within the **Debye length** of another charged particle will its trajectory be dominated by the charges of individual particles, causing a large-angle deflection similar to that of collisions in a neutral fluid. The ratio between the Debye length and the scale of interest determines to what extent the plasma is **collisional** (dominated by interactions with individual charges) or **collisionless** (dominated by collective interactions). Plasma are excellent conductors, and therefore are quickly shorted by currents; hence in many cases one may ignore the electrical field, and focus exclusively on the magnetic field instead (this is called **magneto-hydro-dynamics**, or MHD for short). Many astrophysical plasmas have relatively weak magnetic fields, and we therefore don’t make big errors if we ignore them. In this case, when electromagnetic interactions are not important, plasmas behave very much like neutral fluids. Because of this, most of the material covered in this course will focus on neutral fluids only.
Finally, panel (c) of Figure 1 depicts the trajectory (‘orbit’) of a particle in a self-gravitating, collisionless and neutral fluid (i.e., a star within a galaxy). Here the particle experiences the collective gravitational force of all other particles. Contrary to a plasma, there is no Debye shielding, as there are only ‘positive charges’ (aka mass) for gravity. In principle particles can still undergo large-angle deflections (‘collisions’) with other particles when they have a sufficiently close encounter, but we will see that this can be ignored as long as \( N \), the number of particles of the system, is sufficiently large. Hence, self-gravitating systems in which the forces are purely gravitational are always collisionless, as long as \( N \) is sufficiently large.

In this course, we will start with standard hydrodynamics, which applies mainly to neutral, collisional fluids. We briefly discuss the intriguing realm of collisionless fluid dynamics (‘galactic dynamics’) and end with a discussion of radiative processes, and their interaction between radiation and matter. Unfortunately, we won’t have time to discuss plasma’s and the role of magnetic fields. However, for those interested, these lecture notes do include three ‘extra-curricular’ chapters at the end (chapters P1-P3) that describe some of the basics. These are NOT part of the official course material, but are just included for those interested.

Throughout what follows, we use ‘fluid’ to mean a neutral fluid, and ‘plasma’ to refer to a fluid in which the particles are electrically charged.

**Ideal (Perfect) Fluids and Ideal Gases:**

As we discuss in more detail in Chapter 4, the resistance of fluids to shear distortions is called viscosity, which is a microscopic property of the fluid that depends on the nature of its constituent particles, and on thermodynamic properties such as temperature. Fluids are also conductive, in that the microscopic collisions between the constituent particles cause heat conduction through the fluid. In many fluids encountered in astrophysics, the viscosity and conduction are very small. An ideal fluid, also called a perfect fluid, is a fluid with zero viscosity and zero conduction.

NOTE: An ideal (or perfect) fluid should NOT be confused with an ideal or perfect gas, which is defined as a gas in which the pressure is solely due to the kinetic motions of the constituent particles. As we show in Chapter 11, and as you have probably seen before, this implies that the pressure can be written as \( P = n k_B T \), with \( n \) the particle number density, \( k_B \) the Boltzmann constant, and \( T \) the temperature.
Examples of Fluids in Astrophysics:

- **Stars**: stars are spheres of gas in hydrostatic equilibrium (i.e., gravitational force is balanced by pressure gradients). Densities and temperatures in a given star cover many orders of magnitude. To good approximation, its equation of state is that of an ideal gas.

- **Giant (gaseous) planets**: Similar to stars, gaseous planets are large spheres of gas, albeit with a rocky core. Contrary to stars, though, the gas is typically so dense and cold that it can no longer be described with the equation of state of an ideal gas.

- **Planet atmospheres**: The atmospheres of planets are stratified, gaseous fluids retained by the planet’s gravity.

- **White Dwarfs & Neutron stars**: These objects (stellar remnants) can be described as fluids with a degenerate equation of state.

- **Proto-planetary disks**: the dense disks of gas and dust surrounding newly formed stars out of which planetary systems form.

- **Inter-Stellar Medium (ISM)**: The gas in between the stars in a galaxy. The ISM is typically extremely complicated, and roughly has a three-phase structure: it consists of a dense, cold ($\sim 10$K) molecular phase, a warm ($\sim 10^4$K) phase, and a dilute, hot ($\sim 10^6$K) phase. Stars form out of the dense molecular phase, while the hot phase is (shock) heated by supernova explosions. The reason for this three phase medium is associated with the various cooling mechanisms. At high temperature when all gas is ionized, the main cooling channel is Bremmstrahlung (acceleration of free electrons by positively charged ions). At low temperatures ($< 10^4$K), the main cooling channel is molecular cooling (or cooling through hyperfine transitions in metals).

- **Inter-Galactic Medium (IGM)**: The gas in between galaxies. This gas is typically very, very dilute (low density). It is continuously ‘exposed’ to adiabatic cooling due to the expansion of the Universe, but also is heated by radiation from stars (galaxies) and AGN (active galactic nuclei). The latter, called ‘reionization’, assures that the typical temperature of the IGM is $\sim 10^4$K.
• **Intra-Cluster Medium (ICM):** The hot gas in clusters of galaxies. This is gas that has been shock heated when it fell into the cluster; typically gas passes through an **accretion shock** when it falls into a dark matter halo, converting its infall velocity into thermal motion.

• **Accretion disks:** Accretion disks are gaseous, viscous disks in which the viscosity (enhanced due to turbulence) causes a net rate of radial matter towards the center of the disk, while angular momentum is being transported outwards (accretion)

• **Galaxies (stellar component):** as we will see later, the stellar component of galaxies is a collisionless fluid; to very, very good approximation, two stars in a galaxy will never collide with other.

• **Dark matter halos:** Another example of a collisionless fluid (at least, we assume that dark matter is collisionless)...
A dynamical theory consists of two characteristic elements:

1. a way to describe the state of the system
2. a (set of) equation(s) to describe how the state variables change with time

Consider the following examples:

**Example 1:** a classical dynamical system

This system is described by the position vectors \( \vec{x} \) and momentum vectors \( \vec{p} \) of all the \( N \) particles, i.e., by \( (\vec{x}_1, \vec{x}_2, ..., \vec{x}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N) \).

If the particles are truly classical, in that they can’t emit or absorb radiation, then one can define a Hamiltonian

\[
\mathcal{H}(\vec{x}_i, \vec{p}_i, t) \equiv \mathcal{H}(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N, t) = \sum_{i=1}^{N} \vec{p}_i \cdot \dot{\vec{x}}_i - \mathcal{L}(\vec{x}_i, \dot{\vec{x}}_i, t)
\]

where \( \mathcal{L}(\vec{x}_i, \dot{\vec{x}}_i, t) \) is the system’s Lagrangian, and \( \dot{\vec{x}}_i = \frac{d\vec{x}_i}{dt} \).

The equations that describe the time-evolution of these state-variables are the Hamiltonian equations of motion:

\[
\dot{\vec{x}}_i = \frac{\partial \mathcal{H}}{\partial \vec{p}_i}; \quad \dot{\vec{p}}_i = -\frac{\partial \mathcal{H}}{\partial \vec{x}_i}
\]

**Example 2:** an electromagnetic field

The state of this system is described by the electrical and magnetic fields, \( \vec{E}(\vec{x}) \) and \( \vec{B}(\vec{x}) \), respectively, and the equations that describe their evolution with time are the Maxwell equations, which contain the terms \( \partial \vec{E}/\partial t \) and \( \partial \vec{B}/\partial t \).
Example 3: a quantum system

The state of a quantum system is fully described by the (complex) wavefunction $\psi(\vec{x})$, the time-evolution of which is described by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

where $\hat{H}$ is now the Hamiltonian operator.

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<tr>
<th>Level</th>
<th>Description of state</th>
<th>Dynamical equations</th>
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<td>$\psi(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N)$</td>
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<tr>
<td>1: $N$ classical particles</td>
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<td>3: Continuum model</td>
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Different levels of dynamical theories to describe neutral fluids

The different levels of Fluid Dynamics

There are different ‘levels’ of dynamical theories to describe fluids. Since all fluids are ultimately made up of constituent particles, and since all particles are ultimately ‘quantum’ in nature, the most ‘basic’ level of fluid dynamics describes the state of a fluid in terms of the $N$-particle wave function $\psi(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N)$, which evolves in time according to the Schrödinger equation. We will call this the level-0 description of fluid dynamics. Since $N$ is typically extremely large, this level-0 description is extremely complicated and utterly unfeasible. Fortunately, it is also unnecessary.

According to what is known as Ehrenfest’s theorem, a system of $N$ quantum particles can be treated as a system of $N$ classical particles if the characteristic separation between the particles is large compared to the ‘de Broglie’ wavelength

$$\lambda = \frac{\hbar}{p} \approx \frac{\hbar}{\sqrt{mk_B T}}$$

Here $\hbar$ is the Planck constant, $p$ is the particle’s momentum, $m$ is the particle mass, $k_B$ is the Boltzman constant, and $T$ is the temperature of the fluid. This de Broglie
wavelength indicates the ‘characteristic’ size of the wave-packet that according to quantum mechanics describes the particle, and is typically very small. Except for extremely dense fluids such as white dwarfs and neutron stars, or ‘exotic’ types of dark matter (i.e., ‘fuzzy dark matter’), the de Broglie wavelength is always much smaller than the mean particle separation, and classical, Newtonian mechanics suffices. As we have seen above, a classical, Newtonian system of $N$ particles can be described by a Hamiltonian, and the corresponding equations of motions. We refer to this as the level-1 description of fluid dynamics (see under ‘example 1’ above). Clearly, when $N$ is very large is it unfeasible to solve the $2N$ equations of motion for all the positions and momenta of all particles. We need another approach.

In the level-2 approach, one introduces the distribution function $f(\vec{x}, \vec{p}, t)$, which describes the number density of particles in 6-dimensional ‘phase-space’ $(\vec{x}, \vec{p})$ (i.e., how many particles are there with positions in the 3D volume $\vec{x} + d\vec{x}$ and momenta in the 3D volume $\vec{p} + d\vec{p}$). The equation that describes how $f(\vec{x}, \vec{p}, t)$ evolves with time is called the Boltzmann equation for a neutral fluid, or the Vlasov equation for a plasma.

At the final level-3, the fluid is modelled as a continuum. This means we ignore that fluids are made up of constituent particles, and rather describe the fluid with continuous fields, such as the density and velocity fields $\rho(\vec{x})$ and $\vec{u}(\vec{x})$ which assign to each point in space a scalar quantity $\rho$ and a vector quantity $\vec{u}$, respectively. For a neutral fluid, the state in this level-3 approach is fully described by three fields: the density $\rho(\vec{x})$, the velocity field $\vec{u}(\vec{x})$, and the internal, specific energy $\varepsilon(\vec{x})$ (or, equivalently, the temperature $T(\vec{x})$). In the MHD treatment of plasmas one also needs to specify the magnetic field $\vec{B}(\vec{x})$. The equations that describe the time-evolution of $\rho(\vec{x})$, $\vec{u}(\vec{x})$, and $\varepsilon(\vec{x})$ are called the continuity equation, the Navier-Stokes equations, and the energy equation, respectively. Collectively, we shall refer to these as the hydrodynamic equations or fluid equations. In MHD you have to slightly modify the Navier-Stokes equations, and add an additional induction equation describing the time-evolution of the magnetic field. For an ideal (or perfect) fluid (i.e., no viscosity and/or conductivity), the Navier-Stokes equations reduce to what are known as the Euler equations.

Throughout this course, we mainly focus on the level-3 treatment, to which we refer hereafter as the macroscopic approach. However, it is important to realize that this macroscopic treatment is not always possible. For example, the dynamics of
collisionless fluids, such as galaxies and dark matter haloes, which obviously play an important role in astrophysics, cannot be treated this way, and must rely instead on a microscopic treatment (typically level-2). Yet, as we will see, under the assumption of certain symmetries, one can derive, from the Boltzmann equation, a set of equations (the Jeans equations) that closely resemble (but are subtly different from) the corresponding Euler equations for collisional, inviscid fluids derived in the macroscopic approach.

**Fluid Dynamics: The Macroscopic Approach:**

In the macroscopic approach, the fluid is treated as a *continuum*. It is often useful to think of this continuum as ‘made up’ of fluid elements (FE). These are small fluid volumes that nevertheless contain many particles, that are significantly larger than the mean-free path of the particles, and for which one can define local hydrodynamical variables such as density, pressure and temperature. The requirements are:

1. the FE needs to be much smaller than the characteristic scale in the problem, which is the scale over which the hydrodynamical quantities $Q$ change by an order of magnitude, i.e.,
   $$ l_{FE} \ll l_{scale} \sim \frac{Q}{\nabla Q} $$

2. the FE needs to be sufficiently large that fluctuations due to the finite number of particles (‘discreteness noise’) can be neglected, i.e.,
   $$ n l_{FE}^3 \gg 1 $$
   where $n$ is the number density of particles.

3. the FE needs to be sufficiently large that it ‘knows’ about the local conditions through collisions among the constituent particles, i.e.,
   $$ l_{FE} \gg \lambda $$
   where $\lambda$ is the mean-free path of the fluid particles.

Note that fluid elements can NOT be defined for a collisionless fluid (which has an infinite mean-free path). This is one of the reasons why one cannot use the macroscopic approach to derive the equations that govern a collisionless fluid.
Fluid Dynamics: closure:
In general, a fluid element is characterized by the following six hydro-dynamical variables:

- **mass density** $\rho$ [g/cm$^3$]
- **fluid velocity** $\vec{u}$ [cm/s] (3 components)
- **pressure** $P$ [erg/cm$^3$]
- **specific internal energy** $\varepsilon$ [erg/g]

Note that $\vec{u}$ is the velocity of the fluid element, not to be confused with the velocity $\vec{v}$ of individual fluid particles, used in the Boltzmann distribution function. Rather, $\vec{u}$ is (roughly) a vector sum of all particles velocities $\vec{v}$ that make up the fluid element.

In the case of an **ideal (or perfect) fluid** (i.e., with zero viscosity and conductivity), the Navier-Stokes equations (which are the hydrodynamical momentum equations) reduce to what are called the **Euler equations**. In that case, the evolution of fluid elements is described by the following set of hydrodynamical equations:

1. **continuum equation** relating $\rho$ and $\vec{u}$
2. **3 momentum equations** relating $\rho$, $\vec{u}$ and $P$
3. **1 energy equation** relating $\rho$, $\vec{u}$, $P$ and $\varepsilon$

Thus we have a total of 5 equations for 6 unknowns. One can solve the set (‘close it’) by using a **constitutive relation**. In almost all cases, this is the **equation of state** (EoS) $P = P(\rho, \varepsilon)$.

- Sometimes the EoS is expressed as $P = P(\rho, T)$. In that case another constitutive relation is needed, typically $\varepsilon = \varepsilon(\rho, T)$.

- If the EoS is **barotropic**, i.e., if $P = P(\rho)$, then the energy equation is not needed to close the set of equations. There are two barotropic EoS that are encountered frequently in astrophysics: the **isothermal** EoS, which describes a fluid for which cooling and heating always balance each other to maintain a constant temperature, and the **adiabatic** EoS, in which there is no net heating or cooling (other than adiabatic heating or cooling due to the compression or expansion of volume, i.e., the $P \, dV$ work). We will discuss these cases in more detail later in the course.

- No EoS exists for a **collisionless fluid**. Consequently, for a collisionless fluid one can never close the set of fluid equations, unless one makes a number of simplifying
assumptions (i.e., one postulates various symmetries)

• If the fluid is not ideal, then the momentum equations include terms that contain the (kinetic) \textit{viscosity}, \( \nu \), and the energy equation includes a term that contains the conductivity, \( \mathcal{K} \). Both \( \nu \) and \( \mathcal{K} \) depend on the mean-free path of the constituent particles and therefore depend on the temperature and collisional cross-section of the particles. Closure of the set of hydrodynamic equations then demands additional constitutive equations \( \nu(T) \) and \( \mathcal{K}(T) \). Often, though, \( \nu \) and \( \mathcal{K} \) are simply assumed to be constant (the \( T \)-dependence is ignored).

• In the case the fluid is exposed to an \textbf{external force} (i.e., a gravitational or electrical field), the momentum and energy equations contain an extra force term.

• In the case the fluid is \textbf{self-gravitating} (i.e., in the case of stars or galaxies) there is an additional unknown, the gravitational potential \( \Phi \). However, there is also an additional equation, the \textbf{Poisson equation} relating \( \Phi \) to \( \rho \), so that the set of equations remains closed.

\textbf{Fluid Dynamics: Eulerian vs. Lagrangian Formalism:}

One distinguishes two different formalisms for treating fluid dynamics:

• \textbf{Eulerian Formalism:} in this formalism one solves the fluid equations ‘at fixed positions’: the evolution of a quantity \( Q \) is described by the local (or partial, or Eulerian) derivative \( \partial Q / \partial t \). An Eulerian hydrodynamics code is a ‘grid-based code’, which solves the hydro equations on a fixed grid, or using an adaptive grid, which refines resolution where needed. The latter is called \textbf{Adaptive Mesh Refinement} (AMR).

• \textbf{Lagrangian Formalism:} in this formalism one solves the fluid equations ‘comoving with the fluid’, i.e., either at a fixed particle (collisionless fluid) or at a fixed fluid element (collisional fluid). The evolution of a quantity \( Q \) is described by the substantial (or Lagrangian) derivative \( dQ/dt \) (sometimes written as \( DQ/Dt \)). A Lagrangian hydrodynamics code is a ‘particle-based code’, which solves the hydro equations per simulation particle. Since it needs to smooth over neighboring particles in order to compute quantities such as the fluid density, it is called \textbf{Smoothed Particle Hydrodynamics} (SPH).
To derive an expression for the **substantial derivative** \( \frac{dQ}{dt} \), realize that \( Q = Q(t, x, y, z) \). When the fluid element moves, the scalar quantity \( Q \) experiences a change

\[
\frac{dQ}{dt} = \frac{\partial Q}{\partial t} \, dt + \frac{\partial Q}{\partial x} \, dx + \frac{\partial Q}{\partial y} \, dy + \frac{\partial Q}{\partial z} \, dz
\]

Dividing by \( dt \) yields

\[
\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} u_x + \frac{\partial Q}{\partial y} u_y + \frac{\partial Q}{\partial z} u_z
\]

where we have used that \( \frac{dx}{dt} = u_x \), which is the \( x \)-component of the fluid velocity \( \vec{u} \), etc. Hence we have that

\[
\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \vec{u} \cdot \nabla Q
\]

Using a similar derivation, but now for a vector quantity \( \vec{A}(\vec{x}, t) \), it is straightforward to show that

\[
\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{u} \cdot \nabla) \vec{A}
\]

which, in index-notation, is written as

\[
\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + u_j \frac{\partial A_i}{\partial x_j}
\]

Another way to derive the above relation between the Eulerian and Lagrangian derivatives, is to think of \( \frac{dQ}{dt} \) as

\[
\frac{dQ}{dt} = \left. \lim_{\delta t \to 0} \left[ \frac{Q(\vec{x} + \delta \vec{x}, t + \delta t) - Q(\vec{x}, t)}{\delta t} \right] \right|_{\delta t \to 0}
\]

Using that

\[
\vec{u} = \left. \lim_{\delta t \to 0} \left[ \frac{(\vec{x}(t + \delta t) - \vec{x}(t))}{\delta t} \right] \right|_{\delta t \to 0} = \frac{\delta \vec{x}}{\delta t}
\]

and

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\[ \nabla Q = \lim_{\delta \vec{x} \to 0} \left[ \frac{Q(\vec{x} + \delta \vec{x}, t) - Q(\vec{x}, t)}{\delta \vec{x}} \right] \]

It is straightforward to show that this results in the same expression for the substantial derivative as above.

Figure 2: Streaklines showing laminar flow across an airfoil; made by injecting dye at regular intervals in the flow

**Kinematic Concepts: Streamlines, Streaklines and Particle Paths:**

In fluid dynamics it is often useful to distinguish the following kinematic constructs:

- **Streamlines:** curves that are instantaneously tangent to the velocity vector of the flow. Streamlines show the direction a massless fluid element will travel in at any point in time.

- **Streaklines:** the locus of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline.

- **Particle paths:** (aka pathlines) are the trajectories that individual fluid elements follow. The direction the path takes is determined by the streamlines of the fluid at each moment in time.

Only if the flow is **steady**, which means that all partial time derivatives (i.e., \( \partial \bar{u}/\partial t = \partial \rho/\partial t = \partial P/\partial t \)) vanish, will streamlines be identical to streaklines be identical to particle paths. For a non-steady flow, they will differ from each other.
CHAPTER 3

Continuity & Momentum Equations

In this chapter we derive the continuity equation and the momentum equations for a fluid using the macroscopic continuum approach.

**Continuity Equation:** Consider an Eulerian volume $V$. The mass inside $V$ is given by $M = \int \rho \, dV$. The rate at which this mass increases has to be balanced by the rate at which mass flows in or out of $V$. This implies that

$$\frac{\partial}{\partial t} \int_V \rho \, dV = - \int_S \rho \mathbf{u} \cdot d\mathbf{S} = - \int_V \nabla \cdot (\rho \mathbf{u}) \, dV$$

where the last equality follows from Gauss’ divergence theorem, and the minus sign is required because $d\mathbf{S}$ is directed outwards. Since this has to hold for any volume $V$, we obtain the continuity equation:

<table>
<thead>
<tr>
<th></th>
<th>Vector Notation</th>
<th>Index Notation</th>
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</thead>
<tbody>
<tr>
<td>Lagrangian:</td>
<td>$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0$</td>
<td>$\frac{d\rho}{dt} + \rho \frac{\partial u_i}{\partial x_i} = 0$</td>
</tr>
<tr>
<td>Eulerian:</td>
<td>$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$</td>
<td>$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$</td>
</tr>
</tbody>
</table>

NOTE: A liquid is (to good approximation) an **incompressible fluid** which means that $d\rho/dt = 0$ (i.e., the density of each fluid element is conserved). From the continuity equation we then see that $\nabla \cdot \mathbf{u} = 0$ (i.e., the flow is divergence free). It is important to realize that also **gases** (including air) can often be regarded as incompressible. When air is compressed in some localized region, the excess air tries to spread around quickly (at the **speed of sound**). Hence, only objects moving at speeds comparable or faster than the speed of sound (jet planes, shocks) produce appreciable compression in air. We say that **subsonic flow** of gas is to good approximation incompressible as long as the container that holds the gas is much larger than the region being disturbed.
It is useful to remember the following ‘mnemonic’

\[
\text{Fluid is incompressible} \iff \nabla \cdot \vec{u} = 0 \iff \frac{d\rho}{dt} = 0
\]

**Momentum Equations:** To derive the momentum equations, we start with Newton’s second law of motion

\[
\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}
\]

where \(\vec{p} = m\vec{v}\). Consider a fluid element \(\delta V = \int dV\) in an external force field \(\vec{F}_{\text{ext}}\). In most astrophysical cases of interest to us, this external force will be gravity, which is conservative, so that we can write \(\vec{F}_{\text{ext}} = -m\nabla \Phi\), with \(\Phi\) the Newtonian gravitational potential. The momentum of our fluid element is given by \(\vec{p} = \int \rho \vec{u} dV\). We thus can write Newton’s second law of motion as

\[
\frac{d}{dt} \left[ \int \rho \vec{u} dV \right] = \int \vec{F}' dV
\]

where \(\vec{F}'\) is the total force (including the external one) per unit volume acting on our fluid element.

Let us first consider the term on the left. Note that you may not take the derivative inside of the integral! After all, \(V\) is the Lagrangian volume of our fluid element, and is thus a function of time. Instead, we make the assumption that the fluid element is sufficiently small that we may neglect changes in \(\rho \vec{u}\) across its volume, so that

\[
\frac{d}{dt} \left[ \int \rho \vec{u} dV \right] = \frac{d}{dt} (\rho \vec{u} \delta V) = \rho \delta V \frac{d\vec{u}}{dt}
\]

where the second equality follows from the fact that \(d(\rho \delta V)/dt = 0\) (i.e., the mass of our fluid element is conserved).

Next we work out the \(\int \vec{F}' dV\) term. The total force acting on our fluid element consists of two components. The external body force \(\vec{F}_{\text{ext}}\) and an internal surface force due to the fluid’s pressure. In the case of gravity, we have that \(\vec{F}'_{\text{ext}} = -\rho \nabla \Phi\).
For the contribution due to the fluid’s pressure, we assume for now that the pressure is isotropic (in the next chapter we will relax this assumption).

The pressure force acting on an infinitesimal surface element of our fluid element is $-P d\hat{S}$, where the minus sign arises because $d\hat{S}$ is directed outwards and $\hat{F}$ is the force acting on our fluid element. Hence, the total pressure force acting on our fluid element in (Cartesian) direction $\hat{n}$ is

$$\hat{F} \cdot \hat{n} = - \int_{S} P \hat{n} \cdot \hat{S} = - \int_{V} \nabla \cdot (P \hat{n}) dV$$

where we have used Gauss’ divergence theorem. Hence, per unit volume we have that

$$\hat{F}' \cdot \hat{n} = - \nabla \cdot (P \hat{n}) = - P \nabla \cdot \hat{n} - \nabla P \cdot \hat{n} = - \nabla P \cdot \hat{n}$$

where the last equality follows from the fact that $\hat{n}$ is a unit vector in a constant direction. We thus see that the internal (or ‘surface’) force per unit volume is given by $\hat{F}' = - \nabla P$. Combining all the above, and using that $\int \hat{F}' dV \approx \hat{F}' \delta V$ we obtain that

$$\rho \delta V \frac{d\vec{u}}{dt} = - \rho \nabla \Phi \delta V - \nabla P \delta V$$

Since this must be true for any volume element $\delta V$, the momentum equations are:

<table>
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<td>$\frac{d\vec{u}}{dt} = - \nabla P \rho - \nabla \Phi$</td>
<td>$\frac{d\vec{u}_i}{dt} = - \frac{1}{\rho} \frac{\partial P}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Eulerian:</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = - \nabla P \rho - \nabla \Phi$</td>
<td>$\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_j \frac{\partial \vec{u}_i}{\partial x_j} = - \frac{1}{\rho} \frac{\partial P}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}$</td>
<td></td>
</tr>
</tbody>
</table>

These momentum equations are called the Euler Equations. As we will see, they describe an ideal (or perfect) fluid in which viscosity can be ignored (sometimes called an inviscid fluid).
CHAPTER 4

Viscosity & The Stress Tensor

In deriving the momentum equation, in the previous chapter, we made the simplifying assumption that the force acting on a surface of fluid element is a pure normal force (a force acting along the normal to the surface). However, in general, this force per unit area, called the stress, can have any angle wrt the normal. It is useful to decompose the stress in a normal stress, which is the component of the stress along the normal to the surface, and a shear stress, which is the component along the tangent to the surface.

To see that fluid elements in general are subjected to shear stress, consider the following: Consider a flow (i.e., a river) in which we inject a small, spherical blob (a fluid element) of dye. If the only stress to which the blob is subject is normal stress, the only thing that can happen to the blob is an overall compression or expansion. However, from experience we know that the blob of dye will shear into an extended, ‘spaghetti’-like feature; hence, the blob is clearly subjected to shear stress.

If the dye is injected in a flow of honey, the blob will change shape less easily than in a flow of water. This is because honey has a larger resistance to shear distortions; we say, a larger viscosity. In this chapter we derive the relation between shear stress and viscosity.

**Sign Convention:** The stress $\tilde{\Sigma}(\vec{x}, \hat{n})$ acting at location $\vec{x}$ on a surface with normal $\hat{n}$, is exerted by the fluid on the side of the surface to which the normal points, on the fluid from which the normal points. In other words, a positive stress results in compression. Hence, in the case of pure, normal pressure, we have that $\Sigma = -P$.

**Stress Tensor:** The stress tensor $\sigma_{ij}$ is defined such that $\Sigma_i(\hat{n}) = \sigma_{ij} n_j$. Here $\Sigma_i(\hat{n})$ is the $i$-component of the stress acting on a surface with normal $\hat{n}$, whose $j$-component is given by $n_j$. It can be shown (but will not be done here) that the stress tensor is symmetric. This means that $\sigma_{ji} = \sigma_{ij}$ and thus that there are only 6 (rather than 9) independent stress components. It also means that $\sigma_{ij}$ can be diagonalized, and thus that there exists a coordinate basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ for which $\sigma_{ij} = \sigma_{(ii)} \delta_{ij}$. Here $\delta_{ij}$ is the Kronecker delta, and $\sigma_{(ii)}$ refers to the $ii$-component of the tensor (i.e., the brackets indicate NOT to use the Einstein summation convention). These $\sigma_{(ii)}$
are called the **eigenvalues** of the stress tensor, while the corresponding $\vec{e}_i$ are the **eigendirections**.

NOTE: similar to a vector, whose components depend on the coordinate system, but whose length is an invariant, the components of a tensor also depend on the coordinate system. The invariant of a tensor is its **trace**: $\text{Tr}(\sigma_{ij}) = \sigma_{ii}$, which is thus equal to the sum of its eigenvalues.

*If you are not familiar with Einstein’s summation convention, see Appendix A*

**The issues of closure:** In deriving the Euler equations, we assumed that stress is isotropic, and given by (minus) the hydrostatic equilibrium pressure $P$. The momentum and continuity equation can then be closed by adopting a barotropic equation of state $P = P(\rho)$. However, now we see that in general, one has to replace $-P$ by the stress tensor $\sigma_{ij}$, which contains 6 independent components. Clearly, we run the risk of being able to ‘close’ our set of equations. However, as we will see below, most fluids obey a number of conditions under which $\sigma_{ij}$ contains only 3 independent components, 1 of which is almost always equal to zero. The two remaining components are the hydrostatic equilibrium pressure $P$ and the shear viscosity $\mu$. Both are related to the density and temperature of the fluid, and by using the related **constitutive equations** one can thus achieve closure.

**Pascal’s law for hydrostatics:** In a static fluid, there is no preferred direction, and hence the stress has to be isotropic. Thus, in a static fluid the three eigenvalues are identical, and equal to minus the hydrostatic pressure (the minus sign arises from the sign-convention of the stress):

$$\text{static fluid} \iff \sigma_{ij} = -P \delta_{ij}$$

**Viscous Stress Tensor:** This motivates us to write

$$\sigma_{ij} = -P \delta_{ij} + \tau_{ij}$$

where we have introduced a new tensor, $\tau_{ij}$, which is known as the viscous stress tensor, or the **deviatoric stress tensor**.

**Microscopic Origin of Shear Stress:** So what is the origin of shear stress? As mentioned above, in a static fluid, there is no shear stress. Consequently, shear stress
Figure 3: Illustration of origin of shear stress. The shear in the velocity field shears fluid element 2 due to the transfer of momentum from fluid element 1 to 2 and from 2 to 3. The resulting deformation of 2 can be described as a shear stress acting on its bounding surface.

is only present if there is a gradient in the fluid flow; $\partial u_i/\partial x_j \neq 0$. Note that $\partial u_i/\partial x_j$ is also a tensor; it is called the deformation tensor, $T_{ij}$. Consider the situation depicted in Fig. 1. Three neighboring fluids elements (1, 2 and 3) have different streaming velocities, $\vec{u}$. Due to the microscopic motions and collisions (characterized by a non-zero mean free path), there is a net transfer of momentum from the faster moving fluid elements to the slower moving fluid elements. This net transfer of momentum will tend to erase the shear in $\vec{u}(\vec{x})$, and therefore manifests itself as a shear-resistance, known as viscosity. Due to the transfer of momentum, the fluid elements deform; in our figure, 1 transfers linear momentum to the top of 2, while 3 extracts linear momentum from the bottom of 2. Consequently, fluid element 2 is sheared as depicted in the figure at time $t + \Delta t$. From the perspective of fluid element 2, some internal force (from within its boundaries) has exerted a shear-stress on its bounding surface.

**Viscosity:** a measure of a fluid’s resistance to deformation by shear stress. For liquids, viscosity corresponds to the informal concept of ”thickness”. A fluid with zero viscosity is called inviscid. From the picture above, it is clear that one can think of an inviscid fluid as having zero mean-free path. Such a fluid is called an ideal fluid.

As described in Appendix F (which is NOT part of the curriculum for this course), most (astrophysical) fluids are Newtonian, in that they obey a number of condi-
tions. As detailed in that appendix, for a Newtonian fluid, the relation between the stress tensor and the deformation tensor is given by

\[
\sigma_{ij} = -P\delta_{ij} + \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] + \eta \delta_{ij} \frac{\partial u_k}{\partial x_k}
\]

Here \( P \) is the pressure, \( \delta_{ij} \) is the Kronecker delta function, \( \mu \) is the coefficient of shear viscosity, and \( \eta \) is the coefficient of bulk viscosity (aka the ‘second viscosity’). We thus see that for a Newtonian fluid, the stress tensor only has three independent components: \( P, \mu \) and \( \eta \).

Let’s take a closer look at these three quantities, starting with the pressure \( P \). To be exact, \( P \) is the thermodynamic equilibrium pressure, and is normally computed thermodynamically from some equation of state, \( P = P(\rho, T) \). It is related to the translational kinetic energy of the particles when the fluid, in equilibrium, has reached equipartition of energy among all its degrees of freedom, including (in the case of molecules) rotational and vibrations degrees of freedom.

In addition to the thermodynamic equilibrium pressure, \( P \), we can also define a mechanical pressure, \( P_m \), which is purely related to the translational motion of the particles, independent of whether the system has reached full equipartition of energy. The mechanical pressure is simply the average normal stress and therefore follows from the stress tensor according to

\[
P_m = -\frac{1}{3} \text{Tr}(\sigma_{ij}) = -\frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})
\]

As shown in Appendix F, these two pressures are related according to

\[
P_m = P - \eta \nabla \cdot \mathbf{u}
\]

From this expression it is clear that the bulk viscosity, \( \eta \), is only non-zero if \( P \neq P_m \). This, in turn, can only happen if the constituent particles of the fluid have degrees of freedom beyond position and momentum (i.e., when they are molecules with rotational or vibrational degrees of freedom). Hence, for a fluid of monoatoms, \( \eta = 0 \).

From the fact that \( P = P_m + \eta \nabla \cdot \mathbf{u} \) it is clear that for an incompressible flow \( P = P_m \) and the value of \( \eta \) is irrelevant; bulk viscosity plays no role in incompressible
fluids or flows. The only time when $P_m \neq P$ is when a fluid consisting of particles with internal degrees of freedom (e.g., molecules) has just undergone a large volumetric change (i.e., during a shock). In that case there may be a lag between the time the translational motions reach equilibrium and the time when the system reaches full equipartition in energy among all degrees of freedom. In astrophysics, bulk viscosity can generally be ignored, but be aware that it may be important in shocks. This only leaves the shear viscosity $\mu$, which describes the ability of the fluid to resist shear stress via momentum transport resulting from collisions and the non-zero mean free path of the particles.

Relation to the microscopic picture: In later Chapters we will derive the fluid equations using the microscopic (particle-based) approach. In order to touch base with what we have covered here, we briefly link the stress tensor to the microscopic (‘random’) velocities of the particles in a fluid element.

Velocity of fluid particles: We can split the velocity, $\vec{v}$, of a fluid particle in a streaming velocity, $\vec{u}$, and a ‘random’ velocity, $\vec{w}$:

$$\vec{v} = \vec{u} + \vec{w}$$

where $\langle \vec{v} \rangle = \vec{u}$, $\langle \vec{w} \rangle = 0$ and $\langle . \rangle$ indicates the average over a fluid element. If we define $v_i$ as the velocity in the $i$-direction, we have that

$$\langle v_i \rangle = u_i + \langle w_i \rangle$$

Using these definitions of velocities, we can write the stress tensor as

$$\sigma_{ij} \equiv -\rho \langle w_i w_j \rangle$$

from which it is manifest that $\sigma_{ij}$ is symmetric!
In Chapter 3 we ignored shear stresses, which resulted in the following momentum equations (in Lagrangian index form):

\[
\frac{du_i}{dt} = - \frac{1}{\rho} \frac{\partial P}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}
\]

In the previous chapter, we showed that (for a Newtonian fluid) the stress tensor can be written as

\[
\sigma_{ij} = -P \delta_{ij} + \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] + \eta \delta_{ij} \frac{\partial u_k}{\partial x_k}
\]

We now incorporate this stress tensor in the momentum equations. Using that \(\partial P/\partial x_i = \delta_{ij} \partial P/\partial x_j\) we can rewrite the above form as

\[
\rho \frac{du_i}{dt} = - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) \right] + \frac{\partial}{\partial x_i} \left( \eta \frac{\partial u_k}{\partial x_k} \right) - \rho \frac{\partial \Phi}{\partial x_i}
\]

In order to take the shear into account, all we need to do now is to replace \(-P \delta_{ij}\) with the stress tensor (effectively this means, adding a term that is the gradient of the viscous stress tensor, \(\tau_{ij}\)). The result can be written as

\[
\rho \frac{du_i}{dt} = \frac{\partial(-P \delta_{ij})}{\partial x_j} - \rho \frac{\partial \Phi}{\partial x_i}
\]

These momentum equations are called the Navier-Stokes equations. It is more common, and more useful, to write out the viscous stress tensor, yielding

\[
\rho \frac{du_i}{dt} = - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) \right] + \frac{\partial}{\partial x_i} \left( \eta \frac{\partial u_k}{\partial x_k} \right) - \rho \frac{\partial \Phi}{\partial x_i}
\]

These are the Navier-Stokes equations (in Lagrangian index form) in all their glory, containing both the shear viscosity term and the bulk viscosity term (the latter is often ignored).
Note that $\mu$ and $\eta$ are usually functions of density and temperature so that they have spatial variations. However, it is common to assume that these are sufficiently small so that $\mu$ and $\eta$ can be treated as constants, in which case they can be taken outside the differentials. In what follows we will make this assumption as well.

The Navier-Stokes equations in **Lagrangian vector form** are

$$
\rho \frac{d\vec{u}}{dt} = -\nabla P + \mu \nabla^2 \vec{u} + \left( \eta + \frac{1}{3} \mu \right) \nabla (\nabla \cdot \vec{u}) - \rho \nabla \Phi
$$

If we ignore the bulk viscosity ($\eta = 0$) then this reduces to

$$
\frac{d\vec{u}}{dt} = -\frac{\nabla P}{\rho} + \nu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right] - \nabla \Phi
$$

where we have introduced the **kinetic viscosity** $\nu \equiv \mu/\rho$. Note that these equations reduce to the **Euler equations** in the limit $\nu \to 0$. Also, note that the $\nabla (\nabla \cdot \vec{u})$ term is only significant in the case of flows with **variable compression** (i.e., viscous dissipation of acoustic waves or shocks), and can often be ignored. This leaves the $\nu \nabla^2 \vec{u}$ term as the main addition to the Euler equations. Yet, this simple ‘diffuse’ term dramatically changes the character of the equation, as it introduces a higher spatial derivative. Hence, additional boundary conditions are required to solve the equations. When solving problems with solid boundaries (not common in astrophysics), this condition is typically that the **tangential** (or shear) velocity at the boundary vanishes. Although this may sound ad hoc, it is supported by observation; for example, the blades of a fan collect dust.

Recall that when writing the Navier-Stokes equation in **Eulerian form**, we have that $d\vec{u}/dt \to \partial \vec{u}/\partial t + \vec{u} \cdot \nabla \vec{u}$. It is often useful to rewrite this extra term using the vector calculus identity

$$
\vec{u} \cdot \nabla \vec{u} = \nabla \left( \frac{\vec{u} \cdot \vec{u}}{2} \right) + (\nabla \times \vec{u}) \times \vec{u}
$$

Hence, for an **irrotational flow** (i.e., a flow for which $\nabla \times \vec{u} = 0$), we have that $\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2$, where $u \equiv |\vec{u}|$. 

30
In the previous chapters, we derived the Navier-Stokes equations using a macroscopic (continuum) approach to fluid dynamics, based on the concept of fluid elements. We now rederive the same set of equations, but using a far more rigorous, microscopic, particle-based approach. Throughout we consider a 3-dimensional configuration space, which means that the position vector of an individual particle has three components. Note, though, that everything that follows is trivially generalized to a higher or lower dimensional configuration space.

**Degree of freedom:** an independent physical parameter in the formal description of the state of the physical system. In what follows we use \( n \) or \( n_{\text{dof}} \) to indicate the number of degrees of freedom.

**Phase-Space:** The phase-space of a dynamical system is a space in which all possible states of a system are represented, with each possible state corresponding to one unique point in that phase-space. The dimensionality of phase-space is \( n_{\text{dof}} \).

**Caution:** I will use ‘phase-space’ to refer to both this \( n_{\text{dof}} \)-dimensional space, in which each state is associate with a point in that space, as well as to the 6-dimensional space \((\vec{x}, \vec{v})\) in which each individual particle is associated with a point in that space. In order to avoid confusion, in this chapter I will refer to the former as \( \Gamma \)-space, and the latter as \( \mu \)-space (following the nomenclature in Choudhuri’s textbook).

Let \( N \) be the number of constituent particles in our fluid. In all cases of interests, \( N \) will be a huge number; \( N \gg 10^{20} \). How do you (classically) describe such a system? To completely describe a fluid of \( N \) particles, you need to specify for each particle the following quantities:
position $\vec{x} = (x_1, x_2, x_3)$
momentum $\vec{p} = (p_1, p_2, p_3)$
internal degrees of freedom $\vec{s} = (s_1, s_2, ..., s_K)$

Examples of internal degrees of freedom are electrical charge (in case of a plasma), or the rotation or vibrational modes for molecules, etc. The number of degrees of freedom in the above example is $n_{dof} = N(6 + K)$. In what follows we will only consider particles with zero internal dof (i.e., $K = 0$ so that $n_{dof} = 6N$). Such particles are sometimes called monoatoms, and can be treated as point particles. The **microstate** of a system composed of $N$ monoatoms is completely described by

$$\vec{\Gamma} = (\vec{x}_1, \vec{x}_2, ..., \vec{x}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N)$$

which corresponds to a single point in our $6N$-dimensional phase-space.

As already discussed in Chapter 2, the dynamics of our fluid of $N$ monoatoms is described by its **Hamiltonian**

$$\mathcal{H}(\vec{x}_i, \vec{p}_i, t) \equiv \mathcal{H}(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N, t)$$

and the corresponding **equations of motion** are:

$$\dot{\vec{x}}_i = \frac{\partial \mathcal{H}}{\partial \vec{p}_i}; \quad \dot{\vec{p}}_i = -\frac{\partial \mathcal{H}}{\partial \vec{x}_i}$$

Thus, given $\vec{x}_i$ and $\vec{p}_i$ for all $i = 1, 2, ..., N$, at any given time $t_0$, one can compute the Hamiltonian and solve for the equations of motion to obtain $\vec{x}_i(t)$ and $\vec{p}_i(t)$. They specify a unique **trajectory** $\vec{\Gamma}(t)$ in this phase-space (see panel [a] of Fig. 4). Note that no two trajectories $\vec{\Gamma}_1(t)$ and $\vec{\Gamma}_2(t)$ are allowed to cross each other. If that were the case, it would mean that the same microstate can evolve differently, which would be a violation of the **deterministic character** of classical physics. The Hamiltonian formalism described above basically is a complete treatment of fluid dynamics. In practice, though, it is utterly useless, simply because $N$ is HUGE, making it impossible to specify the complete set of initial conditions. We neither have (nor want) the detailed information that is required to specify a microstate. We are only concerned with (interested in) the average behavior of the **macroscopic** properties of the system, such as density, temperature, pressure, etc. With each
Figure 4: Illustration of evolution in Γ-space. The x- and y-axes represent the 3N-dimensional position-vector and momentum-vector, respectively. Panel (a) shows the evolution of a state (indicated by the red dot). As time goes on, the positions and momentum of all the particles change (according to the Hamiltonian equations of motion), and the state moves around in Γ-space. Panel (b) shows the evolution of an ensemble of microstates (called a macrostate). As neighboring states evolve slightly differently, the volume in Γ-space occupied by the original microstates (the red, oval region) is stretched and sheared into a ‘spaghetti-like’ feature. According to Liouville’s theorem, the volume of this spaghetti-like feature is identical to that of the original macrostate (i.e., the flow in Γ-space is incompressible). Note also, that two trajectories in Γ-space can NEVER cross each other.
such macrostate corresponds a huge number of microstates, called a statistical ensemble.

The ensemble is described statistically by the \(N\)-body distribution function

\[
f^{(N)}(\vec{x}_1, \vec{p}_1) \equiv f^{(N)}(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N, \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N)
\]

which expresses the ensemble’s probability distribution, i.e., \(f^{(N)}(\vec{x}_i, \vec{p}_i)\,dV\) is the probability that the actual microstate is given by \(\vec{\Gamma}(\vec{x}_i, \vec{p}_i)\), where \(dV = \prod_{i=1}^N d^3\vec{x}_i\,d^3\vec{p}_i\).

This implies the following normalization condition

\[
\int dV\,f^{(N)}(\vec{x}_i, \vec{p}_i) = 1
\]

In our statistical approach, we seek to describe the evolution of the \(N\)-body distribution function, \(f^{(N)}(\vec{x}_i, \vec{p}_i, t)\), rather than that of a particular microstate, which instead is given by \(\vec{\Gamma}(\vec{x}_i, \vec{x}_i, t)\). Since probability is locally conserved, it must obey a continuity equation; any change of probability in one part of phase-space must be compensated by a flow of probability into or out of neighboring regions. As we have seen in Chapter 2, the continuity equation of a (continuum) density field, \(\rho(\vec{x})\), is given by

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

which expresses that the local change in the mass enclosed in some volume is balanced by the divergence of the flow out of that volume. In the case of our probability distribution \(f^{(N)}\) we have that \(\nabla\) is in \(6N\)-dimensional phase-space, and includes \(\partial/\partial \vec{x}_i\) and \(\partial/\partial \vec{p}_i\), i.e.,

\[
\nabla = (\partial/\partial \vec{x}_1, \partial/\partial \vec{p}_1) = (\partial/\partial \vec{x}_1, \partial/\partial \vec{x}_2, \ldots, \partial/\partial \vec{x}_N, \partial/\partial \vec{p}_1, \partial/\partial \vec{p}_2, \ldots, \partial/\partial \vec{p}_N)
\]

Similarly, the ‘velocity vector’ in our \(6N\)-dimensional \(\Gamma\)-space is given by

\[
\vec{w} \equiv (\dot{\vec{x}}_i, \dot{\vec{p}}_i) = (\dot{\vec{x}}_1, \dot{\vec{x}}_2, \ldots, \dot{\vec{x}}_N, \dot{\vec{p}}_1, \dot{\vec{p}}_2, \ldots, \dot{\vec{p}}_N)
\]

Hence, the continuity equation for \(f^{(N)}\), which is known as the \textbf{Liouville equation}, can be written as

\[
\frac{\partial f^{(N)}}{\partial t} + \nabla \cdot (f^{(N)} \vec{w}) = 0
\]
Using the fact that the gradient of the product of a vector and a scalar can be written as the sum of the scalar times the divergence of the vector, plus the dot-product of the vector and the gradient of the scalar (see Appendix A), we have that

\[ \nabla \cdot (f^{(N)} \vec{w}) = f^{(N)} \nabla \cdot \vec{w} + \vec{w} \cdot \nabla f^{(N)} \]

If we write out the divergence of \( \vec{w} \) as

\[ \nabla \cdot \vec{w} = \sum_{i=1}^{N} \left[ \frac{\partial \dot{x}_i}{\partial x_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right] \]

and use the Hamiltonian equations of motion to write \( \dot{x}_i \) and \( \dot{p}_i \) as gradients of the Hamiltonian, we find that

\[ \nabla \cdot \vec{w} = \sum_{i=1}^{N} \left[ \frac{\partial}{\partial x_i} \left( \frac{\partial H}{\partial p_i} \right) - \frac{\partial}{\partial p_i} \left( \frac{\partial H}{\partial x_i} \right) \right] = 0 \]

Thus, we obtain the important result that

**In a Hamiltonian system the flow in \( \Gamma \)-space is incompressible**

This is generally known as the **Liouville Theorem**. It implies that the volume in \( \Gamma \)-space occupied by a macrostate does NOT change under Hamiltonian evolution. Although the microstates that make up the macrostate can disperse, the volume they occupy stays connected and constant; it typically will change shape, but its total volume remains fixed (see panel [b] of Fig. 4).
Using this result, we can write our **Liouville equation** in any of the following three forms:

\[
\frac{\partial f^{(N)}}{\partial t} + \vec{w} \cdot \nabla f^{(N)} = 0
\]

\[
\frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \left( \dot{x}_i \cdot \frac{\partial f^{(N)}}{\partial x_i} + \dot{p}_i \cdot \frac{\partial f^{(N)}}{\partial p_i} \right) = 0
\]

\[
\frac{df^{(N)}}{dt} = 0
\]

The second expression follows from the first by simply writing out the terms of the divergence. The third expression follows from the second one upon realizing that \(f^{(N)} = f^{(N)}(t, \vec{x}_1, \vec{x}_2, ..., \vec{x}_3, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N)\) and using the fact that for a function \(f(x, y)\) the infinitessimal \(df = (\partial f/\partial x) \, dx + (\partial f/\partial y) \, dy\).

The Liouville equation is basically a complete ‘level-1’ (see Chapter 2) dynamical theory for fluids. Rather than describing the evolution of a single microstate, \(\Gamma(t)\), it describes the evolution of an ensemble of microstates (a macrostate). If anything, this makes computations even harder; for starters, the \(N\)-point distribution function \(f^{(N)}\) is a function of \(6N\) variables, which is utterly unmanageable. However, the Liouville equation is an important, powerful starting point for the development of a ‘level-2’ dynamical theory, from which in turn we can construct a ‘level-3’ theory.

Recall from Chapter 2, that a level-2 theory seeks to describe the evolution of the phase-space distribution function (DF)

\[
f(\vec{x}, \vec{p}) = \frac{dN^6}{d^3\vec{x} \, d^3\vec{p}}
\]

which describes the density of particles in 6D phase-space \((\vec{x}, \vec{p})\). In what follows, we shall refer to this 6-dimensional phase-space as \(\mu\)-space, to distinguish it from the 6N-dimensional \(\Gamma\)-space. And we shall refer to the above DF as the 1-point DF, \(f^{(1)}\), in order to distinguish it from the \(N\)-point DF, \(f^{(N)}\), which appears in the Liouville equation. Whereas the latter describes the ensemble density of micro-states in \(\Gamma\)-space, the latter describes the density of particles in \(\mu\)-space.

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One can derive an equation for the time-evolution of the 1-point DF, starting from
the Liouville equation, using what is called the BBGKY hierarchy (after Bogoli-
ubov, Born, Green, Kirkwood and Yvon, who independently developed this approach
between 1935 and 1946). A detailed derivation is beyond the scope of this course,
but for the interested reader, a short description is provided in Appendix H (which
is not part of this course’s curriculum). What follows instead, is a more heuristic
‘derivation’.

In deriving the Liouville theorem, we made use of the Hamiltonian equations of
motion based on the $N$-particle Hamiltonian $\mathcal{H}(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N, t)$. If the
$N$ particles are identical (such that we may swap the order of particles $i$ and $j$ in
the Hamiltonian; see Appendix G), and if the $N$ particles do not interact with each
other and move under the influence of an external potential $\Phi(\vec{x})$ alone, then one
can introduce a Hamiltonian

$$\mathcal{H}(\vec{x}, \vec{v}, t) = \frac{1}{2} \vec{v}^2 + \Phi(\vec{x})$$

where $\vec{v} = \vec{p}/m$ with $m$ the particle mass. The equations of motion for any particle
are given by the same Hamiltonian equations as before, but with the $N$-particle
Hamiltonian replaced by this one. Since the density of particles in phase-space
is subject to a continuity equation (i.e., classical particles can not pop in or out of
existence), one can use exactly the same approach as for the derivation of the Liouville
equation, to show that the evolution of the 1-point DF (under the conditions listed
above!) is given by

$$\frac{df^{(1)}}{dt} = 0$$

which is identical to the Liouville equation, but with $f^{(N)}$ replaced by $f^{(1)}$.

This seems all pretty straightforward, except that the assumption that the particles
do not interact with other particles is clearly unrealistic. Particles interact with
each other gravitationally, via Coulomb forces (in the case of an ionized plasma),
or via van der Waals forces. Such a two-particle interaction is typically described
via a potential of the form $\Phi(\vec{x}_1, \vec{x}_2)$. Since this cannot be written in the form
$\Phi(\vec{x})$, it cannot be included in the Hamiltonian for our $\mu$-space description (but it is
straightforward to add it to the $N$-point Hamiltonian used in $\Gamma$-space; see Appendix
G). Consequently, we cannot use the same procedure as for the Liouville equation to derive an expression for the time-evolution of \( f^{(1)} \).

What this means is that a Hamiltonian formulation of the dynamics of \( N \)-particles is always possible in \( \Gamma \)-space, but in \( \mu \)-space it is only possible if the mutual interactions amongst the particles are non-existing (which is basically never the case), or if they can be neglected. In the latter case, we say that the system is \textbf{collisionless}. For a collisionless system, we have that

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i} = 0
\]

which is called the \textbf{collisionless Boltzmann equation} or \textbf{CBE} for short. It expresses that the flow of particles in \( \mu \)-space is incompressible, and that the local phase-space density around any particle is fixed. The evolution of a collisionless system of particles under this CBE is depicted in the left-hand panel of Fig. 5. Note that in the above expression we have omitted the superscript ‘(1)’ on the DF for brevity. We will follow that nomenclature from here on out (i.e., whenever no superscript is given, it is to be understood that this is the 1-point DF). Note also that the subscripts \( i \) here refer to one of the three Cartesian coordinates \( (i = 1, 2, 3) \), whereas in the expression for the Liouville equation above (box with three different expressions) the subscripts \( i \) refer to particles \( (i = 1, 2, ..., N) \).

But how are we to deal with collisional fluids? Since the two-body interactions involve a potential of the form \( \Phi(x_1, x_2) \), we also need knowledge about the probability to have two particles at \( \vec{x}_1 \) and \( \vec{x}_2 \) simultaneously. This is described by the \textbf{two-point} \( f^{(2)}(\vec{x}_1, \vec{v}_1, \vec{x}_2, \vec{v}_2, t) \). Hence, the expression for the evolution of \( f^{(1)} \) necessarily needs to account for \( f^{(2)} \), and one thus needs to complement this with an expression for how \( f^{(2)} \) evolves with time. The latter, though, depends on three-body interactions of the form \( \Phi(\vec{x}_1, \vec{x}_2, \vec{x}_3) \) which obviously requires knowledge of the \textbf{three-point} \( f^{(3)} \), etc, etc. Hence, we need an entire ‘hierarchy’ of expressions for \( \frac{\partial f^{(k)}}{\partial t} \) for \( k = 1, 2, ..., N \). This is known as the BBGKY hierarchy, which is a set of \( N \) coupled differential equations. As for the Liouville equation, it is basically utterly impossible to solve this huge set of equations.

However, we can make progress by making assumptions and/or simplifications that allow us to truncate this hierarchy of equations at some (hopefully low) order. A reasonable assumption to consider is that the fluid is sufficiently dilute that interactions between three or more bodies are extremely rare compared to two-body
Figure 5: Illustration of evolution in μ-space. The x- and y-axes represent the 3-dimensional position-vector and momentum-vector, respectively. Panel (a) shows the evolution of a collection of particles (indicated by the red dots) in a collisionless system governed by the CBE. As time goes on, the positions and momentum of all the particles change (according to the Hamiltonian equations of motion), and the particles move around in μ-space smoothly (no abrupt changes). If the combined potential of all the particle (or the external potential) is time-variable, trajectories of individual particles are allowed to cross each other, unlike trajectories in Γ-space, which can never cross. Panel (b) shows the evolution of a collection of particles in a collisional system (where collisions are highly localized). Collisions cause abrupt changes in momentum. The dynamics of this system is described by the Boltzmann equation.
interactions. This means we don’t need to worry about $f^{(K)}$ for $K \geq 3$. In addition, if we consider a neutral fluid, in which the collisions are extremely short-range (to good approximation the collisions are direct-collisions, which are perfectly localized), then we only need to know $f^{(2)}(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2, t)$ for $\vec{x}_2 = \vec{x}_1$, i.e., we only need to know $f^{(2)}(\vec{x}, \vec{v}_1, \vec{v}_2, t)$.

We can write this 2-point DF as

$$f^{(2)}(\vec{x}, \vec{v}_1, \vec{v}_2, t) = f^{(1)}(\vec{x}, \vec{v}_1, t) f^{(1)}(\vec{x}, \vec{v}_2, t) + g(\vec{x}, \vec{v}_1, \vec{v}_2, t)$$

where we have introduced the two-particle **velocity correlation function** $g(\vec{x}, \vec{v}_1, \vec{v}_2, t)$.

In a bold move, Boltzmann made the assumption that the velocities of two particles that undergo a collision are NOT CORRELATED. This assumption, which is often called the assumption of ‘molecular chaos’, implies that $g(\vec{x}, \vec{v}_1, \vec{v}_2, t) = 0$, and thus that we can write the 2-point DF simply as the product of two 1-point DFs. It also implies that we can write down a closed equation for the time-evolution of $f^{(1)}$ (i.e., the evolution of $f^{(1)}$ no longer needs input from $f^{(2)}$).

This closed equation for $f^{(1)}$ is called the **Boltzmann equation**, and is given by

$$\frac{df^{(1)}}{dt} = \frac{\partial f^{(1)}}{\partial t} + \dot{x}_i \frac{\partial f^{(1)}}{\partial x_i} + \dot{v}_i \frac{\partial f^{(1)}}{\partial v_i} = I[f^{(1)}]$$

where $I[f^{(1)}]$ is called the **collision integral**. It describes how, due to collisions, particles are ‘kicked’ in and out of certain parts of phase-space. The right-hand panel of Fig. 5 shows an illustration of evolution under the Boltzmann equation.

The **Boltzmann equation** is valid under the following conditions:

- dilute gas; density is sufficiently low so that only binary collisions need to be considered
- collisions can be thought of as instantaneous and perfectly localized.
- ”molecular chaos”: velocities of colliding particles are uncorrelated

The first two assumptions are reasonable, but the molecular chaos assumption has a long and interesting history (see text-box on the next page).
The assumption of "molecular chaos" (also known as "Stosszahlansatz") which allows one to write down a closed equation for the time-evolution of the 1-point DF, was used by L. Boltzmann to proof his famous H-Theorem, which basically states that entropy should always increase (i.e., it is supposed to be a proof of the second law of thermodynamics). This in turn implies time-asymmetry, giving rise to the thermodynamic arrow of time. However, as first brought to bear by J. Loschmidt, it should not be possible to deduce an irreversible process from time-symmetric dynamics and a time-symmetric formalism (i.e., the dynamics that result from the Liouville equation, which has no underlying assumptions, is perfectly time-reversible!). The origin of this "Loschmidt paradox", as it is known, is the questionable assumption of "molecular chaos". After all, once the particles are allowed to collide, their velocity directions and positions in fact do become correlated. Molecular chaos basically assumes that the subsequent collisions with all the other particles somehow erases this correlation again. To what extent this is true, and whether the H-theorem really proofs the second law of thermodynamics is a topic of ongoing debate, especially among philosophers of science.

The material in this text-box is not part of the curriculum for this course
CHAPTER 7

From Boltzmann to Navier-Stokes

In the previous chapter we derived the closed Boltzmann equation:
\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i} = I[f]
\]
where \(I[f]\) is the collision integral, and we have used the shorthand notation \(f\) for the 1-particle DF \(f^{(1)}\). In what follows we will adopt that notation throughout, and only use the superscript-notation whenever confusion might arise. The Boltzmann equation describes how the phase-space density around a particle (or fluid element) changes with time due to collisions.

Recall that for a collisionless fluid, \(I[f] = 0\), and the Boltzmann equation reduces to the collisionless Boltzmann equation (CBE):
\[
\frac{df}{dt} = 0
\]
which expresses that the 6D phase-space density of a collisionless fluid is incompressible: the phase-space density around any given particle is conserved.

Using that \(\dot{x}_i = v_i\), and that \(\dot{v}_i = \partial \Phi / \partial x_i\), where \(\Phi\) is the potential appearing in the Hamiltonian, which is responsible for the acceleration of the particles, we can write the Boltzmann equation in vector-form as
\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} = I[f]
\]

Let us now take a closer look at the collision integral \(I[f]\). It basically expresses the Eulerian time-derivative of the DF due to collisions, i.e., \(I[f] = (\partial f/\partial t)_{\text{coll}}\). Recall that we have made the assumption of a dilute gas, so that we only need to consider two-body interactions. In what follows, we make the additional assumption that all collisions are elastic [actually, this is sort of implied by the fact that we assume that the dynamical are Hamiltonian]. An example is shown in Figure 1, where \(\vec{p}_1 + \vec{p}_2 \rightarrow \)
Figure 6: Illustration of ‘collision’ between two particles with momenta \( p_1 \) and \( p_2 \) due to interaction potential \( U(r) \). The impact parameter of the collision is \( b \).

\[ \vec{p}_1' + \vec{p}_2' \]. Since we assumed a short-range, instantaneous and localized interaction, so that the external potential doesn’t significantly vary over the interaction volume (the dashed circle in Fig. 1), we have

\[
\text{momentum conservation: } \vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' \\
\text{energy conservation: } |\vec{p}_1|^2 + |\vec{p}_2|^2 = |\vec{p}_1'|^2 + |\vec{p}_2'|^2
\]

where as throughout we have assumed equal mass particles.

In addition, we have time-reversibility, so that it is equally likely that the inverse process \((-\vec{p}_1' + -\vec{p}_2' \rightarrow -\vec{p}_1 + -\vec{p}_2)\) happens.

We can write the rate at which particles of momentum \( \vec{p}_1 \) at location \( \vec{x} \) experience collisions \( \vec{p}_1 + \vec{p}_2 \rightarrow \vec{p}_1' + \vec{p}_2' \) as

\[ \mathcal{R} = \omega(\vec{p}_1, \vec{p}_2|\vec{p}_1', \vec{p}_2') f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2) d^3\vec{p}_2 d^3\vec{p}_1' d^3\vec{p}_2' \]

Here \( f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2) \) is the 2-particle DF, expressing the probability that at location \( \vec{x} \), you encounter two particles with momenta \( \vec{p}_1 \) and \( \vec{p}_2 \), respectively. The function \( \omega(\vec{p}_1, \vec{p}_2|\vec{p}_1', \vec{p}_2') \) depends on the interaction potential \( U(\vec{r}) \) and can be calculated
Using our assumption of molecular chaos, which states that the momenta of the interacting particles are independent, we have that

\[ f^{(2)}(\bar{x}, \bar{x}, \bar{p}_1, \bar{p}_2) = f^{(1)}(\bar{x}, \bar{p}_1) f^{(1)}(\bar{x}, \bar{p}_2) \]

so that the collision integral can be written as

\[
I[f] = \int d^3\bar{p}_2 d^3\bar{p}_1' d^3\bar{p}_2' \omega(\bar{p}_1', \bar{p}_2'|\bar{p}_1, \bar{p}_2) \left[ f(\bar{x}, \bar{p}_1') f(\bar{x}, \bar{p}_2') - f(\bar{x}, \bar{p}_1) f(\bar{x}, \bar{p}_2) \right]
\]

The first term within the square brackets describes the replenishing collisions, in which particles at \((\bar{x}, \bar{p}_1')\) are scattered into \((\bar{x}, \bar{p}_1)\). The second term with the square brackets describes the depleting collisions, in which particles at \((\bar{x}, \bar{p}_1)\) are kicked out of their phase-space volume into \((\bar{x}, \bar{p}_1')\).

Because of the symmetries in \(\omega(\bar{p}_1', \bar{p}_2'|\bar{p}_1, \bar{p}_2)\) (i.e., time-reversibility, and elasticity of collisions), it is straightforward to show that

\[
\int d^3\bar{p} A(\bar{x}, \bar{p}) \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = 0
\]

if

\[
A(\bar{x}, \bar{p}_1) + A(\bar{x}, \bar{p}_2) = A(\bar{x}, \bar{p}_1') + A(\bar{x}, \bar{p}_2')
\]

Quantities \(A(\bar{x}, \bar{p})\) for which this is the case are called collisional invariants. There are three such quantities of interest to us

\[
\begin{align*}
A &= 1 & \text{particle number conservations} \\
A &= \bar{p} & \text{momentum conservation} \\
A &= \bar{p}^2/(2m) & \text{energy conservation}
\end{align*}
\]
Thus far, we have derived the Boltzmann equation, and we have been able to write down an expression for the collision integral under the assumptions of (i) short-range, elastic collisions and (ii) molecular chaos. How do we proceed from here? Solving the actual Boltzmann equation, i.e. characterizing the evolution of $f$ in 6D phase-space is extremely difficult, and provides little insight. Rather, we are interested what happens to our macroscopic quantities that describe the fluid ($\rho, \vec{u}, P, \varepsilon$, etc). We can use the Boltzmann equation to describe the time-evolution of these macroscopic quantities by considering moment equations of the Boltzmann equation.

In mathematics, the $n^{th}$-moment of a real-valued, continuous function $f(x)$ is

$$
\mu_n = \int x^n f(x) \, dx
$$

If $f(x)$ is normalized, so that it can be interpreted as a probability function, then $\mu_n = \langle x^n \rangle$

In our case, consider the scalar function $Q(\vec{v})$. The expectation value for $Q$ at location $\vec{x}$ at time $t$ is given by

$$
\langle Q \rangle = \langle Q \rangle(\vec{x}, t) = \frac{\int Q(\vec{v}) f(\vec{x}, \vec{v}, t) \, d^3\vec{v}}{\int f(\vec{x}, \vec{v}, t) \, d^3\vec{v}}
$$

Using that

$$
n = n(\vec{x}, t) = \int f(\vec{x}, \vec{v}, t) \, d^3\vec{v}
$$

we thus have that

$$
\int Q(\vec{v}) f(\vec{x}, \vec{v}, t) \, d^3\vec{v} = n \langle Q \rangle
$$

We will use this abundantly in what follows. In particular, define

$$
g(\vec{x}, t) = \int Q(\vec{v}) f(\vec{x}, \vec{v}, t) \, d^3\vec{v}
$$
Then, in relation to fluid dynamics, there are a few functions $Q(\vec{v})$ that are of particular interest:

\[
\begin{align*}
Q(\vec{v}) &= 1 \quad \Rightarrow \quad g(\vec{x}, t) = n(\vec{x}, t) \quad \text{number density} \\
Q(\vec{v}) &= m \quad \Rightarrow \quad g(\vec{x}, t) = \rho(\vec{x}, t) \quad \text{mass density} \\
Q(\vec{v}) &= m \vec{v} \quad \Rightarrow \quad g(\vec{x}, t) = \rho(\vec{x}, t) \vec{u}(\vec{x}, t) \quad \text{momentum flux density} \\
Q(\vec{v}) &= \frac{1}{2} m(\vec{v} - \vec{u})^2 \quad \Rightarrow \quad g(\vec{x}, t) = \rho(\vec{x}, t) \varepsilon(\vec{x}, t) \quad \text{specific energy density}
\end{align*}
\]

where we have used that $\langle \vec{v} \rangle = \vec{u}$, and $\langle (\vec{v} - \vec{u})^2 / 2 \rangle = \varepsilon$.

This indicates that we can obtain dynamical equations for the macroscopic fluid quantities by multiplying the Boltzmann equation with appropriate functions, $Q(\vec{v})$, and integrating over all of velocity space.

Hence, we seek to solve equations of the form

\[
\int Q(\vec{v}) \left[ \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} \right] \, d^3 \vec{v} = \int Q(\vec{v}) \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \, d^3 \vec{v}
\]

In what follows, we restrict ourselves to $Q(\vec{v})$ that are collisional invariants so that the integral on the right-hand side vanishes, and we are left with

\[
\int Q(\vec{v}) \frac{\partial f}{\partial t} \, d^3 \vec{v} + \int Q(\vec{v}) \vec{v} \cdot \nabla f \, d^3 \vec{v} - \int Q(\vec{v}) \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} \, d^3 \vec{v} = 0
\]

Since mass, momentum and energy are all conserved in elastic, short-range collisions we have that the momentum integral over the collision integral will be zero for the zeroth, first and second order moment equations! In other words, although collisional and collisionless systems solve different Boltzmann equations, their zeroth, first and second moment equations are identical!

We now split the above equation in three terms:

\[
\begin{align*}
\text{I} & \quad \int Q(\vec{v}) \frac{\partial f}{\partial t} \, d^3 \vec{v} \\
\text{II} & \quad \int Q(\vec{v}) v_i \frac{\partial f}{\partial x_i} \, d^3 \vec{v} \\
\text{III} & \quad \int Q(\vec{v}) \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \, d^3 \vec{v}
\end{align*}
\]

where we have that $\text{I} + \text{II} - \text{III} = 0$, as long as $Q$ is a collisional invariant.
We now proceed to rewrite each of these three integrals in turn.

**Integral I**
The first integral can be written as
\[
\begin{align*}
\int Q(\vec{v}) \frac{\partial f}{\partial t} \, d^3\vec{v} &= \int \frac{\partial Q}{\partial t} f \, d^3\vec{v} = \frac{\partial}{\partial t} \int Q f \, d^3\vec{v} = \frac{\partial}{\partial t} n\langle Q \rangle
\end{align*}
\]
where we have used that both \( Q(\vec{v}) \) and the integration volume are independent of time.

**Integral II**
Using similar logic, the second integral can be written as
\[
\begin{align*}
\int Q(\vec{v}) v_i \frac{\partial f}{\partial x_i} \, d^3\vec{v} &= \int \frac{\partial Q}{\partial x_i} v_i f \, d^3\vec{v} = \frac{\partial}{\partial x_i} \int Q v_i f \, d^3\vec{v} = \frac{\partial}{\partial x_i} [n \langle Q v_i \rangle]
\end{align*}
\]
Here we have used that
\[
Q v_i \frac{\partial f}{\partial x_i} = \frac{\partial (Q v_i f)}{\partial x_i} - f \frac{\partial Q v_i}{\partial x_i} = \frac{\partial (Q v_i f)}{\partial x_i}
\]
where the last step follows from the fact that neither \( v_i \) nor \( Q \) depend on \( x_i \).

**Integral III**
For the third, and last integral, we are going to define \( \vec{F} = \nabla \Phi \) and \( \nabla_v \equiv (\partial/\partial v_x, \partial/\partial v_y, \partial/\partial v_z) \), i.e., \( \nabla_v \) is the equivalent of \( \nabla \) but in velocity space. This allows us to write
\[
\begin{align*}
\int Q \vec{F} \cdot \nabla_v f \, d^3\vec{v} &= \int \nabla_v \cdot (Q f \vec{F}) \, d^3\vec{v} - \int f \nabla_v \cdot (Q \vec{F}) \, d^3\vec{v} \\
&= \int Q f \vec{F} \, d^2S_v - \int f \frac{\partial QF_i}{\partial v_i} \, d^3\vec{v} \\
&= -\int f Q \frac{\partial F_i}{\partial v_i} \, d^3\vec{v} - \int F_i \frac{\partial Q}{\partial v_i} \, d^3\vec{v} \\
&= -\int f \frac{\partial \Phi}{\partial x_i} \frac{\partial Q}{\partial v_i} \, d^3\vec{v} = -\frac{\partial \Phi}{\partial x_i} n \langle \frac{\partial Q}{\partial v_i} \rangle
\end{align*}
\]
Here we have used Gauss’ divergence theorem, and the fact that the integral of \( Q f \vec{F} \) over the surface \( S_v \) (which is a sphere with radius \( |\vec{v}| = \infty \)) is equal to zero. This follows from the ‘normalization’ requirement that \( \int f \, d^3 \vec{v} = n \). We have also used that \( F_i = \partial \Phi / \partial x_i \) is independent of \( v_i \).

Combining the above expressions for I, II, and III, we obtain that

\[
\frac{\partial}{\partial t} n\langle Q \rangle + \frac{\partial}{\partial x_i} \left[ n\langle Q v_i \rangle \right] + \frac{\partial \Phi}{\partial x_i} n \left\langle \frac{\partial Q}{\partial v_i} \right\rangle = 0
\]

In what follows we refer to this as the master-moment-equation.

Now let us consider \( Q = m \), which is indeed a collisional invariant, as required. Substitution in the master-moment equation, and using that \( \langle m \rangle = m \), that \( mn = \rho \) and that \( \langle mv_i \rangle = m\langle v_i \rangle = mu_i \), we obtain

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0
\]

which we recognize as the continuity equation in Eulerian index form.

Next we consider \( Q = mv_j \), which is also a collisional invariant. Using that \( n\langle mv_j v_i \rangle = \rho \langle v_i v_j \rangle \) and that

\[
\frac{\partial \Phi}{\partial x_i} n \left\langle \frac{\partial mv_j}{\partial v_i} \right\rangle = \frac{\partial \Phi}{\partial x_i} \rho \left\langle \frac{\partial v_j}{\partial v_i} \right\rangle = \frac{\partial \Phi}{\partial x_i} \rho \delta_{ij} = \rho \frac{\partial \Phi}{\partial x_j}
\]

substitution of \( Q = mv_j \) in the master-moment equation yields

\[
\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho \langle v_iv_j \rangle}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} = 0
\]

Next we use that

\[
\frac{\partial \rho u_j}{\partial t} = \rho \frac{\partial u_j}{\partial t} + u_j \frac{\partial \rho}{\partial t} = \rho \frac{\partial u_j}{\partial t} - u_j \frac{\partial \rho u_k}{\partial x_k}
\]

where, in the last step, we have used the continuity equation. Substitution in the above equation, and using that \( k \) is a mere dummy variable (which can therefore be replaced by \( i \)), we obtain that
If we now restrict ourselves to \textbf{collisional fluids}, and use that the \textbf{stress tensor} can be written as
\[
\sigma_{ij} = -\rho \langle w_i w_j \rangle = -\rho \langle v_i v_j \rangle + \rho u_i u_j = -P \delta_{ij} + \tau_{ij}
\]
then the equation above can be rewritten as
\[
\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = \frac{1}{\rho} \frac{\partial P}{\partial x_j} + \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial \Phi}{\partial x_j}
\]
which we recognize as the \textbf{momentum equations (Navier-Stokes)} in Eulerian index form. As we have seen in Chapter 3, as long as the fluid is \textbf{Newtonian}, the \textbf{viscous stress tensor}, \(\tau_{ij}\), can be described by two parameters only: the coefficient of shear viscosity, \(\mu\), and the coefficient of bulk viscosity, \(\eta\) (which can typically be ignored).

If instead we assume a \textbf{collisionless fluid}, then
\[
\sigma_{ij} = -\rho \langle w_i w_j \rangle = -\rho \langle v_i v_j \rangle + \rho u_i u_j
\]
and the momentum equations (now called the \textbf{Jeans equations}) reduce to
\[
\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial \Phi}{\partial x_j}
\]
In this case, we have no constraints on \(\sigma_{ij}\) other than that it is \textbf{manifest symmetric}; for a collisionless fluid the stress tensor \(\sigma_{ij} = \sigma_{ij}(\vec{x}, t)\) has a total of 6 unknowns. The Jeans equations form the basis for building dynamical models of galaxies. However, since they contain many more unknowns than the number of equations, they can in
general not be solved unless one makes a number of highly oversimplified assumptions (i.e., the system is spherically symmetric, the velocity structure is isotropic, etc.). This is the topic of Galactic Dynamics. Note that adding higher order moment equations \( Q(v) \propto v^a \) with \( a \geq 3 \) doesn’t help in achieving closure since the new equations also add new unknowns, such as \( \langle v_i v_j v_k \rangle \), etc. Ultimately, the problem is that collisionless fluids do not have constitutive equations such as the equation of state for a collisionless fluid.
Summary

For collisionless systems we have:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \langle v_i \rangle)}{\partial x_i} = 0 \\
\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\frac{\partial \Phi}{\partial x_j} + \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_i} \\
\nabla^2 \Phi = 4\pi G \rho
\]

This is a set of 5 equations with 11 unknowns (\(\rho, \Phi, \langle v_i \rangle [3] \) and \(\sigma_{ij} [6] \)!! Closure can only be achieved by making a number of simplifying assumptions.

For collisional systems we have:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \langle v_i \rangle)}{\partial x_i} = 0 \\
\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\frac{\partial \Phi}{\partial x_j} + \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_i} \\
\nabla^2 \Phi = 4\pi G \rho \\
P = P(\rho, T)
\]

Using that the stress tensor is entirely described by the pressure \(P\), and the coefficients of shear and bulk viscosity, \(\mu\) and \(\eta\), this constitutes a set of 6 equations with 9 unknowns: (\(\rho, \Phi, \langle v_i \rangle [3], P, \mu, \eta, T\)). If the fluid is inviscid then \(\mu = \eta = 0\), which reduces the number of unknowns to 7. Closure can then be achieved by either assuming a barotropic EoS, \(P = P(\rho)\), or by including one extra equation (the energy equation; see Chapter 14).
Figure 7: Flowchart of the origin of the dynamical equations describing fluids.
CHAPTER 8

Vorticity & Circulation

Vorticity: The vorticity of a flow is defined as the curl of the velocity field:

\[ \vec{\omega} = \nabla \times \vec{u} \]

It is a microscopic measure of rotation (vector) at a given point in the fluid, which can be envisioned by placing a paddle wheel into the flow. If it spins about its axis at a rate \( \Omega \), then \( \omega = |\vec{\omega}| = 2\Omega \).

Circulation: The circulation around a closed contour \( C \) is defined as the line integral of the velocity along that contour:

\[ \Gamma_C = \oint_C \vec{u} \cdot d\vec{l} = \int_S \vec{\omega} \cdot d\vec{S} \]

where \( S \) is an arbitrary surface bounded by \( C \). The circulation is a macroscopic measure of rotation (scalar) for a finite area of the fluid.

Irrotational fluid: An irrotational fluid is defined as being curl-free; hence, \( \vec{\omega} = 0 \) and therefore \( \Gamma_C = 0 \) for any \( C \).

Vortex line: a line that points in the direction of the vorticity vector. Hence, a vortex line relates to \( \vec{\omega} \), as a streamline relates to \( \vec{u} \).

Vortex tube: a bundle of vortex lines. The circularity of a curve \( C \) is proportional to the number of vortex lines that thread the enclosed area.

In an inviscid fluid the vortex lines/tubes move with the fluid: a vortex line anchored to some fluid element remains anchored to that fluid element.
Figure 8: Evolution of a vortex tube. Solid dots correspond to fluid elements. Due to the shear in the velocity field, the vortex tube is stretched and tilted. However, as long as the fluid is inviscid and barotropic Kelvin's circularity theorem assures that the circularity is conserved with time. In addition, since vorticity is divergence-free ('solenoidal'), the circularity along different cross sections of the same vortex-tube is the same.

Vorticity equation: The Navier-Stokes momentum equations, in the absence of bulk viscosity, in Eulerian vector form, are given by

\[
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi + \nu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right]
\]

Using the vector identity \((\vec{u} \cdot \nabla) \vec{u} = \frac{1}{2} \nabla u^2 + (\nabla \times \vec{u}) \times \vec{u} = \nabla (u^2/2) - \vec{u} \times \vec{w}\) allows us to rewrite this as

\[
\frac{\partial \vec{u}}{\partial t} - \vec{u} \times \vec{w} = -\frac{\nabla P}{\rho} - \nabla \Phi - \frac{1}{2} \nabla u^2 + \nu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right]
\]

If we now take the curl on both sides of this equation, and we use that curl(\(\nabla S\)) = 0 for any scalar field \(S\), and that \(\nabla \times (\nabla^2 \vec{A}) = \nabla^2 (\nabla \times \vec{A})\), we obtain the vorticity equation:
\[
\frac{\partial \vec{w}}{\partial t} = \nabla \times (\vec{u} \times \vec{w}) - \nabla \times \left( \frac{\nabla P}{\rho} \right) + \nu \nabla^2 \vec{w}
\]

To write this in Lagrangian form, we first use that \( \nabla \times (S \vec{A}) = \nabla S \times \vec{A} + S(\nabla \times \vec{A}) \) [see Appendix A] to write

\[
\nabla \times \left( \frac{1}{\rho} \nabla P \right) = \nabla \left( \frac{1}{\rho} \right) \times \nabla P + \frac{1}{\rho} (\nabla \times \nabla P) = \frac{\rho \nabla (1) - 1 \nabla \rho}{\rho^2} \times \nabla P = \frac{\nabla P \times \nabla \rho}{\rho^2}
\]

where we have used, once more, that \( \text{curl(grad} S) = 0 \). Next, using the vector identities from Appendix A, we write

\[
\nabla \times (\vec{w} \times \vec{u}) = \vec{w} (\nabla \cdot \vec{u}) - (\vec{w} \cdot \nabla) \vec{u} - (\vec{u} \cdot \nabla) \vec{w} + (\vec{u} \cdot \nabla) \vec{w}
\]

The third term vanishes because \( \nabla \cdot \vec{w} = \nabla \cdot (\nabla \times \vec{u}) = 0 \). Hence, using that \( \partial \vec{w}/\partial t + (\vec{u} \cdot \nabla) \vec{w} = d\vec{w}/dt \) we finally can write the vorticity equation in Lagrangian form:

\[
\frac{d\vec{w}}{dt} = (\vec{w} \cdot \nabla) \vec{u} - \vec{w} (\nabla \cdot \vec{u}) + \frac{\nabla \rho \times \nabla P}{\rho^2} + \nu \nabla^2 \vec{w}
\]

This equation describes how the vorticity of a fluid element evolves with time. We now describe the various terms of the \( \text{rhs} \) of this equation in turn:

- \( (\vec{w} \cdot \nabla) \vec{u} \): This term represents the stretching and tilting of vortex tubes due to velocity gradients. To see this, we pick \( \vec{w} \) to be pointing in the \( z \)-direction. Then

\[
(\vec{w} \cdot \nabla) \vec{u} = w_z \frac{\partial \vec{u}}{\partial z} = w_z \frac{\partial u_x}{\partial z} \vec{e}_x + w_z \frac{\partial u_y}{\partial z} \vec{e}_y + w_z \frac{\partial u_z}{\partial z} \vec{e}_z + \]

The first two terms on the \( \text{rhs} \) describe the tilting of the vortex tube, while the third term describes the stretching.

- \( \vec{w} (\nabla \cdot \vec{u}) \): This term describes stretching of vortex tubes due to flow compressibility. This term is zero for an incompressible fluid or flow \( (\nabla \cdot \vec{u} = 0) \). Note that, again under the assumption that the vorticity is pointing in the \( z \)-direction,
\( \vec{w}(\nabla \cdot \vec{u}) = w_z \left[ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right] \vec{e}_z \)

- \((\nabla \rho \times \nabla P)/\rho^2\): This is the **baroclinic** term. It describes the production of vorticity due to a misalignment between pressure and density gradients. This term is zero for a **barotropic** EoS: if \( P = P(\rho) \) the pressure and density gradients are parallel so that \( \nabla P \times \nabla \rho = 0 \). Obviously, this baroclinic term also vanishes for an incompressible fluid (\( \nabla \rho = 0 \)) or for an isobaric fluid (\( \nabla P = 0 \)). The baroclinic term is responsible, for example, for creating vorticity in pyroclastic flows (see Fig. 4).

- \( \nu \nabla^2 \vec{w} \): This term describes the **diffusion** of vorticity due to **viscosity**, and is obviously zero for an inviscid fluid (\( \nu = 0 \)). Typically, viscosity generates/creates vorticity at a bounding surface: due to the **no-slip** boundary condition shear arises giving rise to vorticity, which is subsequently diffused into the fluid by the viscosity. In the interior of a fluid, no new vorticity is generated; rather, viscosity diffuses and dissipates vorticity.

- \( \nabla \times \vec{F} \): There is a fifth term that can create vorticity, which however does not appear in the vorticity equation above. The reason is that we assumed that the only external force is gravity, which is a conservative force and can therefore be written as the gradient of a (gravitational) potential. More generally, though, there may be non-conservative, external body forces present, which would give rise to a \( \nabla \times \vec{F} \) term in the rhs of the vorticity equation. An example of a non-conservative force creating vorticity is the **Coriolis force**, which is responsible for creating hurricanes.
Using the definition of circulation, it can be shown (here without proof) that

$$\frac{d\Gamma}{dt} = \int_S \left[ \frac{\partial \vec{w}}{\partial t} + \nabla \times (\vec{w} \times \vec{u}) \right] \cdot d\vec{S}$$

Using the vorticity equation, this can be rewritten as

$$\frac{d\Gamma}{dt} = \int_S \left[ \nabla \rho \times \nabla P \frac{\rho^2}{\rho^2} + \nu \nabla^2 \vec{w} + \nabla \times \vec{F} \right] \cdot d\vec{S}$$

where, for completeness, we have added in the contribution of an external force $\vec{F}$ (which vanishes if $\vec{F}$ is conservative). Using Stokes’ Curl Theorem (see Appendix B) we can also write this equation in a line-integral form as

$$\frac{d\Gamma}{dt} = -\oint \nabla P \rho \cdot d\vec{l} + \nu \oint \nabla^2 \vec{u} \cdot d\vec{l} + \oint \vec{F} \cdot d\vec{l}$$

which is the form that is more often used.
NOTE: By comparing the equations expressing $d\vec{w}/dt$ and $d\Gamma/dt$, it is clear that the stretching and tilting terms present in the equation describing $d\vec{w}/dt$, are absent in the equation describing $d\Gamma/dt$. This implies that stretching and tilting changes the vorticity, but keeps the circularity invariant. This is basically the first theorem of Helmholtz described below.

**Kelvin’s Circulation Theorem:** The number of vortex lines that thread any element of area that moves with the fluid (i.e., the circulation) remains unchanged in time for an inviscid, barotropic fluid, in the absence of non-conservative forces.

The proof of Kelvin’s Circulation Theorem is immediately evident from the above equation, which shows that $d\Gamma/dt = 0$ if the fluid is both inviscid ($\nu = 0$), barotropic ($P = P(\rho) \Rightarrow \nabla \rho \times \nabla P = 0$), and there are no non-conservative forces ($\vec{F} = 0$).

We end this chapter on vorticity and circulation with the three theorems of Helmholtz, which hold in the absence of non-conservative forces (i.e., $\vec{F} = 0$).

**Helmholtz Theorem 1:** The strength of a vortex tube, which is defined as the circularity of the circumference of any cross section of the tube, is constant along its length. This theorem holds for any fluid, and simply derives from the fact that the vorticity field is divergence-free (we say solenoidal): $\nabla \cdot \vec{w} = \nabla \cdot (\nabla \times \vec{u}) = 0$. To see this, use Gauss’ divergence theorem to write that

$$\int_V \nabla \cdot \vec{w} \, dV = \int_S \vec{w} \cdot d^2 S = 0$$

Here $V$ is the volume of a subsection of the vortex tube, and $S$ is its bounding surface. Since the vorticity is, by definition, perpendicular to $S$ along the sides of the tube, the only non-vanishing components to the surface integral come from the areas at the top and bottom of the vortex tube; i.e.

$$\int_S \vec{w} \cdot d^2 \vec{S} = \int_{A_1} \vec{w} \cdot (-\hat{n}) \, dA + \int_{A_2} \vec{w} \cdot \hat{n} \, dA = 0$$

where $A_1$ and $A_2$ are the areas of the cross sections that bound the volume $V$ of the vortex tube. Using Stokes’ curl theorem, we have that
\[ \int_A \vec{w} \cdot \hat{n} \, dA = \oint_C \vec{u} \cdot d\vec{l} \]

Hence we have that \( \Gamma_{C_1} = \Gamma_{C_2} \) where \( C_1 \) and \( C_2 \) are the curves bounding \( A_1 \) and \( A_2 \), respectively.

**Helmholtz Theorem 2:** A vortex line cannot end in a fluid. Vortex lines and tubes must appear as closed loops, extend to infinity, or start/end at solid boundaries.

**Helmholtz Theorem 3:** A barotropic, inviscid fluid that is initially irrotational will remain irrotational in the absence of rotational (i.e., non-conservative) external forces. Hence, such a fluid does not and cannot create vorticity (except across curved shocks, see Chapter 11).

The proof of Helmholtz’ third theorem is straightforward. According to Kelvin’s circulation theorem, a barotropic, inviscid fluid has \( d\Gamma/dt = 0 \) everywhere. Hence,

\[
\frac{d\Gamma}{dt} = \int_S \left[ \frac{\partial \vec{w}}{\partial t} + \nabla \times (\vec{w} \times \vec{u}) \right] \cdot d^2\vec{S} = 0
\]

Since this has to hold for any \( S \), we have that \( \partial \vec{w}/\partial t = \nabla \times (\vec{u} \times \vec{w}) \). Hence, if \( \vec{w} = 0 \) initially, the vorticity remains zero for ever.

**Potential Flow:** An implication of Helmholtz’ third theorem is that if an inviscid fluid is incompressible (\( \nabla \cdot \vec{u} = 0 \)) and irrotational (\( \nabla \times \vec{u} = 0 \)), then it will remain irrotational. Such a flow is called **Potential Flow**, and obeys **Laplace’s equation**:

\[
\nabla^2 \Phi_u = 0
\]

where \( \Phi_u \) is called the **velocity potential**, defined according to \( \vec{u} = \nabla \Phi_u \). Although there is no such thing as an inviscid liquid, viscosity typically only manifests itself in thin **boundary layers** (where Kelvin’s circulation theorem doesn’t apply). Outside of the boundary layer, flow is often accurately described by potential flow.
Figure 10: A beluga whale demonstrating Kelvin’s circulation theorem and Helmholtz’ second theorem by producing a closed vortex tube under water, made out of air.
We start this chapter on the Bernoulli equation by introducing an important thermodynamic quantity, called the **enthalpy**, $H$, which is a measure for the total energy of a thermodynamic system that includes the internal energy, $U$, and the amount of energy required to make room for it by displacing its environment and establishing its volume and pressure:

$$H = U + PV$$

The associated **specific enthalpy** is given by

$$h = \frac{H}{m} = \varepsilon + \frac{P}{\rho}$$

with $\varepsilon$ the specific energy. The differential of the enthalpy can be written as

$$dH = dU + P \, dV + V \, dP$$

Using the first law of thermodynamics, according to which $dU = dQ - PdV$, and the second law of thermodynamics, according to which $dQ = TdS$, we can rewrite this as

$$dH = T \, dS + V \, dP$$

which, in specific form, becomes

$$dh = T \, ds + \frac{dP}{\rho}$$

with $s \equiv S/m$ the **specific entropy**. This relation is one of the **Gibbs relations** frequently encountered in thermodynamics. NOTE: we are ignoring changes in the chemical potential here (see Appendix I).
The natural variables of $h$ are the specific entropy, $s$, and the pressure $P$. Hence, $h = h(s, P)$, and we thus have that

$$dh = \frac{\partial h}{\partial s} ds + \frac{\partial h}{\partial P} dP$$

From a comparison with the previous expression for $dh$, we see that

$$\frac{\partial h}{\partial s} = T, \quad \frac{\partial h}{\partial P} = \frac{1}{\rho}$$

Now let’s find an expression for the spatial gradient in the specific enthalpy, $\nabla h$:

$$\nabla h = \frac{\partial h}{\partial x} \vec{e}_x + \frac{\partial h}{\partial y} \vec{e}_y + \frac{\partial h}{\partial z} \vec{e}_z$$

$$= \left( \frac{\partial h}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial h}{\partial P} \frac{\partial P}{\partial x} \right) \vec{e}_x + \left( \frac{\partial h}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial h}{\partial P} \frac{\partial P}{\partial y} \right) \vec{e}_y + \left( \frac{\partial h}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial h}{\partial P} \frac{\partial P}{\partial z} \right) \vec{e}_z$$

$$= \frac{\partial h}{\partial s} \left( \frac{\partial s}{\partial x} \vec{e}_x + \frac{\partial s}{\partial y} \vec{e}_y + \frac{\partial s}{\partial z} \vec{e}_z \right) + \frac{\partial h}{\partial P} \left( \frac{\partial P}{\partial x} \vec{e}_x + \frac{\partial P}{\partial y} \vec{e}_y + \frac{\partial P}{\partial z} \vec{e}_z \right)$$

$$= T \nabla s + \frac{1}{\rho} \nabla P$$

We thus see that we can write the term $\nabla P/\rho$, which appears in our Navier-Stokes equation, in terms of a gradient in the enthalpy, and a gradient in the specific entropy! Note that for an isentropic flow, which is defined as a flow that conserves (specific) entropy (see below), $ds = 0$. Taking that into account in the above, we see that for an isentropic flow, $\nabla h = \nabla P/\rho$.

**Intermezzo: isentropic vs. adiabatic**

According to second law of thermodynamics: $T dS \geq dQ$. The equality only holds for a **reversible** process.

- $dS = 0 \implies$ isentropic
- $dQ = 0 \implies$ adiabatic

Note that only if a process is reversible AND adiabatic, can we call it isentropic.

An irreversible, adiabatic process, however, can still create entropy.
In what follows, throughout the remainder of this chapter, we will only consider ideal fluids. This means that we will ignore viscosity, and thus that our Navier-Stokes momentum equations reduce to the Euler equations:

\[
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi
\]

Using the vector identity

\[
(\vec{u} \cdot \nabla) \vec{u} = \nabla \left( \frac{u^2}{2} \right) + (\nabla \times \vec{u}) \times \vec{u} = \nabla \left( \frac{u^2}{2} \right) + \vec{w} \times \vec{u}
\]

(see also previous chapter on vorticity), we can recast the Euler equation as

\[
\frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{u^2}{2} + \Phi \right) + \nabla P \frac{1}{\rho} - (\vec{u} \times \vec{w}) = 0
\]

This is the most general form of what is called Bernoulli’s theorem, which holds for any ideal fluids.

In an ideal fluid, there is no heat flow and no dissipation. One of the consequences is that fluid flow in an ideal fluid conserves entropy: \( ds/dt = 0 \). We say that the flow is isentropic.

Now let’s consider a steady flow of an ideal fluid: Steady means that all partial time derivatives vanish. Since

\[
\frac{ds}{dt} = \frac{\partial s}{\partial t} + (\vec{u} \cdot \nabla) s
\]

we thus have that, for a steady flow of an ideal fluid, \( (\vec{u} \cdot \nabla) s = 0 \).

Recall that \( dh = T ds + dP/\rho \). Hence we have that

\[
\frac{dh}{dt} = T \frac{ds}{dt} + \frac{1}{\rho} \frac{dP}{dt}
\]

and since \( ds/dt = 0 \), we have that

\[
\frac{\partial h}{\partial t} + (\vec{u} \cdot \nabla) h = \frac{1}{\rho} \left[ \frac{\partial P}{\partial t} + (\vec{u} \cdot \nabla) P \right]
\]
where we have used the relation between Eulerian and Lagrangian derivatives. Since
the partial time derivatives for a steady flow vanish, we thus see that

\[
(\vec{u} \cdot \nabla) h = \frac{1}{\rho} (\vec{u} \cdot \nabla) P
\]

Let’s take the most general form of the Bernoulli’s theorem above, applied to a steady
flow, so that the first term vanishes, and dot the equation with \( \vec{u} \) to obtain

\[
\vec{u} \cdot \nabla \left( \frac{1}{2} u^2 + \Phi \right) + \frac{1}{\rho} (\vec{u} \cdot \nabla) P - \vec{u} \cdot \vec{u} \times \vec{w} = 0
\]

Using that the last term vanishes (the cross product of a vector with another vector
is always perpendicular to that vector), and substituting the relation \((\vec{u} \cdot \nabla) h = \frac{1}{\rho} (\vec{u} \cdot \nabla) P\), we can recast this in the form

\[
(\vec{u} \cdot \nabla) B = 0
\]

where we have defined the Bernoulli function

\[
B \equiv \frac{u^2}{2} + \Phi + h
\]

Hence, for a steady flow of an ideal fluid, the gradient in the Bernoulli function is
always perpendicular to the streamlines.

Since

\[
\frac{dB}{dt} = \frac{\partial B}{\partial t} + (\vec{u} \cdot \nabla) B
\]

we see that a steady flow of ideal fluid always obeys

\[
\frac{dB}{dt} = 0
\]

Hence, in a steady flow of an ideal fluid, the Bernoulli function \( B \), just like the
entropy, is conserved moving with the fluid.

NOTE: for a steady flow, the streamlines are the same as the particle paths. Hence
\((\vec{u} \cdot \nabla) B = 0\) implies that their can be no gradient (no change) in \( B \) along a particle
path, and thus \( dB = 0 \).
Let’s go back to the most general form of Bernoulli’s theorem

$$\frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{1}{2} u^2 + \Phi \right) + \frac{\nabla P}{\rho} - (\vec{u} \times \vec{w}) = 0$$

Using that $\nabla P/\rho = \nabla h - T \nabla s$, we can recast this in the following form

$$\frac{\partial \vec{u}}{\partial t} + \nabla B - T \nabla s - (\vec{u} \times \vec{w}) = 0$$

which implies that for a steady flow of an ideal fluid we have that

$$\nabla B = T \nabla s + (\vec{u} \times \vec{w})$$

This is known as Crocco’s theorem. It implies that an isentropic fluid (meaning $\nabla s = 0$) that is irrotational (meaning $\vec{w} = 0$ everywhere) has $\nabla B = 0$:

In a steady, isentropic, irrotational, ideal fluid the Bernoulli function is everywhere the same.

**Intermezzo: isentropic flow vs. isentropic fluid**

If a flow is isentropic, then $ds/dt = 0$

If a fluid is isentropic, then $\nabla s = 0$

The flow of an ideal fluid is always isentropic, but an ideal fluid is not necessarily isentropic (there can be gradients in $s$ perpendicular to the streamlines).

From Crocco’s theorem we see that if $\nabla B = \nabla s = 0$, a steady inviscid flow cannot have any vorticity. It also implies that we can create vorticity in a fluid by creating a gradient in $s$ (or $B$). This can be achieved across a curved shock. In a shock the entropy increases (the fluid is moved to another ‘adiabat’). In a curved shock, that gradient in $s$ is not perpendicular to $\vec{u}$ (i.e., $\nabla s \cdot \vec{u} \neq 0$), and thus in the post-shocked gas there will likely be a gradient in $s$ (as well as in $B$) and thus non-zero vorticity.

Note that in astrophysics one rarely encounters solid boundaries, which can create vorticity in their boundary layers due to viscosity (which is anyways oftern very small for astrophysical fluids). As we have seen in the previous chapter, the only
way to create vorticity in the absence of viscosity is via the baroclinic process, which would imply that one could never create vorticity in a barotropic, astrophysical fluid. However, curved shocks can, and are indeed one of the most important mechanisms in astrophysics for creating vorticity.

We have seen above that in a steady flow of an ideal fluid, the Bernoulli function is conserved moving with the flow \( (dB/dt = 0) \). Note that \( B \) is a kind of energy measure. It is the sum of the specific kinetic energy, \( u^2/2 \), the specific potential energy, \( \Phi \), and the specific enthalpy, \( h \), which itself is the sum of the specific internal energy, \( \varepsilon \) and the term \( P/\rho \) which expresses the specific work needed to create the system’s volume (\( P/\rho \) is sometimes called the ‘pressure potential’).

Bernoulli’s theorem states thus states that in a steady flow of an ideal fluid, the ‘energy’ \( B \) is conserved along streamlines (particle paths). However, along those streamlines ordered motion \( (\vec{u}) \) can be converted into random motion \( (\varepsilon, \text{related to } P) \), and vice versa. Kinetic energy can also be converted into potential energy, and vice versa. In fact, we have that

\[
\frac{dB}{dt} = \vec{u} \cdot \frac{d\vec{u}}{dt} + \frac{d\Phi}{dt} + \frac{1}{\rho} \frac{dP}{dt}
\]

where we have used that \( ds/dt = 0 \) for an ideal fluid. The most often encountered form of Bernoulli’s theorem is the case where the gravitational energy doesn’t vary significantly along the flow, so that \( d\Phi/dt = 0 \). Bernoulli’s theorem than implies that

\[
\vec{u} \cdot \frac{d\vec{u}}{dt} = -\frac{1}{\rho} \frac{dP}{dt}
\]

and thus that if the speed of a flow increases, its internal pressure must decrease (this holds for any steady flow of ideal fluid where changes in \( \Phi \) can be ignored). Applications of Bernoulli’s equation discussed in class are the shower curtain which moves inwards when the shower is on, and the lift experienced by an airplane wing (but be aware of the Equal Transit Time Fallacy), the pitot tube used in airplanes to measure speed, and the Laval nozel Another application is the venturi meter (see problem set 3).
CHAPTER 10

Turbulence

Non-linearity: The Navier-Stokes equation is non-linear. This non-linearity arises from the convective (material) derivative term

\[
\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2 - \vec{u} \times \vec{w}
\]

which describes the “inertial acceleration” and is ultimately responsible for the origin of the chaotic character of many flows and of turbulence. Because of this non-linearity, we cannot say whether a solution to the Navier-Stokes equation with nice and smooth initial conditions will remain nice and smooth for all time (at least not in 3D).

Laminar flow: occurs when a fluid flows in parallel layers, without lateral mixing (no cross currents perpendicular to the direction of flow). It is characterized by high momentum diffusion and low momentum convection.

Turbulent flow: is characterized by chaotic and stochastic property changes. This includes low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time.

The Reynold’s number: In order to gauge the importance of viscosity for a fluid, it is useful to compare the ratio of the inertial acceleration \((\vec{u} \cdot \nabla \vec{u})\) to the viscous acceleration \((\nu [\nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u})])\). This ratio is called the Reynold’s number, \(\mathcal{R}\), and can be expressed in terms of the typical velocity scale \(U \sim |\vec{u}|\) and length scale \(L \sim 1/\nabla\) of the flow, as

\[
\mathcal{R} = \frac{\vec{u} \cdot \nabla \vec{u}}{\nu [\nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u})]} \sim \frac{U^2/L}{\nu U/L^2} = \frac{U L}{\nu}
\]

If \(\mathcal{R} \gg 1\) then viscosity can be ignored (and one can use the Euler equations to describe the flow). However, if \(\mathcal{R} \ll 1\) then viscosity is important.
**Similarity:** Flows with the same Reynold’s number are similar. This is evident from rewriting the Navier-Stokes equation in terms of the following dimensionless variables

\[ \tilde{u} = \frac{\bar{u}}{U} \quad \tilde{x} = \frac{x}{L} \quad \tilde{t} = t \frac{U}{L} \quad \tilde{p} = \frac{P}{\rho U^2} \quad \tilde{\Phi} = \frac{\Phi}{U^2} \quad \tilde{\nabla} = L \nabla \]

This yields (after multiplying the Navier-Stokes equation with \( L/U^2 \)):

\[ \frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \tilde{\nabla} \tilde{u} + \tilde{\nabla} \tilde{p} + \tilde{\nabla} \tilde{\Phi} = \frac{1}{R} \left[ \tilde{\nabla}^2 \tilde{u} + \frac{1}{3} \tilde{\nabla}(\tilde{\nabla} \cdot \tilde{u}) \right] \]

which shows that the form of the solution depends only on \( R \). This principle is extremely powerful as it allows one to making scale models (i.e., when developing airplanes, cars etc). NOTE: the above equation is only correct for an incompressible fluid, i.e., a fluid that obeys \( \nabla \rho = 0 \). If this is not the case the term \( \tilde{P} (\nabla \rho / \rho) \) needs to be added at the rhs of the equation, braking its scale-free nature.

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As a specific example, consider fluid flow past a cylinder of diameter $L$:

- $\mathcal{R} \ll 1$: "creeping flow". In this regime the flow is viscously dominated and (nearly) symmetric upstream and downstream. The inertial acceleration $(\ddot{u} \cdot \nabla \ddot{u})$ can be neglected, and the flow is (nearly) time-reversible.

- $\mathcal{R} \sim 1$: Slight asymmetry develops

- $10 \leq \mathcal{R} \leq 41$: Separation occurs, resulting in two counter-rotating vortices in the wake of the cylinder. The flow is still steady and laminar, though.

- $41 \leq \mathcal{R} \leq 10^3$: "von Kármán vortex street"; unsteady laminar flow with counter-rotating vortices shed periodically from the cylinder. Even at this stage the flow is still ‘predictable’.

- $\mathcal{R} > 10^3$: vortices are unstable, resulting in a turbulent wake behind the cylinder that is ‘unpredictable’.

Figure 12: Illustration of flows at different Reynolds number.
Figure 13: The image shows the von Kármán Vortex street behind a 6.35 mm diameter circular cylinder in water at Reynolds number of 168. The visualization was done using hydrogen bubble technique. Credit: Sanjay Kumar & George Laughlin, Department of Engineering, The University of Texas at Brownsville

The following movie shows a $R = 250$ flow past a cylinder. Initially one can witness separation, and the creation of two counter-rotating vortices, which then suddenly become ‘unstable’, resulting in the von Kármán vortex street:

http://www.youtube.com/watch?v=IDeGDFZSYo8
Locomotion at Low-Reynolds number: Low Reynolds number corresponds to high kinetic viscosity for a given $U$ and $L$. In this regime of ‘creeping flow’ the flow past an object is (nearly) time-reversible. Imagine trying to move (swim) in a highly viscous fluid (take honey as an example). If you try to do so by executing time-symmetric movements, you will not move. Instead, you need to think of a symmetry-breaking solution. Nature has found many solutions for this problem. If we make the simplifying “rule-of-thumb” assumption that an animal of size $L$ meters moves roughly at a speed of $U = L$ meters per second (yes, this is very, very rough, but an ant does move close to 1 mm/s, and a human at roughly 1 m/s), then we have that $\mathcal{R} = UL/\nu \simeq L^2/\nu$. Hence, with respect to a fixed substance (say water, for which $\nu \sim 10^{-2}\text{cm}^2/\text{s}$), smaller organisms move at lower Reynolds number (effectively in a fluid of higher viscosity). Scaling down from a human to bacteria and single-cell organisms, the motion of the latter in water has $\mathcal{R} \sim 10^{-5} – 10^{-2}$. Understanding the locomotion of these organisms is a fascinating sub-branch of bio-physics.
Boundary Layers: Even when $R \gg 1$, viscosity always remains important in thin boundary layers adjacent to any solid surface. This boundary layer must exist in order to satisfy the no-slip boundary condition. If the Reynolds number exceeds a critical value, the boundary layer becomes turbulent. Turbulent layers and their associated turbulent wakes exert a much bigger drag on moving bodies than their laminar counterparts.

Momentum Diffusion & Reynolds stress: This gives rise to an interesting phenomenon. Consider flow through a pipe. If you increase the viscosity (i.e., decrease $R$), then it requires a larger force to achieve a certain flow rate (think of how much harder it is to push honey through a pipe compared to water). However, this trend is not monotonic. For sufficiently low viscosity (large $R$), one finds that the trend reverses, and that is becomes harder again to push the fluid through the pipe. This is a consequence of turbulence, which causes momentum diffusion within the flow, which acts very much like viscosity. However, this momentum diffusion is not due to the viscous stress tensor, $\tau_{ij}$, but rather to the Reynolds stress tensor $R_{ij}$. To understand the ‘origin’ of the Reynolds stress tensor, consider the following:

For a turbulent flow, $\vec{u}(t)$, it is advantageous to decompose each component of $\vec{u}$ into a ‘mean’ component, $\bar{u}_i$, and a ‘fluctuating’ component, $u'_i$, according to

$$u_i = \bar{u}_i + u'_i$$

This is know as the Reynolds decomposition. The ‘mean’ component can be a time-average, a spatial average, or an ensemble average, depending on the detailed characteristics of the flow. Note that this is reminiscent of how we decomposed the microscopic velocities of the fluid particles in a ‘mean’ velocity (describing the fluid elements) and a ‘random, microscopic’ velocity ($\vec{v} = \bar{u} + \vec{w}$).

Substituting this into the Navier-Stokes equation, and taking the average of that, we obtain

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[ \tau_{ij} - \rho u'_i u'_j \right]$$

where, for simplicity, we have ignored gravity (the $\nabla \Phi$-term). This equation looks identical to the Navier-Stokes equation (in absence of gravity), except for the $-\rho u'_i u'_j$ term, which is what we call the Reynolds stress tensor.
\[ R_{ij} = -\rho u'_i u'_j \]

Note that \( u'_i u'_j \) means the same averaging (time, space or ensemble) as above, but now for the product of \( u'_i \) and \( u'_j \). Note that \( \overline{u'}_i = 0 \), by construction. However, the expectation value for the product of \( u'_i \) and \( u'_j \) is generally not. As is evident from the equation, the Reynolds stresses (which reflect momentum diffusion due to turbulence) act in exactly the same way as the viscous stresses. However, they are only present when the flow is turbulent.

Note also that the Reynolds stress tensor is related to the \textbf{two-point correlation tensor}

\[ \xi_{ij}(\vec{r}) \equiv \overline{u'_i(\vec{x}, t) u'_j(\vec{x} + \vec{r}, t)} \]

in the sense that \( R_{ij} = \xi_{ij}(0) \). At large separations, \( \vec{r} \), the fluctuating velocities will be uncorrelated so that \( \lim_{r \to \infty} \xi_{ij} = 0 \). But on smaller scales the fluctuating velocities will be correlated, and there will be a ‘characteristic’ scale associated with these correlations, called the \textbf{correlation length}.

**Turbulence:** Turbulence is still considered as one of the last "unsolved problems of classical physics" [Richard Feynman]. What we technically mean by this is that we do not yet know how to calculate \( \xi_{ij}(\vec{r}) \) (and higher order correlation functions, like the three-point, four-point, etc) in a particular situation from a fundamental theory. Salmon (1998) nicely sums up the challenge of defining turbulence:

Every aspect of turbulence is controversial. Even the definition of fluid turbulence is a subject of disagreement. However, nearly everyone would agree with some elements of the following description:

- Turbulence requires the presence of vorticity; irrotational flow is smooth and steady to the extent that the boundary conditions permit.
- Turbulent flow has a complex structure, involving a broad range of space and time scales.
- Turbulent flow fields exhibit a high degree of apparent randomness and disorder. However, close inspection often reveals the presence of embedded coherent flow structures

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• Turbulent flows have a high rate of viscous energy dissipation.
• Advected tracers are rapidly mixed by turbulent flows.

However, one further property of turbulence seems to be more fundamental than all of these because it largely explains why turbulence demands a statistical treatment...turbulence is chaotic.

The following is a brief, qualitative description of turbulence:

Turbulence kicks in at sufficiently high Reynolds number (typically $R > 10^3 - 10^4$). Turbulent flow is characterized by irregular and seemingly random motion. Large vortices (called eddies) are created. These contain a large amount of kinetic energy. Due to vortex stretching these eddies are stretched thin until they ‘break up’ in smaller eddies. This results in a cascade in which the turbulent energy is transported from large scales to small scales. This cascade is largely inviscid, conserving the total turbulent energy. However, once the length scale of the eddies becomes comparable to the mean free path of the particles, the energy is dissipated; the kinetic energy associated with the eddies is transformed into internal energy. The scale at which this happens is called the Kolmogorov length scale. The length scales between the scale of turbulence ‘injection’ and the Kolomogorov length scale at which it is dissipated is called the inertial range. Over this inertial range turbulence is believed/observed to be scale invariant. The ratio between the injection scale, $L$, and the dissipation scale, $l$, is proportional to the Reynolds number according to $L/l \propto R^{3/4}$. Hence, two turbulent flows that look similar on large scales (comparable $L$), will dissipate their energies on different scales, $l$, if their Reynolds numbers are different.

Molecular clouds: an example of turbulence in astrophysics are molecular clouds. These are gas clouds of masses $10^5 - 10^6 M_\odot$, densities $n_H \sim 100 - 500 \text{cm}^{-3}$, and temperatures $T \sim 10\text{K}$. They consist mainly of molecular hydrogen and are the main sites of star formation. Observations show that their velocity linewidths are $\sim 6-10\text{km/s}$, which is much higher than their sound speed ($c_s \sim 0.2\text{km/s}$). Hence, they are supported against (gravitational) collapse by supersonic turbulence. On small scales, however, the turbulent motions compress the gas to high enough densities that stars can form. A numerical simulation of a molecular cloud with supersonic turbulence is available here:

http://www.youtube.com/watch?v=3z9ZKAkbMhY
CHAPTER 11

Equations of State

Closure: The continuity and momentum (Euler) equations are 4 equations with 6 unknowns ($\rho$, $\vec{u}$, $P$, and $\Phi$). With the Poisson equation, which relates $\rho$ and $\Phi$, we are still one equation short for closure. This equation can either be an equation of state (but only if it is barotropic, i.e., $P = P(\rho)$), or the energy equation (see Chapter 11).

Equation of State (EoS): a thermodynamic equation describing the state of matter under a given set of physical conditions. In what follows we will always write our EoS in the form $P = P(\rho, T)$. Other commonly used forms are $P = P(\rho, \varepsilon)$ or $P = P(\rho, S)$.

Ideal Gas: a hypothetical gas that consists of identical point particles (i.e. of zero volume) that undergo perfectly elastic collisions and for which interparticle forces can be neglected.

An ideal gas obeys the ideal gas law: $PV = Nk_B T$.

Here $N$ is the total number of particles, $k_B$ is Boltzmann’s constant, and $V$ is the volume occupied by the fluid. Using that $\rho = N \mu m_p / V$, where $\mu$ is the mean molecular weight in units of the proton mass $m_p$, we have that the EoS for an ideal gas is given by

$$P = \frac{k_B T}{\mu m_p} \rho$$

NOTE: astrophysical gases are often well described by the ideal gas law. Even for a fully ionized gas, the interparticle forces (Coulomb force) can typically be neglected (i.e., the potential energies involved are typically < 10% of the kinetic energies). Ideal gas law breaks down for dense, and cool gases, such as those present in gaseous planets.
Maxwell-Boltzmann Distribution: the distribution of particle momenta, $\vec{p} = m\vec{v}$, of an ideal gas follows the Maxwell-Boltzmann distribution.

$$P(\vec{p}) d^3\vec{p} = \left(\frac{1}{2\pi m k_B T}\right)^{3/2} \exp\left(-\frac{\vec{p}^2}{2m k_B T}\right) d^3\vec{p}$$

where $p^2 = \vec{p} \cdot \vec{p}$. This distribution follows from maximizing entropy under the following assumptions:

1. all magnitudes of velocity are *a priori* equally likely
2. all directions are equally likely (isotropy)
3. total energy is constrained at a fixed value
4. total number of particles is constrained at a fixed value

Using that $E = p^2/2m$ we thus see that $P(\vec{p}) \propto e^{-E/k_B T}$.

Pressure: pressure arises from (elastic) collisions of particles. A particle hitting a wall head on with momentum $p = mv$ results in a transfer of momentum to the wall of $2mv$. Using this concept, and assuming isotropy for the particle momenta, it can be shown that

$$P = \zeta n \langle E \rangle$$

where $\zeta = 2/3$ ($\zeta = 1/3$) in the case of a non-relativistic (relativistic) fluid, and

$$\langle E \rangle = \int_0^\infty E P(E) dE$$

is the average, translational energy of the particles. In the case of our ideal (non-relativistic) fluid,

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} = \int_0^\infty \frac{p^2}{2m} P(p) dp = \frac{3}{2} k_B T$$
Hence, we find that the **EoS for an ideal gas** is indeed given by

\[
P = \frac{2}{3} n \langle E \rangle = n k_B T = \frac{k_B T}{\mu m_p} ρ
\]

**Specific Internal Energy:** the internal energy per unit mass for an ideal gas is

\[
ε = \frac{\langle E \rangle}{\mu m_p} = \frac{3}{2} k_B T
\]

Actually, the above derivation is only valid for a true ‘ideal gas’, in which the particles are point particles. More generally,

\[
ε = \frac{1}{\gamma - 1} \frac{k_B T}{\mu m_p}
\]

where \( γ \) is the **adiabatic index**, which for an ideal gas is equal to \( γ = (q+5)/(q+3) \), with \( q \) the internal degrees of freedom of the fluid particles: \( q = 0 \) for point particles (resulting in \( γ = 5/3 \)), while diatomic particles have \( q = 2 \) (at sufficiently low temperatures, such that they only have rotational, and no vibrational degrees of freedom). The fact that \( q = 2 \) in that case arises from the fact that a diatomic molecule only has two relevant rotation axes; the third axis is the symmetry axis of the molecule, along which the molecule has negligible (zero in case of point particles) moment of inertia. Consequently, rotation around this symmetry axis carries no energy.

**Photon gas:** Having discussed the EoS of an ideal gas, we now focus on a gas of photons. Photons have energy \( E = hν \) and momentum \( p = E/c = hν/c \), with \( h \) the Planck constant.

**Black Body:** an idealized physical body that absorbs all incident radiation. A black body (BB) in thermal equilibrium emits electro-magnetic radiation called **black body radiation**.

The **spectral number density distribution** of BB photons is given by

\[
n_γ(ν, T) = \frac{8πν^2}{c^3} \frac{1}{e^{hν/k_BT} - 1}
\]
which implies a spectral energy distribution

\[ u(\nu, T) = n_\gamma(\nu, T) h\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_BT} - 1} \]

and thus an energy density of

\[ u(T) = \int_0^\infty u(\nu, T) d\nu = \frac{4\sigma_{SB}}{c} T^4 \equiv a_r T^4 \]

where

\[ \sigma_{SB} = \frac{2\pi^5 k_B^4}{15h^3c^2} \]

is the Stefan-Boltzmann constant and \( a_r \simeq 7.6 \times 10^{-15} \text{erg cm}^{-3} \text{K}^{-4} \) is called the radiation constant.

**Radiation Pressure:** when the photons are reflected off a wall, or when they are absorbed and subsequently re-emitted by that wall, they transfer twice their momentum in the normal direction to that wall. Since photons are relativistic, we have that the EoS for a photon gas is given by

\[
P = \frac{1}{3} n \langle E \rangle = \frac{1}{3} n_\gamma \langle h\nu \rangle = \frac{1}{3} u(T) = \frac{aT^4}{3}
\]

where we have used that \( u(T) = n_\gamma \langle E \rangle \).

**Quantum Statistics:** according to quantum statistics, a collection of many indistinguishable elementary particles in thermal equilibrium has a momentum distribution given by

\[
f(\vec{p}) d^3\vec{p} = \frac{g}{h^3} \left[ \exp \left( \frac{E(\vec{p}) - \mu}{k_BT} \right) \pm 1 \right]^{-1} d^3\vec{p}
\]

where the signature \( \pm \) takes the positive sign for fermions (which have half-integer spin), in which case the distribution is called the Fermi-Dirac distribution, and the negative sign for bosons (particles with zero or integer spin), in which case the distribution is called the Bose-Einstein distribution. The factor \( g \) is the spin degeneracy factor, which expresses the number of spin states the particles can have (\( g = 1 \) for neutrinos, \( g = 2 \) for photons and charged leptons, and \( g = 6 \)).
for quarks). Finally, \( \mu \) is called the **chemical potential**, and is a form of potential energy that is related (in a complicated way) to the number density and temperature of the particles (see Appendix I).

**Classical limit:** In the limit where the mean interparticle separation is much larger than the de Broglie wavelength of the particles, so that quantum effects (e.g., Heisenberg’s uncertainty principle) can be ignored, the above distribution function of momenta can be accurately approximated by the **Maxwell-Boltzmann distribution**.

**Heisenberg’s Uncertainty Principle:** \( \Delta x \Delta p_x > h \) (where \( h = 6.63 \times 10^{-27} \text{g cm}^2 \text{s}^{-1} \) is Planck’s constant). One interpretation of this quantum principle is that phase-space is quantized; no particle can be localized in a phase-space element smaller than the fundamental element

\[
\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3
\]

**Pauli Exclusion Principle:** no more than one fermion of a given spin state can occupy a given phase-space element \( h^3 \). Hence, for electrons, which have \( g = 2 \), the maximum phase-space density is \( 2/h^3 \).

**Degeneracy:** When compressing and/or cooling a fermionic gas, at some point all possible low momentum states are occupied. Any further compression therefore results in particles occupying high (but the lowest available) momentum states. Since particle momentum is ultimately responsible for pressure, this degeneracy manifests itself as an extremely high pressure, known as **degeneracy pressure**.

**Fermi Momentum:** Consider a fully degenerate gas of electrons of electron density \( n_e \). It will have fully occupied the part of phase-space with momenta \( p \leq p_F \). Here \( p_F \) is the maximum momentum of the particles, and is called the **Fermi momentum**. The energy corresponding to the Fermi momentum is called the **Fermi energy**, \( E_F \) and is equal to \( p_F^2 / 2m \) in the case of a non-relativistic gas, and \( p_F c \) in the case of a relativistic gas.

Let \( V_x \) be the volume occupied in configuration space, and \( V_p = \frac{4}{3}\pi p_F^3 \) the volume occupied in momentum space. If the total number of particles is \( N \), and the gas is
fully degenerate, then

\[ V_x V_p = \frac{N}{2} h^3 \]

Using that \( n_e = N/V_x \), we find that

\[ p_F = \left( \frac{3}{8\pi n_e} \right)^{1/3} h \]

**EoS of Non-Relativistic, Degenerate Gas:** Using the information above, it is relatively straightforward (see Problem Set 4) to compute the EoS for a fully degenerate gas. Using that for a non-relativistic fluid \( E = p^2/2m \) and \( P = \frac{2}{3} n \langle E \rangle \), while degeneracy implies that

\[
\langle E \rangle = \frac{1}{N} \int_0^{E_i} E N(E) \, dE = \frac{1}{N} \int_0^{p_F} \frac{p^2}{2m} \frac{2}{h^3} V_x 4\pi p^2 \, dp = \frac{3}{5} \frac{p_F^2}{2m}
\]

we obtain that

\[
P = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2}{m^{8/3}} \rho^{5/3}
\]

**EoS of Relativistic, Degenerate Gas:** In the case of a relativistic, degenerate gas, we use the same procedure as above. However, this time we have that \( P = \frac{1}{3} n \langle E \rangle \) while \( E = p c \), which results in

\[
P = \frac{1}{8} \left( \frac{3}{\pi} \right)^{1/3} \frac{c h}{m^{4/3}} \rho^{4/3}
\]
White Dwarfs and the Chandrasekhar limit: White dwarfs are the end-states of stars with mass low enough that they don’t form a neutron star. When the pressure support from nuclear fusion in a star comes to a halt, the core will start to contract until degeneracy pressure kicks in. The star consists of a fully ionized plasma. Assume for simplicity that the plasma consists purely of hydrogen, so that the number density of protons is equal to that of electrons: \( n_p = n_e \). Because of equipartition

\[
\frac{p_p^2}{2m_p} = \frac{p_e^2}{2m_e}
\]

Since \( m_p \gg m_e \) we have also that \( p_p \gg p_e \) (in fact \( p_p/p_e = \sqrt{m_p/m_e} \approx 43 \)). Consequently, when cooling or compressing the core of a star, the electrons will become degenerate well before the protons do. Hence, white dwarfs are held up against collapse by the degeneracy pressure from electrons. Since the electrons are typically non-relativistic, the EoS of the white dwarf is: \( P \propto \rho^{5/3} \). If the white dwarf becomes more and more massive (i.e., because it is accreting mass from a companion star), the Pauli-exclusion principle causes the Fermi momentum, \( p_F \), to increase to relativistic values. This softens the EoS towards \( P \propto \rho^{4/3} \). Such an equation of state is too soft to stabilize the white dwarf against gravitational collapse; the white dwarf collapses until it becomes a neutron star, at which stage it is supported against further collapse by the degeneracy pressure from neutrons. This happens when the mass of the white dwarf reaches \( M_{\text{lim}} \approx 1.44 M_\odot \), the so-called Chandrasekhar limit.

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Summary of equations of state for different kind of fluids

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Heat Transfer: In order to close the fluid equations, we need to add an equation that describes how the internal energy (heat) of a fluid element changes as a function of time. There are four fundamental modes of heat transfer:

- **Radiation**: the transfer of energy to and from a fluid element by means of absorption or emission of electro-magnetic radiation.
- **Advection**: the transfer of energy from one location to another as a side effect of physically moving a fluid element containing that energy.
- **Conduction**: the transfer of energy between fluid elements that are in physical contact due to microscopic diffusion (requires temperature gradients).
- **Convection**: the transfer of energy between a fluid element and its environment due to bulk motion plus diffusion (i.e., convection is simply a combination of advection and conduction). Convection occurs whenever the temperature gradient becomes too large (Schwarzschild’s stability criterion; see Chapter 16).

Another mode of energy transfer that is relevant for astronomy is the heating due to cosmic rays, which are energetic elementary particles (mainly protons) that have been accelerated to relativistic speeds by shocks from supernova etc. In what follows, we will treat cosmic ray heating as a component of radiative heating.

**Energy Density**: The energy density, $E$, of a fluid consists of three components: kinetic energy, potential energy, and internal energy:

$$E = \rho \left( \frac{1}{2} u^2 + \Phi + \varepsilon \right)$$

where $\varepsilon$ is the specific internal energy of the fluid. Note that $E$ as defined here is the energy per unit volume.
Energy equation: The Lagrangian derivative of the energy density is given by

\[
\frac{dE}{dt} = \frac{E}{\rho} \frac{d\rho}{dt} + \rho \vec{u} \cdot \frac{d\vec{u}}{dt} + \rho \frac{d\Phi}{dt} + \rho \frac{d\varepsilon}{dt}
\]

which simply follows from applying the chain rule to \( E = \rho \left( \frac{1}{2} u^2 + \Phi + \varepsilon \right) \).

We now treat each of these four terms in turn:

1st term: Using the continuity equation we have that

\[
\frac{E}{\rho} \frac{d\rho}{dt} = -E \nabla \cdot \vec{u}
\]

2nd term: Using the (Euler) momentum equation we have that

\[
\rho \vec{u} \cdot \frac{d\vec{u}}{dt} = \vec{u} \cdot \rho \frac{d\vec{u}}{dt} = -\vec{u} \cdot (\nabla P + \rho \nabla \Phi)
\]

3rd term: Using the expression for the substantial (Lagrangian) derivative we have that

\[
\rho \frac{d\Phi}{dt} = \rho \frac{\partial \Phi}{\partial t} + \rho \vec{u} \cdot \nabla \Phi
\]

4th term: Using the first law of thermodynamics, \( d\varepsilon = dQ - dW \), where \( dQ \) is the specific heat absorbed and \( dW = P d(1/\rho) \) is the specific work done by the fluid, we have that

\[
\rho \frac{d\varepsilon}{dt} = \rho \frac{dQ}{dt} + P \frac{d\rho}{\rho} \frac{d\rho}{dt}
\]
Combining all the above, and using that
\[
\frac{dE}{dt} = \frac{\partial E}{\partial t} + (\vec{u} \cdot \nabla) E
\]
we finally obtain the energy equation for an inviscid fluid:

\[
\frac{\partial E}{\partial t} + \nabla \cdot [(E + P)\vec{u}] = -\mathcal{L} + \rho \frac{\partial \Phi}{\partial t}
\]

where we have defined the net heating rate per unit volume

\[
\mathcal{L} \equiv \rho \frac{dQ}{dt} \equiv C - \mathcal{H}
\]

where \(C\) and \(\mathcal{H}\) are the volumetric cooling and heating rates, respectively, which express heat transfer due to the emission and/or absorption of radiation (and cosmic rays).

Note that the external (gravitational) potential only enters with a partial time-derivative. Hence, only when the external potential varies with time, does it have an impact on the evolution of the total energy density of fluid elements. If the potential is steady (i.e., \(\frac{\partial \Phi}{\partial t} = 0\), then the presence of the gravitational potential can cause the conversion of kinetic energy into potential energy (and vice versa), but it does not change the total energy density. Changes in the energy of individual fluid elements due to a time-variable gravitational potential is called violent relaxation, and is the main relaxation mechanisms for collisionless systems.

Using that \(E = \rho \left( \frac{1}{2} u^2 + \Phi + \varepsilon \right)\), the energy equation can also be written as:

\[
\frac{\partial}{\partial t} \left[ \rho \left( \frac{u^2}{2} + \varepsilon \right) \right] + \frac{\partial}{\partial x_k} \left[ \rho \left( \frac{u^2}{2} + \varepsilon \right) u_k + P \delta_{jk} u_j \right] = -\mathcal{L} - \rho u_k \frac{\partial \Phi}{\partial x_k}
\]

In deriving the above form of the energy equation we have used that
\[ \rho \frac{\partial \Phi}{\partial t} - \frac{\partial \rho \Phi}{\partial t} - \frac{\partial \rho \Phi u_k}{\partial x_k} = -\Phi \frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \Phi \vec{u}) \]

\[ = -\Phi \frac{\partial \rho}{\partial t} - \Phi \nabla \cdot \rho \vec{u} - \rho \vec{u} \nabla \Phi \]

\[ = -\Phi \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} \right] - \rho \vec{u} \nabla \Phi = -\rho \vec{u} \nabla \Phi \]

where, in the final step, we have used the **continuity equation**. One of the advantages of this index-form, is that it is easier to incorporate the effects of **viscosity**. By replacing \(-P \, \delta_{ij}\) with the **stress tensor** \(\sigma_{ij} = -P \, \delta_{ij} + \tau_{ij}\), and adding a term describing **conduction**, we obtain the fully general **energy equation for a viscous fluid**:

\[
\frac{\partial}{\partial t} \left[ \rho \left( \frac{u^2}{2} + \varepsilon \right) \right] =
- \frac{\partial}{\partial x_k} \left[ \rho \left( \frac{u^2}{2} + \varepsilon \right) u_k + (P \, \delta_{jk} - \tau_{jk}) u_j + F_{\text{cond}, k} \right] - L - \rho u_k \frac{\partial \Phi}{\partial x_k}
\]

The \(\rho \left( \frac{u^2}{2} + \varepsilon \right) u_k\) term on the rhs describes **advection**, the \(P \, \delta_{jk} \, u_j\) term describes the **work done**, the \(\tau_{jk} \, u_j\) term describes **viscous dissipation** (i.e., the conversion of ordered bulk motion into disordered random motion), \(F_{\text{cond}, k}\) is the **conduction flux** in the \(k\)-direction, \(L\) describes the change in (internal) energy due to the absorption or emission of radiation (or cosmic rays), and the last term on the rhs describes the change of energy due to motion in a gravitational potential.

**Conduction**: to first order in the ratio of the mean free path \(l\) of the particles and the length scale \(L\) of the physical system, the **conduction heat flux** can be written as

\[
\vec{F}_{\text{cond}} = -\mathcal{K} \nabla T
\]

where \(\mathcal{K}\) is called the **thermal conductivity** and has units of \(\text{erg s}^{-1} \text{cm}^{-1} \text{K}^{-1}\). It is roughly given by \(\mathcal{K} \sim \frac{3}{2} k_B n v_{\text{th}} l\), where \(v_{\text{th}} \propto T^{1/2}\) is the thermal (microscopic) velocity of the particles. Using that the mean free path \(l = (n \sigma)^{-1}\), with \(\sigma\) the collision cross section, we thus see that \(\mathcal{K} \propto T^{1/2}/\sigma\). As expected, conduction increases...
with temperature (particles move faster) and decreases with increasing cross section (particles move less far).

To see another ‘representation’ of the conductivity, which links it directly to the microscopic motion of the fluid particles, we now (for the sake of completeness) derive the above energy equation starting from the master moment equation

\[
\frac{\partial}{\partial t} \left[ n\langle Q \rangle \right] + \frac{\partial}{\partial x_k} \left[ n\langle v_k Q \rangle \right] + n \frac{\partial \Phi}{\partial x_k} \left\langle \frac{\partial Q}{\partial v_k} \right\rangle = 0
\]

derived in Chapter 7 from the Boltzmann equation. For the energy equation, we need to set

\[ Q = \frac{1}{2}mv^2 = \frac{m}{2}v_i v_i = \frac{m}{2}(u_i + w_i)(u_i + w_i) = \frac{m}{2}(u^2 + 2u_i w_i + w^2) \]

Hence, we have that \[ \langle Q \rangle = \frac{1}{2}mu^2 + \frac{1}{2}m\langle w^2 \rangle \] where we have used that \[ \langle u \rangle = u \] and \[ \langle w \rangle = 0. \] Using that \[ \rho = mn, \] the first term in the master moment equation thus becomes

\[ \frac{\partial}{\partial t} \left[ n\langle Q \rangle \right] = \frac{\partial}{\partial t} \left[ \rho \frac{u^2}{2} + \rho \varepsilon \right] \]

where we have used that the specific internal energy \[ \varepsilon = \frac{1}{2}\langle w^2 \rangle. \] For the second term, we use that

\[
\begin{align*}
n\langle v_k Q \rangle &= \frac{\rho}{2}\langle (u_k + w_k)(u^2 + 2u_i w_i + w^2) \rangle \\
&= \frac{\rho}{2}\langle u^2 u_k + 2u_i u_k w_i + w^2 u_k + u^2 w_k + 2u_i w_i w_k + w^2 w_k \rangle \\
&= \frac{\rho}{2}\left[ u^2 u_k + u_k \langle w^2 \rangle + 2u_i \langle w_i w_k \rangle + \langle w^2 w_k \rangle \right] \\
&= \frac{\rho}{2}u^2 u_k + \rho \varepsilon u_k + \rho u_i \langle w_i w_k \rangle + F_{\text{cond},k}
\end{align*}
\]

Here we have defined the conductivity

\[ F_{\text{cond},k} \equiv \rho \langle w_k \frac{1}{2}w^2 \rangle = \langle \rho \varepsilon w_k \rangle \]

This makes it clear that conduction describes how internal energy is dispersed due to the random motion of the fluid particles. Using that \[ \rho \langle w_i w_k \rangle = -\sigma_{ik} = P \delta_{ik} - \tau_{ik}, \] the second term of the master moment equation becomes
\[
\frac{\partial}{\partial x_k} [n\langle v_k Q \rangle] = \frac{\partial}{\partial x_k} \left[ \frac{u^2}{2} u_k + \rho \varepsilon u_k + (P \delta_{ik} - \tau_{ik}) u_i + F_{\text{cond},k} \right]
\]

Finally, for the third term we use that

\[
\frac{\partial Q}{\partial v_k} = m \frac{\partial v^2}{2 \partial v_k} = mv_k
\]

To understand the last step, note that in Cartesian coordinates \( v^2 = v_x^2 + v_y^2 + v_z^2 \).

Hence, we have that

\[
n \frac{\partial \Phi}{\partial x_k} \langle \frac{\partial Q}{\partial v_k} \rangle = \rho \frac{\partial \Phi}{\partial x_k} \langle v_k \rangle = \rho \frac{\partial \Phi}{\partial x_k} u_k
\]

Combining the three terms in the master moment equation, we finally obtain the following **energy equation**:

\[
\frac{\partial}{\partial t} \left[ \rho \left( \frac{u^2}{2} + \varepsilon \right) \right] = -\frac{\partial}{\partial x_k} \left[ \rho \left( \frac{u^2}{2} + \varepsilon \right) u_k + (P \delta_{jk} - \tau_{jk}) u_j + F_{\text{cond},k} \right] - \rho u_k \frac{\partial \Phi}{\partial x_k}
\]

which is exactly the same as that derived above, except for the \(-\mathcal{L}\) term, which is absent from the derivation based on the Boltzmann equation, since the later does not include the effects of radiation.

The final task of this lecture on the energy equation is to derive an equation that describes the evolution of the **internal energy**. This is obtained by subtracting \( u_i \) times the **Navier-Stokes equation** in conservative, Eulerian form from the energy equation derived above.

The Navier-Stokes equation in Eulerian index form is

\[
\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = \frac{1}{\rho} \frac{\partial \sigma_{ik}}{\partial x_k} - \frac{\partial \Phi}{\partial x_i}
\]

Using the **continuity equation**, this can be rewritten in the so-called **conservation form** as

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_k - \sigma_{ik}}{\partial x_k} = -\rho \frac{\partial \rho}{\partial x_i}
\]
Next we multiply this equation with \( u_i \). Using that
\[
\frac{\partial \rho u_i}{\partial t} = \frac{\partial u^2}{\partial t} - \rho u_i \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{u^2}{2} \right] + \frac{\partial}{\partial t} \left[ \frac{\rho u^2}{2} \right] - \rho u_i \frac{\partial u_i}{\partial t}
\]
where we have used that \( \frac{\partial u^2}{\partial t} = 2u_i \frac{\partial u_i}{\partial t} \). Similarly, we have that
\[
\frac{\partial}{\partial x_k} \left[ \rho u_i u_k \right] = \frac{\partial}{\partial x_k} \left[ \frac{u^2}{2} u_k \right] + \frac{\partial}{\partial x_k} \left[ \frac{\rho u^2}{2} u_k \right] - \rho u_i \frac{\partial u_i}{\partial x_k}
\]
Combining the above two terms, and using the \textit{continuity equation} to dispose of the two terms containing the factor \( u^2 / 2 \), the Navier-Stokes equation in conservation form multiplied by \( u_i \) becomes
\[
\frac{\partial}{\partial t} \left[ \frac{u^2}{2} \right] + \frac{\partial}{\partial x_k} \left[ \frac{u^2}{2} u_k \right] = u_i \frac{\partial \sigma_{ik}}{\partial x_k} - \rho u_i \frac{\partial \Phi}{\partial x_i}
\]
Subtracting this from the energy equation ultimately yields the \textit{internal energy equation} in Eulerian index form:
\[
\frac{\partial}{\partial t} (\rho \varepsilon) + \frac{\partial}{\partial x_k} (\rho \varepsilon u_k) = -P \frac{\partial u_k}{\partial x_k} + \mathcal{V} - \frac{\partial F_{\text{cond},k}}{\partial x_k} - \mathcal{L}
\]
where
\[
\mathcal{V} \equiv \tau_{ik} \frac{\partial u_i}{\partial x_k}
\]
is the \textit{rate of viscous dissipation} which describes the rate at which the work done against viscous forces is irreversibly converted into internal energy. In words, the internal energy equation states that the internal energy at some fixed location in space
changes due to \textbf{advection}, (described by the $\partial(\rho \varepsilon u_k)/\partial x_k$ term), due to the \textbf{work done} (described by the $P(\partial u_k/\partial x_k)$ term), due to \textbf{radiation} (described by the $-\mathcal{L}$ term), due to \textbf{conduction} (described by the $\partial F_{\text{cond},k}/\partial x_k$ term) and due to \textbf{viscous dissipation} (described by the $\mathcal{V}$ term). The latter term describes the rate at which heat is added to the internal energy budget via viscous conversion of ordered energy in differential fluid motions to disordered energy in random particle motions. Finally, we mention that the Lagrangian vector form of the \textbf{internal energy equation} is given by

$$
\rho \frac{d \varepsilon}{dt} = -P \nabla \cdot \vec{u} - \nabla \cdot \vec{F}_{\text{cond}} - \mathcal{L} + \mathcal{V}
$$

Note that in this Lagrangian form, there is no term describing \textbf{advection}; after all, we are moving \textbf{with} the fluid.

\textbf{Full Set of Equations of Gravitational, Radiative Hydrodynamics}

\begin{align*}
\frac{d \rho}{dt} &= -\rho \nabla \cdot \vec{u} \\
\rho \frac{d \vec{u}}{dt} &= -\nabla P + \mu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right] - \rho \nabla \Phi \\
\rho \frac{d \varepsilon}{dt} &= -P \nabla \cdot \vec{u} - \nabla \cdot \vec{F}_{\text{cond}} - \mathcal{L} + \mathcal{V} \\
\nabla^2 \Phi &= 4\pi G \rho \\
P &= P(\rho, \varepsilon), \quad \mathcal{V} \equiv \tau_{ik} \frac{\partial u_i}{\partial x_k}, \quad F_{\text{cond},k} = \langle \rho \varepsilon w_k \rangle, \quad \mathcal{L} \equiv \mathcal{C} - \mathcal{H}
\end{align*}
CHAPTER 13

Sound Waves

If a (compressible) fluid in equilibrium is perturbed, and the perturbation is sufficiently small, the perturbation will propagate through the fluid as a sound wave, which is a mechanical, longitudinal wave (i.e., a displacement in the same direction as that of propagation). If the perturbation is small, we may assume that the velocity gradients are so small that viscous effects are negligible (i.e., we can set υ = 0). In addition, we assume that the time scale for conductive heat transport is large, so that energy exchange due to conduction can also safely be ignored. In the absence of these dissipative processes, the wave-induced changes in gas properties are adiabatic. If we then also assume that the undisturbed medium is homogeneous, the resulting flow is isentropic (meaning that the specific entropy is conserved).

Let (ρ₀, P₀, ⃗u₀) be a uniform, equilibrium solution of the Euler fluid equations (i.e., ignore viscosity). Also, in what follows we will ignore gravity (i.e., ∇Φ = 0).

Uniformity implies that ∇ρ₀ = ∇P₀ = ∇⃗u₀ = 0. In addition, since the only allowed motion is uniform motion of the entire system, we can always use a Galilean coordinate transformation so that ⃗u₀ = 0, which is what we adopt in what follows.

Substitution into the continuity and momentum equations, one obtains that ∂ρ₀/∂t = ∂⃗u₀/∂t = 0, indicative of an equilibrium solution as claimed.

**Perturbation Analysis:** Consider a small perturbation away from the above equilibrium solution:

\[
\begin{align*}
ρ₀ &\rightarrow ρ₀ + ρ₁ \\
P₀ &\rightarrow P₀ + P₁ \\
⃗u₀ &\rightarrow ⃗u₀ + ⃗u₁ = ⃗u₁
\end{align*}
\]

where |ρ₁/ρ₀| ≪ 1, |P₁/P₀| ≪ 1 and ⃗u₁ is small (compared to the sound speed, to be derived below).
Substitution in the continuity and momentum equations yields

\[
\frac{\partial}{\partial t}(\rho_0 + \rho_1) + \nabla(\rho_0 + \rho_1)\vec{u}_1 = 0
\]

\[
\frac{\partial \vec{u}_1}{\partial t} + \vec{u}_1 \cdot \nabla \vec{u}_1 = -\frac{\nabla(P_0 + P_1)}{(\rho_0 + \rho_1)}
\]

which, using that \(\nabla \rho_0 = \nabla P_0 = \nabla \vec{u}_0 = 0\) reduces to

\[
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \vec{u}_1 + \nabla (\rho_1 \vec{u}_1) = 0
\]

\[
\frac{\partial \vec{u}_1}{\partial t} + \frac{\rho_1}{\rho_0} \frac{\partial \vec{u}_1}{\partial t} + \vec{u}_1 \cdot \nabla \vec{u}_1 + \frac{\rho_1}{\rho_0} \vec{u}_1 \cdot \nabla \vec{u}_1 = -\frac{\nabla P_1}{\rho_0}
\]

The latter follows from first multiplying the momentum equations with \((\rho_0 + \rho_1)/\rho_0\).

Note that we don’t need to consider the energy equation; this is because (i) we have assumed that conduction is negligible, and (ii) the disturbance is adiabatic (meaning \(dQ = 0\), and there is thus no heating or cooling). And since our flow is isentropic, variations in material properties can be related through derivatives taken at constant entropy.

Next we linearize these equations, which means we use that the perturbed values are all small such that terms that contain products of two or more of these quantities are always negligible compared to those that contain only one such quantity. Hence, the above equations reduce to

\[
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \vec{u}_1 = 0
\]

\[
\frac{\partial \vec{u}_1}{\partial t} + \frac{\nabla P_1}{\rho_0} = 0
\]

These equations describe the evolution of perturbations in an inviscid and uniform fluid. As always, these equations need an additional equation for closure. As mentioned above, we don’t need the energy equation: instead, we can use that the flow is isentropic, which implies that \(P \propto \rho^\gamma\) (i.e., the equation-of-state is barotropic).

Using Taylor series expansion, we then have that

\[
P(\rho_0 + \rho_1) = P(\rho_0) + \left(\frac{\partial P}{\partial \rho}\right)_0 \rho_1 + \mathcal{O}(\rho_1^2)
\]
where we have used \((\partial P/\partial \rho)_0\) as shorthand for the partial derivative of \(P(\rho)\) at \(\rho = \rho_0\). And since the flow is isentropic, we have that the partial derivative is for constant entropy. Using that \(P(\rho_0) = P_0\) and \(P(\rho_0 + \rho_1) = P_0 + P_1\), we find that, when linearized,
\[
P_1 = \left(\frac{\partial P}{\partial \rho}\right)_0 \rho_1
\]
Note that \(P_1 \neq P(\rho_1)\); rather \(P_1\) is the perturbation in pressure associated with the perturbation \(\rho_1\) in the density.

Substitution in the fluid equations of our perturbed quantities yields
\[
\begin{align*}
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \mathbf{u}_1 &= 0 \\
\frac{\partial \mathbf{u}_1}{\partial t} + \left(\frac{\partial P}{\partial \rho}\right)_0 \frac{\nabla \rho_1}{\rho_0} &= 0
\end{align*}
\]
Taking the partial time derivative of the above continuity equation, and using that \(\partial \rho_0/\partial t = 0\), gives
\[
\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial \mathbf{u}_1}{\partial t} = 0
\]
Substituting the above momentum equation, and realizing that \((\partial P/\partial \rho)_0\) is a constant, then yields
\[
\frac{\partial^2 \rho_1}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)_0 \nabla^2 \rho_1 = 0
\]
which we recognize as a wave equation, whose solution is a plane wave:
\[
\rho_1 \propto e^{i(k \cdot \mathbf{x} - \omega t)}
\]
with \(\mathbf{k}\) the wavevector, \(k = |\mathbf{k}| = 2\pi/\lambda\) the wavenumber, \(\lambda\) the wavelength, \(\omega = 2\pi \nu\) the angular frequency, and \(\nu\) the frequency.

To gain some insight, consider the 1D case: \(\rho_1 \propto e^{i(kx - \omega t)} \propto e^{i(k(x - v_p t))}\), where we have defined the phase velocity \(v_p \equiv \omega/k\). This is the velocity with which the wave pattern propagates through space. For our perturbation of a compressible fluid, this phase velocity is called the sound speed, \(c_s\). Substituting the solution \(\rho_1 \propto e^{i(kx - \omega t)}\)
into the wave equation, we see that

$$c_s = \frac{\omega}{k} = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}$$

where we have made it explicit that the flow is assumed to be isentropic. Note that the partial derivative is for the unperturbed medium. This sound speed is sometimes called the **adiabatic speed of sound**, to emphasize that it relies on the assumption of an adiabatic perturbation. If the fluid is an ideal gas, then

$$c_s = \sqrt{\frac{k_B T}{\mu m_p}}$$

which shows that the adiabatic sound speed of an ideal fluid increases with temperature.

We can repeat the above derivation by relaxing the assumption of isentropic flow, and assuming instead that (more generally) the flow is **polytropic**. In that case, $P \propto \rho^\Gamma$, with $\Gamma$ the **polytropic index** (Note: a polytropic EoS is an example of a **barotropic** EoS). The only thing that changes is that now the sound speed becomes

$$c_s = \sqrt{\Gamma \frac{P}{\rho}}$$

which shows that the sound speed is larger for a stiffer EoS (i.e., a larger value of $\Gamma$).

Note also that, for our barotropic fluid, the sound speed is independent of $\omega$. This implies that all waves move equally fast; the shape of a wave packet is preserved as it moves. We say that an ideal (inviscid) fluid with a barotropic EoS is a **non-dispersive medium**.

To gain further insight, let us look once more at the (1D) solution for our perturbation:

$$\rho_1 \propto e^{i(kx-\omega t)} \propto e^{ikx} e^{-i\omega t}$$

Recalling **Euler’s formula** ($e^{i\theta} = \cos \theta + i \sin \theta$), we see that:
• The $e^{ikx}$ part describes a periodic, spatial oscillation with wavelength $\lambda = 2\pi/k$.
• The $e^{-i\omega t}$ part describes the time evolution:
  
  – If $\omega$ is real, then the solution describes a sound wave which propagates through space with a sound speed $c_s$.
  
  – If $\omega$ is imaginary then the perturbation is either exponentially growing (‘unstable’) or decaying (‘damped’) with time.

We will return to this in Chapter 15, when we discuss the Jeans stability criterion.
When discussing sound waves (see Chapter 13), we considered small (linear) perturbations. In this Chapter we consider the case in which the perturbations are large (non-linear). Typically, a large disturbance results in an abrupt discontinuity in the fluid, called a shock. Note: not all discontinuities are shocks, but all shocks are discontinuities.

Consider a polytropic EoS:

\[ P \propto \rho^\Gamma = P_0 \left( \frac{\rho}{\rho_0} \right)^\Gamma \]

The sound speed is given by

\[ c_s = \left( \frac{\partial P}{\partial \rho} \right)^{1/2} = \sqrt{\frac{\Gamma P}{\rho}} = c_{s,0} \left( \frac{\rho}{\rho_0} \right)^{(\Gamma-1)/2} \]

If \( \Gamma = 1 \), i.e., the EoS is isothermal, then the sound speed is a constant, independent of density of pressure. However, if \( \Gamma \neq 1 \), then the sound speed varies with the local density. An important example, often encountered in (astro)physics is the adiabatic EoS, for which \( \Gamma = \gamma \) (\( \gamma = 5/3 \) for a mono-atomic gas). In that case we have that \( c_s \) increases with density (and pressure, and temperature).

In our discussion of sound waves (Chapter 13), we used perturbation theory, in which we neglected the \( \tilde{u}_1 \cdot \nabla \tilde{u}_1 \) term. However, when the perturbations are not small, this term is no longer negligible, and causes non-linearities to develop. The most important of those, is the fact that the sound speed itself varies with density (as we have seen above). This implies that the wave-form of the acoustic wave changes with time; the wave-crest is moving faster than the wave-trough, causing an overall steepening of the wave-form. This steepening continues until the wave-crest tries to overtake the wave-trough, which is not allowed, giving rise to a shock front.
**Mach Number:** if $v$ is the flow speed of the fluid, and $c_s$ is the sound speed, then the Mach number of the flow is defined as

\[
M = \frac{v}{c_s}
\]

Note: *simply accelerating a flow to supersonic speeds does not necessarily generate a shock.* Shocks only arise when an obstruction in the flow causes a deceleration of fluid moving at supersonic speeds. The reason is that disturbances cannot propagate upstream, so that the flow cannot ‘adjust itself’ to the obstacle because there is no way of propagating a signal (which always goes at the sound speed) in the upstream direction. Consequently, the flow remains undisturbed until it hits the obstacle, resulting in a discontinuous change in flow properties; a shock.

**Structure of a Shock:** Fig. 15 shows the structure of a planar shock. The shock has a finite, non-zero width (typically a few mean-free paths of the fluid particles), and separates the ‘up-stream’, pre-shocked gas, from the ‘down-stream’, shocked gas.

For reasons that will become clear in what follows, it is useful to split the downstream region in two sub-regions; one in which the fluid is out of thermal equilibrium, with net cooling $\mathcal{L} > 0$, and, further away from the shock, a region where the downstream gas is (once again) in thermal equilibrium (i.e., $\mathcal{L} = 0$). If the transition between these two sub-regions falls well outside the shock (i.e., if $x_3 \gg x_2$) the shock is said to be adiabatic. In that case, we can derive a relation between the upstream (pre-shocked) properties $(\rho_1, P_1, T_1, u_1)$ and the downstream (post-shocked) properties $(\rho_2, P_2, T_2, u_2)$; these relations are called the **Rankine-Hugoniot jump conditions.** Linking the properties in region three $(\rho_3, P_3, T_3, u_3)$ to those in the pre-shocked gas is in general not possible, except in the case where $T_3 = T_1$. In this case one may consider the shock to be isothermal.

**Rankine-Hugoniot jump conditions:** We now derive the relations between the up- and down-stream quantities, under the assumption that the shock is adiabatic. Consider a rectangular volume $V$ that encloses part of the shock; it has a thickness $dx > (x_2 - x_1)$ and is centered in the $x$-direction on the middle of shock. At fixed $x$ the volume is bounded by an area $A$. If we ignore variations in $\rho$ and $\vec{u}$ in the $y$-
and z-directions, the **continuity equation** becomes

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u_x) = 0
\]

If we integrate this equation over our volume \( V \) we obtain

\[
\int \int \int \frac{\partial \rho}{\partial t} \, dx \, dy \, dz + \int \int \int \frac{\partial}{\partial x} (\rho u_x) \, dx \, dy \, dz = 0
\]

\[
\Leftrightarrow \frac{\partial}{\partial t} \int \rho \, dx \, dy \, dz + A \int \frac{\partial}{\partial x} (\rho u_x) \, dx = 0
\]

\[
\Leftrightarrow \frac{\partial M}{\partial t} + \int d(\rho u_x) = 0
\]

Since there is no mass accumulation in the shock, and mass does not disappear in the shock, we have that

\[
\rho u_x |_{+dx/2} = \rho u_x |_{-dx/2}
\]

In terms of the upstream (index 1) and downstream (index 2) quantities:

\[
\rho_1 u_1 = \rho_2 u_2
\]

This equation describes **mass conservation** across a shock.

The momentum equation in the \( x \)-direction, ignoring viscosity, is given by
\[
\frac{\partial}{\partial t}(\rho u_x) = - \frac{\partial}{\partial x}(\rho u_x u_x + P) - \rho \frac{\partial \Phi}{\partial x}
\]

Integrating this equation over \(V\) and ignoring any gradient in \(\Phi\) across the shock, we obtain

\[
\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2
\]

This equation describes how the shock converts ram pressure into thermal pressure.

Finally, applying the same to the energy equation under the assumption that the shock is adiabatic (i.e., \(dQ/dt = 0\)), one finds that \((E + P)u\) has to be the same on both sides of the shock, i.e.,

\[
\left[ \frac{1}{2} u^2 + \frac{\Phi}{\rho} + \frac{P}{\rho} \right] \rho u = \text{constant}
\]

We have already seen that \(\rho u\) is constant. Hence, if we once more ignore gradients in \(\Phi\) across the shock, we obtain that

\[
\frac{1}{2} u_1^2 + \frac{\varepsilon_1}{\rho_1} + \frac{P_1}{\rho_1} = \frac{1}{2} u_2^2 + \frac{\varepsilon_2}{\rho_2} + \frac{P_2}{\rho_2}
\]

This equation describes how the shock converts kinetic energy into enthalpy. Qualitatively, a shock converts an ordered flow upstream into a disordered (hot) flow downstream.

The three equations in the rectangular boxes are known as the Rankine-Hugoniot (RH) jump conditions for an adiabatic shock. Using straightforward but tedious algebra, these RH jump conditions can be written in a more useful form using the Mach number \(M_1\) of the upstream gas:

\[
\begin{align*}
\frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \left[ \frac{1}{M_1^2} + \frac{\gamma - 1}{\gamma + 1} \left( 1 - \frac{1}{M_1^2} \right) \right]^{-1} \\
P_2 &= \frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1} \\
T_2 &= \frac{P_2 \rho_2}{P_1 \rho_1} = \frac{\gamma - 1}{\gamma + 1} \left[ \frac{2}{\gamma + 1} \left( \gamma M_1^2 - \frac{1}{M_1^2} \right) + \frac{4\gamma}{\gamma - 1} - \frac{\gamma - 1}{\gamma + 1} \right]
\end{align*}
\]
Here we have used that for an ideal gas

\[ P = (\gamma - 1) \rho \varepsilon = \frac{k_B T}{\mu m_p} \rho \]

Given that \( \mathcal{M}_1 > 1 \), we see that \( \rho_2 > \rho_1 \) (shocks compress), \( u_2 < u_1 \) (shocks decelerate), \( P_2 > P_1 \) (shocks increase pressure), and \( T_2 > T_1 \) (shocks heat). The latter may seem surprising, given that the shock is considered to be adiabatic: although the process has been adiabatic, in that \( dQ/dt = 0 \), the gas has changed its adiabat; its entropy has increased as a consequence of the shock converting kinetic energy into thermal, internal energy. In general, in the presence of viscosity, a change that is adiabatic does not imply that the states before and after are simply linked by the relation \( P = K \rho^\gamma \), with \( K \) some constant. Shocks are always viscous, which causes \( K \) to change across the shock, such that the entropy increases; it is this aspect of the shock that causes irreversibility, thus defining an “arrow of time”.

Back to the RH jump conditions: in the limit \( \mathcal{M}_1 \gg 1 \) we have that

\[ \rho_2 = \frac{\gamma + 1}{\gamma - 1} \rho_1 = 4 \rho_1 \]

where we have used that \( \gamma = 5/3 \) for a monoatomic gas. Thus, with an adiabatic shock you can achieve a maximum compression in density of a factor four! Physically, the reason why there is a maximal compression is that the pressure and temperature of the downstream fluid diverge as \( \mathcal{M}_1^2 \). This huge increase in downstream pressure inhibits the amount of compression of the downstream gas. However, this is only true under the assumption that the shock is adiabatic. The downstream, post-shocked gas is out of thermal equilibrium, and in general will be cooling (i.e., \( \mathcal{L} > 0 \)). At a certain distance past the shock (i.e., when \( x = x_3 \) in Fig. 15), the fluid will re-establish thermal equilibrium (i.e., \( \mathcal{L} = 0 \)). In some special cases, one can obtain the properties of the fluid in the new equilibrium state; one such case is the example of an isothermal shock, for which the downstream gas has the same temperature as the upstream gas (i.e., \( T_3 = T_1 \)).

In the case of an isothermal shock, the first two Rankine-Hugoniot jump con-
ditions are still valid, i.e.,
\[
\begin{align*}
\rho_1 u_1 &= \rho_3 u_3 \\
\rho_1 u_1^2 + P_1 &= \rho_3 u_3^2 + P_3
\end{align*}
\]

However, the third condition, which derives from the energy equation, is no longer valid. After all, in deriving that one we had assumed that the shock was adiabatic. In the case of an isothermal shock we have to replace the third RH jump condition with \( T_1 = T_3 \). The latter implies that \( c_s^2 = P_3/\rho_3 = P_1/\rho_1 \), and allows us to rewrite the second RH condition as

\[
\begin{align*}
\rho_1 (u_1^2 + c_s^2) &= \rho_3 (u_3^2 + c_s^2) \\
\iff u_1^2 - \frac{c_s^2}{\rho_1} &= \frac{c_s^2}{\rho_1} - c_s^2 \\
\iff u_1^2 - u_1 u_3 &= (\frac{u_1}{u_3} - 1) c_s^2 \\
\iff u_1 u_3 (u_1 - u_3) &= (u_1 - u_3) c_s^2 \\
\iff c_s^2 &= u_1 u_3
\end{align*}
\]

Here the second step follows from using the first RH jump condition. If we now substitute this result back into the first RH jump condition we obtain that

\[
\frac{\rho_2}{\rho_1} = \frac{u_1}{u_3} = \left( \frac{u_1}{c_s} \right)^2 = \mathcal{M}_1^2
\]

Hence, in the case of isothermal shock (or an adiabatic shock, but sufficiently far behind the shock in the downstream fluid), we have that there is no restriction to how much compression the shock can achieve; depending on the Mach number of the shock, the compression can be huge.
Supernova Blastwave: An important example of a shock in astrophysics are supernova blastwaves. When a supernova explodes, it blasts a shell of matter (the ‘ejecta’) at high (highly supersonic) speed into the surrounding medium. The kinetic energy of this shell material is roughly $E_{\text{SN}} = 10^{51}$ erg. This is roughly 100 times larger than the amount of energy emitted in radiation by the supernova explosion (which is what we ‘see’). For comparison, the entire Milky Way has a luminosity of $\sim 10^{10} L_\odot \simeq 4 \times 10^{43}$ erg s$^{-1}$, which amounts to an energy emitted by stars over an entire year that is of the order of $1.5 \times 10^{51}$ erg. Hence, the kinetic energy released by a single SN is larger than the energy radiated by stars, by the entire galaxy, in an entire year!

The mass of the ejecta is of the order of 1 Solar mass, which implies (using that $E_{\text{SN}} = \frac{1}{2} M_{\odot} v_{\odot}^2$), that the ejecta have a velocity of $\sim 10,000$ km s$^{-1}$!! Initially, this shell material has a mass that is much larger than the mass of the surroundings swept up by the shock, and to lowest order the shell undergoes free expansion. This phase is therefore called the **free-expansion phase**. As the shock moves out, it engulfs more and more interstellar material, which is heated (and compressed) by the shock. Hence, the interior of the shell (=shock) is a super-hot bubble of over-pressurized...
gas, which ‘pushes’ the shock outwards. As more and more material is swept-up, and accelerated outwards, the mass of the shell increases, which causes the velocity of the shell to decelerate. At the early stages, the cooling of the hot bubble is negligible, and the blastwave is said to be in the \textbf{adiabatic phase}, also known as the \textbf{Sedov-Taylor phase}. At some point, though, the hot bubble starts to cool, radiating awaiting the kinetic energy of the supernova, and lowering the interior pressure up to the point that it no longer pushes the shell outwards. This is called the \textbf{radiative phase}. From this point on, the shell expands purely by its inertia, being slowed down by the work it does against the surrounding material. This phase is called the \textbf{snow-plow phase}. Ultimately, the velocity of the shell becomes comparable to the sound speed of the surrounding material, after which it continues to move outward as a sound wave, slowly dissipating into the surroundings.

During the adiabatic phase, we can use a simple dimensional analysis to solve for the evolution of the shock radius, $r_{sh}$, with time. Since the only physical parameters that can determine $r_{sh}$ in this adiabatic phase are time, $t$, the initial energy of the SN explosion, $\varepsilon_0$, and the density of the surrounding medium, $\rho_0$, we have that

$$r_{sh} = f(t, \varepsilon_0, \rho_0) = A t^{\eta} \varepsilon_0^{\alpha} \rho_0^{\beta}$$

It is easy to check that there is only one set of values for $\eta$, $\alpha$ and $\beta$ for which the product on the right has the dimensions of length (which is the dimension of $r_{sh}$). This solution has $\eta = 2/5$, $\alpha = 1/5$ and $\beta = -1/5$, such that

$$r_{sh} = A \left( \frac{\varepsilon}{\rho_0} \right)^{1/5} t^{2/5}$$

and thus

$$v_{sh} = \frac{dr_{sh}}{dt} = \frac{2A}{5} \left( \frac{\varepsilon}{\rho_0} \right)^{1/5} t^{-3/5}$$

which shows that indeed the shock decelerates as it moves outwards.
In this Chapter we discuss the following instabilities:

- convective instability (Schwarzschild criterion)
- interface instabilities (Rayleigh-Taylor & Kelvin-Helmholtz)
- gravitational instability (Jeans criterion)
- thermal instability (Field criterion)

**Convective Instability:** In astrophysics we often need to consider fluids heated from "below" (e.g., stars, Earth’s atmosphere where Sun heats surface, etc.). This results in a temperature gradient: hot at the base, colder further "up". Since warmer fluids are more buoyant ('lighter'), they like to be further up than colder ('heavier') fluids. The question we need to address is under what conditions this adverse temperature gradient becomes unstable, developing "overturning" motions known as thermal convection.

Consider a blob with density \( \rho_b \) and pressure \( P_b \) embedded in an ambient medium of density \( \rho \) and pressure \( P \). Suppose the blob is displaced by a small distance \( \delta z \) upward. After the displacement the blob will have conditions \( (\rho^*_b, P^*_b) \) and its new ambient medium is characterized by \( (\rho', P') \), where

\[
\rho' = \rho + \frac{d\rho}{dz} \delta z \quad \quad P' = P + \frac{dP}{dz} \delta z
\]

Initially the blob is assumed to be in mechanical and thermal equilibrium with its ambient medium, so that \( \rho_b = \rho \) and \( P_b = P \). After the displacement the blob needs to re-establish a new mechanical and thermal equilibrium. In general, the time scale on which it re-establishes mechanical (pressure) equilibrium is the sound crossing.

\[\text{Here and in what follows, ‘up’ refers to the direction opposite to that of gravity.}\]
time, $\tau_s$, while re-establishing thermal equilibrium proceeds much slower, on the conduction time, $\tau_c$. Given that $\tau_s \ll \tau_c$ we can assume that $P_b^* = P'$, and treat the displacement as adiabatic. The latter implies that the process can be described by an adiabatic EoS: $P \propto \rho^{\gamma}$. Hence, we have that

$$
\rho_b^* = \rho_b \left( \frac{P_b^*}{P_b} \right)^{1/\gamma} = \rho_b \left( \frac{P'}{P} \right)^{1/\gamma} = \rho_b \left[ 1 + \frac{1}{\gamma} \frac{dP}{P} \frac{dz}{\delta z} \right]^{1/\gamma}
$$

In the limit of small displacements $\delta z$, we can use Taylor series expansion to show that, to first order,

$$
\rho_b^* = \rho + \frac{\rho}{\gamma P} \frac{dP}{dz} \delta z
$$

where we have used that initially $\rho_b = \rho$, and that the Taylor series expansion, $f(x) \simeq f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + ...$, of $f(x) = [1+x]^{1/\gamma}$ is given by $f(x) \simeq 1 + \frac{1}{\gamma} x + ....$

Suppose we have a stratified medium in which $d\rho/dz < 0$ and $dP/dz < 0$. In that case, if $\rho_b^* > \rho'$ the blob will be heavier than its surrounding and it will sink back to its original position; the system is stable to convection. If, on the other hand, $\rho_b^* < \rho'$ then the displacement has made the blob more buoyant, resulting in instability. Hence, using that $\rho' = \rho + (d\rho/dz) \delta z$ we see that stability requires that

$$
\frac{d\rho}{dz} < \frac{\rho}{\gamma P} \frac{dP}{dz}
$$

This is called the **Schwarzschild criterion for convective stability**.

It is often convenient to rewrite this criterion in a form that contains the temperature. Using that

$$
\rho = \rho(P, T) = \frac{\mu m_p}{k_B T} P
$$

it is straightforward to show that

$$
\frac{d\rho}{dz} = \frac{\rho}{P} \frac{dP}{dz} - \frac{\rho}{T} \frac{dT}{dz}
$$

Substitution in $\rho' = \rho + (d\rho/dz) \delta z$ then yields that

$$
\rho_b^* - \rho' = \left[ -\frac{1}{\gamma} \frac{\rho}{P} \frac{dP}{dz} + \frac{\rho}{T} \frac{dT}{dz} \right] \delta z
$$
Since stability requires that $\rho_* - \rho' > 0$, and using that $\delta z > 0$, $dP/dz < 0$ and $dT/dz < 0$ we can rewrite the above Schwarzschild criterion for stability as

$$\left| \frac{dT}{dz} \right| < \left( 1 - \frac{1}{\gamma} \right) \frac{T}{P} \left| \frac{dP}{dz} \right|$$

This shows that if the temperature gradient becomes too large the system becomes convectively unstable: blobs will rise up until they start to loose their thermal energy to the ambient medium, resulting in convective energy transport that tries to “overturn” the hot (high entropy) and cold (low entropy) material. In fact, without any proof we mention that in terms of the specific entropy, $s$, one can also write the Schwarzschild criterion for convective stability as $ds/dz > 0$.

To summarize, the Schwarzschild criterion for convective stability is given by either of the following three expressions:

$$\left| \frac{dT}{dz} \right| < \left( 1 - \frac{1}{\gamma} \right) \frac{T}{P} \left| \frac{dP}{dz} \right|$$

$$\frac{d\rho}{dz} < \frac{\rho}{\gamma P} \frac{dP}{dz}$$

$$\frac{ds}{dz} > 0$$

**Rayleigh-Taylor Instability:** The Rayleigh-Taylor (RT) instability is an instability of an interface between two fluids of different densities that occurs when one of the fluids is accelerated into the other. Examples include supernova explosions in which expanding core gas is accelerated into denser shell gas and the common terrestrial example of a denser fluid such as water suspended above a lighter fluid such as oil in the Earth’s gravitational field.

It is easy to see where the RT instability comes from. Consider a fluid of density $\rho_2$ sitting on top of a fluid of density $\rho_1 < \rho_2$ in a gravitational field that is pointing in the downward direction. Consider a small perturbation in which the initially horizontal interface takes on a small amplitude, sinusoidal deformation. Since this
implies moving a certain volume of denser material down, and an equally large volume of the lighter material up, it is immediately clear that the potential energy of this ‘perturbed’ configuration is lower than that of the initial state, and therefore energetically favorable. Simply put, the initial configuration is unstable to small deformations of the interface.

Stability analysis (i.e., perturbation analysis of the fluid equations) shows that the dispersion relation corresponding to the RT instability is given by

$$\omega = \pm i k \sqrt{\frac{g}{k} \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

where $g$ is the gravitational acceleration, and the factor $(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ is called the Atwood number. Since the wavenumber of the perturbation $k > 0$ we see that $\omega$ is imaginary, which implies that the perturbations will grow exponentially (i.e., the system is unstable). If $\rho_1 > \rho_2$ though, $\omega$ is real, and the system is stable (perturbations to the interface propagate as waves).

**Kelvin-Helmholtz Instability:** the Kelvin-Helmholtz (KH) instability is an interface instability that arises when two fluids with different densities have a velocity difference across their interface. Similar to the RT instability, the KH instability manifests itself as a small wavy pattern in the interface which develops into turbulence and which causes mixing. Examples where KH instability plays a role are wind blowing over water, (astrophysical) jets, the cloud bands on Jupiter (in particular the famous red spot), and clouds of denser gas falling through the hot, low density intra-cluster medium (ICM).
Stability analysis (i.e., perturbation analysis of the fluid equations) shows that the dispersion relation corresponding to the KH instability is given by

$$\frac{\omega}{k} = \frac{(\rho_1 u_1 + \rho_2 u_2) \pm i (u_1 - u_2) (\rho_1 \rho_2)^{1/2}}{\rho_1 + \rho_2}$$

Note that this dispersion relation has both real and imaginary parts, given by

$$\omega_R \equiv \frac{(\rho_1 u_1 + \rho_2 u_2)}{\rho_1 + \rho_2}$$

and

$$\omega_I \equiv \frac{(u_1 - u_2) (\rho_1 \rho_2)^{1/2}}{\rho_1 + \rho_2}$$

Since the imaginary part is non-zero, except for \(u_1 = u_2\), we have that, in principle, any velocity difference across an interface is KH unstable. In practice, surface tension can stabilize the short wavelength modes so that typically KH instability kicks in above some velocity threshold.

As an example, consider a cold cloud of radius \(R_c\) falling into a cluster of galaxies. The latter contains a hot intra-cluster medium (ICM), and as the cloud moves through this hot ICM, KH instabilities can develop on its surface. If the cloud started out at a large distance from the cluster with zero velocity, then at infall it has a velocity \(v \sim v_{\text{esc}} \sim c_{s,h}\), where the latter is the sound speed of the hot ICM, assumed to be in hydrostatic equilibrium. Defining the cloud’s overdensity \(\delta = \rho_c/\rho_h - 1\), we can write the (imaginary part of the) dispersion relation as

$$\omega = \frac{\rho_h (\rho_c/\rho_h)^{1/2}}{\rho_h [1 + (\rho_c/\rho_h)]} c_{s,h} k = \frac{(\delta + 1)^{1/2}}{\delta + 2} c_{s,h} k$$
The mode that will destroy the cloud has \( k \sim 1/R_c \), so that the time-scale for cloud destruction is

\[
\tau_{KH} \simeq \frac{1}{\omega} \simeq \frac{R_c}{c_{s,h}} \frac{\delta + 2}{(\delta + 1)^{1/2}}
\]

Assuming pressure equilibrium between cloud and ICM, and adopting the EoS of an ideal gas, implies that \( \rho_h T_h = \rho_c T_c \), so that

\[
\frac{c_{s,h}}{c_{s,c}} \frac{T_h^{1/2}}{T_c^{1/2}} = \frac{\rho_c^{1/2}}{\rho_h^{1/2}} = (\delta + 1)^{1/2}
\]

Hence, one finds that the Kelvin-Helmholtz time for cloud destruction is

\[
\tau_{KH} \simeq \frac{1}{\omega} \simeq \frac{R_c}{c_{s,c}} \frac{\delta + 2}{\delta + 1}
\]

Note that \( \tau_{KH} \sim \zeta (R_c/c_{s,c}) = \zeta \tau_s \), with \( \zeta = 1(2) \) for \( \delta \gg 1(\ll 1) \). Hence, the Kelvin-Helmholtz instability will typically destroy clouds falling into a hot ”atmosphere” on a time scale between one and two sound crossing times, \( \tau_s \), of the cloud. Note, though, that magnetic fields and/or radiative cooling at the interface may stabilize the clouds.

**Gravitational Instability:** In our discussion of sound waves we used perturbation analysis to derive a dispersion relation \( \omega^2 = k^2 c_s^2 \). In deriving that equation we ignored gravity by setting \( \nabla \Phi = 0 \) (see Chapter 13). If you do not ignore gravity, then you add one more perturbed quantity; \( \Phi = \Phi_0 + \Phi_1 \) and one more equation, namely the Poisson equation \( \nabla^2 \Phi = 4\pi G\rho \).

It is not difficult to show that this results in a modified dispersion relation:

\[
\omega^2 = k^2 c_s^2 - 4\pi G\rho_0 = c_s^2 \left( k^2 - k_J^2 \right)
\]

where we have introduced the Jeans wavenumber

\[
k_J = \frac{\sqrt{4\pi G\rho_0}}{c_s}
\]

to which we can also associate a Jeans length

\[
\lambda_J \equiv \frac{2\pi}{k_J} = \sqrt{\frac{\pi}{G\rho_0}} c_s
\]
and a **Jeans mass**

\[ M_J = \frac{4}{3} \pi \rho_0 \left( \frac{\lambda_J}{2} \right)^3 = \frac{7}{6} \rho_0 \lambda_J^3 \]

From the dispersion relation one immediately sees that the system is **unstable** (i.e., \( \omega \) is imaginary) if \( k < k_J \) (or, equivalently, \( \lambda > \lambda_J \) or \( M > M_J \)). This is called the **Jeans criterion for gravitational instability**. It expresses when pressure forces (which try to disperse matter) are no longer able to overcome gravity (which tries to make matter collapse), resulting in exponential gravitational collapse on a time scale

\[ \tau_{\text{ff}} = \sqrt{\frac{3 \pi}{32 G \rho}} \]

known as the **free-fall time** for gravitational collapse.

The Jeans stability criterion is of utmost importance in astrophysics. It is used to describes the formation of galaxies and large scale structure in an expanding space-time (in this case the growth-rate is not exponential, but only power-law), to describe the formation of stars in molecular clouds within galaxies, and it may even play an important role in the formation of planets in protoplanetary disks.

In deriving the Jeans Stability criterion you will encounter a somewhat puzzling issue. Consider the **Poisson equation** for the unperturbed medium (which has density \( \rho_0 \) and gravitational potential \( \Phi_0 \)):

\[ \nabla^2 \Phi_0 = 4\pi G \rho_0 \]

Since the initial, unperturbed medium is supposed to be homogeneous there can be no gravitational force; hence \( \nabla \Phi_0 = 0 \) everywhere. The above Poisson equation then implies that \( \rho_0 = 0 \). In other words, an unperturbed, homogeneous density field of non-zero density does not seem to exist. Sir James Jeans ‘ignored’ this ‘nuisance’ in his derivation, which has since become known as the **Jeans swindle**. The problem arises because Newtonian physics is not equipped to deal with systems of infinite extent (a requirement for a perfectly homogeneous density distribution).

See Kiessling (1999; arXiv:9910247) for a detailed discussion, including an elegant demonstration that the Jeans swindle is actually vindicated!
Thermal Instability: Let $\mathcal{L} = \mathcal{L}(\rho, T) = \mathcal{C} - \mathcal{H}$ be the net cooling rate. If $\mathcal{L} = 0$ the system is said to be in thermal equilibrium (TE), while $\mathcal{L} > 0$ and $\mathcal{L} < 0$ correspond to cooling and heating, respectively.

The condition $\mathcal{L}(\rho, T) = 0$ corresponds to a curve in the $(\rho, T)$-plane with a shape similar to that shown in Fig. 11. It has flat parts at $T \sim 10^6K$, at $T \sim 10^4K$, at $T \sim 10 - 100K$. This can be understood from simple atomic physics, and will be discussed in some detail in the lectures on radiative processes (see § 8.5.1 of Mo, van den Bosch & White, 2010 for a detailed discussion). Above the TE curve we have that $\mathcal{L} > 0$ (net cooling), while below it $\mathcal{L} < 0$ (net heating). The dotted curve indicates a line of constant pressure ($T \propto \rho^{-1}$). Consider a blob in thermal and mechanical (pressure) equilibrium with its ambient medium, and with a pressure indicated by the dashed line. There are five possible solutions for the density and temperature of the blob, two of which are indicated by $P_1$ and $P_2$; here confusingly the $P$ refers to ‘point’ rather than ‘pressure’. Suppose I have a blob located at point $P_2$. If I heat the blob, displacing it from TE along the constant pressure curve (i.e., the blob is assumed small enough that the sound crossing time, on which the blob re-established mechanical equilibrium, is short). The blob now finds itself in the region where $\mathcal{L} > 0$ (i.e, net cooling), so that it will cool back to its original location on the TE-curve; the blob is stable. For similar reasons, it is easy to see that a blob located at point $P_1$ is unstable. This instability is called thermal instability, and it explains why the ISM is a three-phase medium, with gas of three different temperatures ($T \sim 10^6K, 10^4K$, and $\sim 10 - 100K$) coexisting in pressure equilibrium. Gas at any other temperature but in pressure equilibrium is thermally unstable.
It is easy to see that the requirement for thermal instability translates into

\[
\left( \frac{\partial L}{\partial T} \right)_P < 0
\]

which is known as the Field criterion for thermal instability (after American astrophysicist George B. Field).

**Fragmentation and Shattering:** Consider the Jeans criterion, expressing a balance between gravity and pressure. Using that the Jeans mass \( M_J \propto \rho \lambda_J^3 \) and that \( \lambda_J \propto \rho^{-1/2} c_s \), we see that

\[
M_J \propto \rho^{-1/2} T^{3/2}
\]

where we have used that \( c_s \propto T^{1/2} \). Now consider a polytropic equation of state, which has \( P \propto \rho^\Gamma \), with \( \Gamma \) the polytropic index. Assuming an ideal gas, such that

\[
P = \frac{k_B T}{\mu m_p} \rho
\]

we thus see that a polytropic ideal gas must have that \( T \propto \rho^{\Gamma^{-1}} \). Substituting that in the expression for the Jeans mass, we obtain that

\[
M_J \propto \rho^{2\Gamma^{-2} - 2} = \rho^{2(\Gamma-\frac{4}{3})}
\]

Thus, we see that for \( \Gamma > 4/3 \) the Jeans mass will increase with increasing density, while the opposite is true for \( \Gamma < 4/3 \). Now consider a system that is (initially) larger than the Jeans mass. Since pressure can no longer support it against its own gravity, the system will start to collapse, which increases the density. If \( \Gamma < 4/3 \), the Jeans mass will becomes smaller as a consequence of the collapse, and now small subregions of the system will find themselves having a mass larger than the Jeans mass \( \Rightarrow \) the system will start to fragment.

If the collapse is adiabatic (i.e., we can ignore cooling), then \( \Gamma = \gamma = 5/3 > 4/3 \) and there will be no fragmentation. However, if cooling is very efficient, such that while the cloud collapses it maintains the same temperature, the EoS is now isothermal, which implies that \( \Gamma = 1 < 4/3 \): the cloud will fragment into smaller collapsing clouds. Fragmentation is believed to underly the formation of star clusters.
A very similar process operates related to the thermal instability. In the discussion of the Field criterion we had made the assumption "the blob is assumed small enough that the sound crossing time, on which the blob re-established mechanical equilibrium, is short". Here 'short' means compared to the cooling time of the cloud. Let’s define the cooling length \( l_{\text{cool}} \equiv c_s \tau_{\text{cool}} \), where \( c_s \) is the cloud’s sound speed and \( \tau_{\text{cool}} \) is the cooling time (the time scale on which it radiates away most of its internal energy). The above assumption thus implies that the size of the cloud, \( l_{\text{cloud}} \ll l_{\text{cool}} \). As a consequence, whenever the cloud cools somewhat, it can immediately re-establish pressure equilibrium with its surrounding (i.e., the sound crossing time, \( \tau_s = l_{\text{cloud}} / c_s \) is much smaller than the cooling time \( \tau_{\text{cool}} = l_{\text{cool}} / c_s \)).

Now consider a case in which \( l_{\text{cloud}} \gg l_{\text{cool}} \) (i.e., \( \tau_{\text{cool}} \ll \tau_s \)). As the cloud cools, it cannot maintain pressure equilibrium with its surroundings; it takes too long for mechanical equilibrium to be established over the entire cloud. What happens is that smaller subregions, of order the size \( l_{\text{cool}} \), will fragment. The smaller fragments will be able to maintain pressure equilibrium with their surroundings. But as the small cloudlets cool further, the cooling length \( l_{\text{cool}} \) shrinks. To see this, realize that when \( T \) drops this lowers the sound speed and decreases the cooling time; after all, we are in the regime of thermal instability, so \( (\partial L / \partial T)_{P} < 0 \). As a consequence, \( l_{\text{cool}} = c_s \tau_{\text{cool}} \) drops as well. So the small cloudlett soon finds itself larger than the cooling length, and it in turn will fragment. This process of shattering continues until the cooling time becomes sufficiently long and the cloudlets are no longer thermally unstable.

This process of shattering is believed to play an important role in the inter-galactic medium (IGM) in between galaxies, and the circum-galactic medium (CGM) in the halos of galaxies.
Hydrostatic Equilibrium: A fluid is said to be in hydrostatic equilibrium (HE) when it is at rest. This occurs when external forces such as gravity are balanced by the forces that arise due to pressure gradients.

Starting from the Navier-Stokes equation (see Chapter 7)

\[ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi + \nu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right] \]

we see that \( \vec{u} = 0 \) implies that

\[ \nabla P = -\rho \nabla \Phi \]

which is the equation of hydrostatic equilibrium.

Stellar Structure: stars are gaseous spheres in hydrostatic equilibrium (except for radial pulsations, which may be considered perturbations away from HE). The structure of stars is therefore largely governed by the above equation. If we assume spherical symmetry (a fairly accurate assumption in the absence of rotation), we have that \( \nabla P = \frac{dP}{dr} \) and \( \nabla \Phi = \frac{d\Phi}{dr} = \frac{GM(r)}{r^2} \), where \( M(r) \) is the mass enclosed within radius \( r \). The equation of HE therefore can be written as

\[ \frac{dP}{dr} = -\frac{GM(r) \rho(r)}{r^2} \]

In addition, we also have that

\[ \frac{dM}{dr} = 4\pi \rho(r) r^2 \]
These are two differential equations with three unknowns; $P$, $\rho$ and $M$. If the equation of state is barotropic, i.e., $P = P(\rho)$, then our set of equations is closed, and the density profile of the star can be solved for (given proper boundary conditions).

However, in general the equation of state is of the form $P = P(\rho, T, \{X_i\})$, where $\{X_i\}$ is the set of the abundances of all elements $i$. The temperature structure of a star and its abundance ratios are governed by nuclear physics (which provides the source of energy) and the various heat transport mechanisms.

**Polytropic Spheres:** A barotropic equation of state of the form $P \propto \rho^\Gamma$ is called a polytropic equation of state, and $\Gamma$ is called the polytropic index. Note that $\Gamma = 1$ and $\Gamma = \gamma$ for an isothermal and adiabatic equations of state, respectively. A spherically symmetric, polytropic fluid in HE is called a polytropic sphere.

**Lane-Emden equation:** Upon substituting the polytropic EoS in the equation of hydrostatic equilibrium and using the Poisson equation, one obtains a single differential equation that completely describes the structure of the polytropic sphere, known as the Lane-Emden equation:

$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$

Here $n = 1/(\Gamma - 1)$ is related to the polytropic index (in fact, confusingly, some texts refer to $n$ as the polytropic index),

$\xi = \left( \frac{4\pi G \rho_c}{\Phi_0 - \Phi_c} \right)^{1/2} r$

is a dimensionless radius,

$\theta = \left( \frac{\Phi_0 - \Phi(r)}{\Phi_0 - \Phi_c} \right)$

with $\Phi_c$ and $\Phi_0$ the values of the gravitational potential at the center ($r = 0$) and at the surface of the star (where $\rho = 0$), respectively. The density is related to $\theta$ according to $\rho = \rho_c \theta^n$ with $\rho_c$ the central density.

Solutions to the Lane-Emden equation are called polytropes of index $n$. In general, the Lane-Emden equation has to be solved numerically subject to the boundary
conditions $\theta = 1$ and $d\theta/d\xi = 0$ at $\xi = 0$. Analytical solutions exist, however, for $n = 0, 1, \text{ and } 5$. Examples of polytropes are stars that are supported by degeneracy pressure. For example, a non-relativistic, degenerate equation of state has $P \propto \rho^{5/3}$ and is therefore describes by a polytrope of index $n = 3/2$. In the relativistic case $P \propto \rho^{4/3}$ which results in a polytrope of index $n = 3$.

Another polytrope that is often encountered in astrophysics is the isothermal sphere, which has $P \propto \rho$ and thus $n = \infty$. It has $\rho \propto r^{-2}$ at large radii, which implies an infinite total mass. If one truncates the isothermal sphere at some radius and embeds it in a medium with external pressure (to prevent the sphere from expanding), it is called a Bonnor-Ebert sphere, which is a structure that is frequently used to describe molecular clouds.

**Heat transport in stars:** Typically, ignoring abundance gradients, stars have an equation of state $P = P(\rho, T)$. The equations of stellar structure, equations (2) and (3), are therefore complemented by a third equation

$$\frac{dT}{dr} = F(r)$$

Since $T$ is a measure of the internal energy, the rhs of this equation describes the heat flux, $F(r)$.

The main heat transport mechanisms in a star are:

- conduction
- convection
- radiation

Note that the fourth heat transport mechanism, advection, is not present in the case of hydrostatic equilibrium, because $\vec{u} = 0$.

Recall that the thermal conductivity $K \propto l v_{th}$, where $l$ is the mean free path of the fluid particles, and $v_{th}$ is the thermal (microscopic) velocity (see Chapter 12). Since radiative heat transport in a star is basically the conduction of photons, and since $c \gg v_e$ and $l_{\text{photon}} \gg l_e$ (where ‘e’ refers to electrons), we have that
in stars radiation is a far more efficient heat transport mechanism than conduction. One therefore is allowed to ignore conduction when describing stellar structure. An exception are relativistic, degenerate cores, for which $v_e \sim c$ and $l_e \sim l_{\text{photon}}$.

**Convection:** convection only occurs if the **Schwarzschild Stability Criterion** is violated, which happens when the temperature gradient $dT/dr$ becomes too large (i.e., larger than the temperature gradient that would exist if the star was adiabatic; see Chapter 15). If that is the case, convection always dominates over radiation as the most efficient heat transport mechanism. In general, more massive stars are more radiative and less convective.

**Trivia:** On average it takes $\sim 200,000$ years for a photon created at the core of the Sun in nuclear burning to make its way to the Sun’s photosphere; from there it only takes $\sim 8$ minutes to travel to the Earth.

**Hydrostatic Mass Estimates:** Consider once more the case of an ideal gas with EoS

$$P = \frac{k_B T}{\mu m_p \rho}$$

We have that

$$\frac{dP}{dr} = \frac{\partial P \, d\rho}{\partial \rho \, dr} + \frac{\partial P \, dT}{\partial T \, dr} = \frac{P \, d\rho}{\rho \, dr} + \frac{P \, dT}{T \, dr} = \frac{P}{r} \left[ r \frac{d\rho}{\rho \, dr} + r \frac{dT}{T \, dr} \right] = \frac{P}{r} \left[ \frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r} \right]$$

Substitution of this equation in the equation for Hydrostatic equilibrium (HE) yields

$$M(r) = -\frac{k_B T(r) r}{\mu m_p G} \left[ \frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r} \right]$$

This equation is often used to measure the ‘hydrostatic’ mass of a galaxy cluster; X-ray measurements can be used to infer $\rho(r)$ and $T(r)$ (after deprojection, which is
analytical in the case of spherical symmetry). Substitution of these two radial dependencies in the above equation then yields an estimate for the cluster’s mass profile, $M(r)$. Note, though, that this mass estimate is based on three crucial assumptions: (i) sphericity, (ii) hydrostatic equilibrium, and (iii) an ideal-gas EoS. Clusters typically are not spherical, often are turbulent (such that $\vec{u} \neq 0$, violating the assumption of HE), and can have significant contributions from non-thermal pressure due to magnetic fields, cosmic rays and/or turbulence. Including these non-thermal pressure sources the above equation becomes

$$M(r) = -\frac{k_B T(r) r}{\mu m_p G} \left[ \frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r} + \frac{P_{\text{nt}}}{P_{\text{th}}} \frac{d \ln P_{\text{nt}}}{d \ln r} \right]$$

were $P_{\text{nt}}$ and $P_{\text{th}}$ are the non-thermal and thermal contributions to the total gas pressure. Unfortunately, it is extremely difficult, if not impossible, to properly measure $P_{\text{nt}}$, which is therefore often ignored. This may result in systematic biases of the inferred cluster mass.
CHAPTER 17

Astrophysical Gases

Most of the baryonic matter in the Universe is in a gaseous state, made up of \( \sim 75\% \) Hydrogen (H), \( \sim 25\% \) Helium (He) and only small amounts of other elements (called ‘metals’). Gases can be neutral, ionized, or partially ionized. The degree of ionization of any given element is specified by a Roman numeral after the element name. For example HI and HII refer to neutral and ionized hydrogen, respectively, while CI is neutral carbon, CII is singly-ionized carbon, and CIV is triply-ionized carbon. A gas that is highly (largely) ionized is called a \textit{plasma}.

**Thermodynamic Equilibrium:** a system is said to be in thermodynamic equilibrium (TE) when it is in
- thermal equilibrium
- mechanical equilibrium
- radiative equilibrium
- chemical equilibrium
- statistical equilibrium

Equilibrium means a state of balance. In a state of thermodynamic equilibrium, there are no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. A system that is in thermodynamic equilibrium experiences no changes when it is isolated from its surroundings.

In a system in TE the energy in the radiation field is in equilibrium with the kinetic energy of the particles. If the system is isolated (to both matter and radiation), and in mechanical equilibrium, then over time TE will be established. For a gas in TE, the \textit{radiation temperature}, \( T_R \), is equal to the \textit{kinetic temperature}, \( T \), is equal to the \textit{excitation temperature}, \( T_{ex} \) (see below for definitions). Since no photons are allowed to escape from a system in TE (this would correspond to energy loss, and thus violate TE), the photons that are produced in the gas (represented by \( T_R \)) therefore must be tightly coupled to the random motions of the particles (represented by \( T \)). This coupling implies that, for bound states, the excitation and de-excitation of the atoms and ions \textit{must} be dominated by \textit{collisions}. In other words, the collision timescales must be shorter than the timescales associated with photon interactions.
or spontaneous de-excitations. If that were not the case, $T$ could not remain equal to $T_R$.

**Local Thermodynamic Equilibrium (LTE):** True TE is rare (in almost all cases energy does escape the system in the form of radiation, i.e., the system cools), and often temperature gradients are present. A good, albeit imperfect, example of TE is the Universe as a whole prior to decoupling. Although true TE is rare, in many systems (stars, gaseous spheres, ISM), we may apply local TE (LTE), which implies that the gas is in TE, but only *locally*. In a system in LTE, there will typically be gradients in temperature, density, pressure, etc, but they are sufficiently small over the mean-free path of a gas particle. For stellar interiors, the fact that radiation is ‘locally trapped’ explains why it takes so long for photons to diffuse from the center (where they are created in nuclear reactions) to the surface (where they are emitted into space). In the case of the Sun, this timescale is of the order of 200,000 years.

**Thermal Equilibrium:** For a system to be in thermal equilibrium with itself, it must not contain temperature gradients. For two systems to be in thermal equilibrium with each other requires an actual or implied thermal connection between them, through a path that is permeable only to heat, and that no net energy is transferred through that path.

For a gas in thermal equilibrium at some uniform temperature, $T$, and uniform number density, $n$, the number density of particles with speeds between $v$ and $v + dv$ is given by the Maxwell-Boltzmann (aka ‘Maxwellian’) velocity distribution:

$$n(v) \, dv = n \left( \frac{m}{2 \pi k_B T} \right)^{3/2} \exp \left( - \frac{m v^2}{2 k_B T} \right) 4 \pi v^2 \, dv$$

where $m$ is the mass of a gas particle. The temperature $T$ is called the kinetic temperature, and is related to the mean-square particle speed, $\langle v^2 \rangle$, according to

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T$$

The most probable speed of the Maxwell-Boltzmann distribution is $v_{\text{mp}} = \sqrt{2 k_B T/m}$, while the mean speed is $v_{\text{mean}} = \sqrt{8 k_B T/m}$. 

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Setting up a Maxwellian velocity distribution requires many elastic collisions. In the limit where most collisions are elastic, the system will typically very quickly equilibrate to thermal equilibrium. However, depending on the temperature of the gas, collisions can also be inelastic: examples of the latter are collisional excitations, in which part of the kinetic energy of the particle is used to excite its target to an excited state (i.e., the kinetic energy is now temporarily stored as a potential energy). If the de-excitation is collisional, the energy is given back to the kinetic energy of the gas (i.e., no photon is emitted). However, if the de-excitation is spontaneous or via stimulated emission, a photon is emitted. If the gas is optically thin to the emitted photon, the energy will escape the gas. The net outcome of the collisional excitation is then one of dissipation, i.e., cooling (kinetic energy of the gas is being radiated away). In the optically thick limit, the photon will be absorbed by another atom (or ion). If the system is in (L)TE, the radiation temperature of these photons being emitted and absorbed is equal to the kinetic temperature of the gas. Although in LTE a subset of the collisions are inelastic, this subset is typically small, and one may still use the Maxwell-Boltzmann distribution to characterize the velocities of the gas particles.

**Mechanical Equilibrium:** A system is said to be in mechanical equilibrium if (i) the vector sum of all external and internal forces is zero and (ii) the sum of the moments of all external and internal forces about any line is zero. The internal forces arise from pressure gradients. If a system is out of mechanical equilibrium, it will try to re-establish mechanical equilibrium, the characteristic time-scale of which is the sound-crossing time.

**Radiative Equilibrium:** A system is said to be in radiative equilibrium if any two randomly selected subsystems of it exchange by radiation equal amounts of heat with each other.

**Chemical Equilibrium:** In a chemical reaction, chemical equilibrium is the state in which both reactants and products are present at concentrations which have no further tendency to change with time. Usually, this state results when the forward reaction proceeds at the same rate as the reverse reaction. The reaction rates of the forward and reverse reactions are generally not zero but, being equal, there are no net changes in the concentrations of the reactant and product.
Statistical Equilibrium: A system is said to be in statistical equilibrium if the level populations of its constituent atoms and ions do not change with time (i.e., if the transition rate into any given level equals the rate out).

For a system in statistical equilibrium, the population of states is described by the Boltzmann law for level populations:

\[
\frac{N_j}{N_i} = \frac{g_j}{g_i} \exp \left( -\frac{h \nu_{ij}}{k_B T} \right)
\]

Here \( N_i \) is the number of atoms in which electrons are in energy level \( i \) (here \( i \) reflects the principal quantum number, with \( i = 1 \) referring to the ground state), \( \nu_{ij} \) is the frequency corresponding to the energy difference \( \Delta E_{ij} = h \nu_{ij} \) of the energy levels, \( h \) is the Planck constant, and \( g_i \) is the statistical weight of population \( i \).

Statistical weight: the statistical weight (aka ‘degeneracy’) indicates the number of states at a given principal quantum number, \( n \). In quantum mechanics, a total of four quantum numbers is needed to describe the state of an electron: the principal quantum number, \( n \), the orbital angular momentum quantum number, \( l \), the magnetic quantum number \( m_l \), and the electron spin quantum number, \( m_s \). For hydrogen \( l \) can take on the values \( 0, 1, ..., n - 1 \), the quantum number \( m_l \) can take on the values \( -l, -l + 1, -1, 0, 1, ..., l \), and \( m_s \) can take on the values \(+1/2\) (‘up’) and \(-1/2\) (‘down’). Hence, \( g_n = 2 n^2 \).

Excitation Temperature: The above Boltzmann law defines the excitation temperature, \( T_{ex} \), as the temperature which, when put into the Boltzmann law for level populations, results in the observed ratio of \( N_j/N_i \). For a gas in (local) TE, all levels in an atom can be described by the same \( T_{ex} \), which is also equal to the kinetic temperature, \( T \), and the radiation temperature, \( T_R \). Under non-LTE conditions, each pair of energy levels can have a different \( T_{ex} \).

Radiation Temperature: the radiation temperature, or \( T_R \), is a temperature that specifies the energy density in the radiation field, \( u_\gamma \), according to \( u_\gamma = a_\gamma T_R^4 \). In TE \( T_R = T \), while for a Black Body, \( T_R \) is the temperature that appears in the Planck function that describes the specific intensity of the radiation field (see Appendix J for more details).
**Partition Function:** Using the above Boltzmann law, it is straightforward to show that

\[
\frac{N_n}{N} = g_n \frac{U}{U} \exp \left( -\frac{\Delta E_n}{k_B T} \right)
\]

where \( N = \sum N_i \) is the total number of atoms (of all energy levels) and \( \Delta E_n \) is the energy difference between state \( n \) and the ground-state, and

\[
U = \sum g_i e^{-\Delta E_i / k_B T}
\]

is called the **partition function**. (Note: the summation is over all principal quantum numbers up to some maximum \( n_{\text{max}} \), which is required to prevent divergence; see Irwin 2007 or Rybicki & Lightman 1979 for details).

**Saha equation:** the Saha equation expresses the ratios of atoms/ions in different ionization states. In particular, the number of atoms/ions in the \( (K+1) \)th ionization state versus those in the \( K \)th ionization state is given by

\[
\frac{N_{K+1}}{N_K} = \frac{2}{n_e} \frac{U_{K+1}}{U_K} \left( \frac{2 \pi m_e k_B T}{\hbar^2} \right)^{3/2} \exp \left( -\frac{\chi_K}{k_B T} \right)
\]

where \( U_n \) is the **partition function** of the \( n \)th ionization state, \( n_e \) is the electron number density, and \( \chi_K \) is the energy required to remove an electron from the ground state of the \( K \)th ionization state. Note that unlike the Boltzmann equation, the Saha equation for the ionization fractions has a dependence on electron density. This reflects that if there is a higher density of free electrons, there is a greater probability that an electron will recombine with the ion, lowering the ionization state of the gas.

**Hydrogen** Since Hydrogen is the most common element in the Universe, it is important to have some understanding of its structure. Fig. 18 shows the energy levels of a hydrogen atom and some of its most important transitions. The Balmer transitions have wavelengths that fall in the optical, and are therefore well known to (optical) astronomers. The Lyman lines typically fall in the UV, and can only be observed from space (or for high-redshift objects, for which the rest-frame UV is redshifted into the optical).
Using the Saha equation, we can compute the ionization fraction for hydrogen as a function of temperature and electron density:

\[
\frac{N_{\text{HII}}}{N_{\text{HI}}} = 2.41 \times 10^{15} \frac{T^{3/2}}{n_{e}} \exp \left( - \frac{1.58 \times 10^{5}}{T} \right)
\]

where we have used that \( \chi_{\text{HI}} = 13.6 \text{eV}, U_{\text{HI}} = 2 \) and \( U_{\text{HII}} = 1 \) (i.e., the ionized hydrogen atom is just a free proton and only exists in a single state). We can also compute, using the Boltzmann law, the ratio of hydrogen atoms in the first excited state compared to those in the ground state. The latter shows that the temperature must exceed 30,000K in order for there to be an appreciable (10 percent) number of hydrogen atoms in the first excited state. However, at such high temperatures, one typically has that most of the hydrogen will be ionized (unless the electron density is unrealistically high). This somewhat unintuitive result arises from the fact that there are many more possible states available for a free electron than for a bound electron in the first excited state. In conclusion, neutral hydrogen in LTE will have virtually all its atoms in the ground state.

If densities are sufficiently low, which is typically the case in the ISM, the collisional excitation rate (which scales with \( n_{e}^{2} \)) is lower than the spontaneous de-excitation rate. If that is the case, the hydrogen gas is no longer in LTE, and once again,
virtually all neutral hydrogen will find itself in the ground state. *Neutral hydrogen in the ISM is in the ground-state and is typically NOT in LTE.* One important consequence of the fact that neutral hydrogen is basically always observed in the ground state, is that observations of hydrogen emission lines (Lyman, Balmer, Paschen, etc) indicates that the hydrogen must be ionized; the lines arise from recombinations, and are therefore called *recombination lines.* Note that Balmer lines are often observed in absorption (for example, Balmer lines are evident in a spectrum of the Sun). This implies that there must be hydrogen present in the first excited state, which seems at odds with the conclusions reached above. A small fraction of excited hydrogen atoms, though, can still produce strong absorption lines, simply because hydrogen is so abundant.

**21cm line emission:** The ground state of hydrogen is split into two hyperfine states due to the two possible orientations of the proton and electron spins: The state in which the spins of proton and electron are aligned has slightly higher energy than the one in which they are anti-aligned. The energy difference between these two hyperfine states corresponds to a photon with a wavelength of 21cm (which falls in the radio). The excitation temperature of this *spin-flip* transition is called the *spin temperature.* Since, for typical interstellar medium conditions, the spin-flip transition is *collisionally induced* (i.e., the rate for spontaneous de-excitation is extremely low), the spin-temperature is typically equal to the kinetic temperature of the hydrogen gas. The 21cm line is an important emission line to probe the distribution (and temperature) of neutral hydrogen gas in the Universe.
CHAPTER 18

Gravity: Poisson Equation & Virial Theorem

Gravity in Astrophysical Fluids: Many of the fluids encountered in astrophysics are self-gravitating, which means that the gravitational force due to the fluid itself exceeds the gravitational force from the external mass distribution. Arguably the most important example of self-gravitating, astrophysical fluids are stars. But Cold Dark Matter halos are also examples of self-gravitating fluids (albeit collisionless). The interstellar medium (ISM) can and cannot be self-gravitating, depending on the conditions. The intra-cluster medium (ICM) is generally not self-gravitating; rather the gravitating potential is dominated by the dark matter.

Gravitational Potential: Gravity is a conservative force, which means that it can be written as the gradient of a scalar field. Newton’s gravitational potential, \( \Phi(\vec{x}) \), is defined such that the gravitational force per unit mass 

\[
\vec{F}_g = -\nabla \Phi
\]

Note that the absolute normalization of \( \Phi \) has no physical relevance; only the gradients of \( \Phi \) matter.

Consider a density distribution \( \rho(\vec{x}) \). What is the gravitational force \( \vec{F}_g \) acting on a particle of mass \( m \) at location \( \vec{x} \)? We can sum the small contributions \( \delta \vec{F}_g \) from different regions \( \vec{x} \pm d^3 \vec{x}' \), given by

\[
\delta \vec{F}_g(\vec{x}) = G \frac{m \delta m(\vec{x}')}{|\vec{x}' - \vec{x}|^2} \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|} = Gm \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \rho(\vec{x}') d^3 \vec{x}'
\]

Adding up all the small contributions yields \( \vec{F}_g(\vec{x}) = \int \delta \vec{F}_g(\vec{x}) = m \vec{g}(\vec{x}) \), where

\[
\vec{g}(\vec{x}) = G \int d^3 \vec{x}' \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \rho(\vec{x}')
\]

is the gravitational field (i.e., the force per unit mass). Using that
we can rewrite $g(\vec{x})$ as
\[
\vec{g}(\vec{x}) = G \int d^3\vec{x}' \nabla_x \left( \frac{1}{|\vec{x}' - \vec{x}|} \right) \rho(\vec{x}') = \nabla_x \int d^3\vec{x}' G \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|} \equiv -\nabla_x \Phi
\]
where in the last step we have defined the gravitational potential
\[
\Phi(\vec{x}) = -G \int d^3\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|}
\]

**Poisson Equation:** It can be shown that the gravitational potential obeys the Poisson equation:
\[
\nabla^2 \Phi = 4\pi G \rho
\]
For a derivation, see Section 3.2 of *Astrophysical Fluid Dynamics* by Clarke & Carswell, or Section 2.1 of *Galactic Dynamics* by Binney & Tremaine.

In general, it is extremely complicated to solve the **Poisson equation** for $\Phi(\vec{x})$ given $\rho(\vec{x})$ [see Chapter 2 of *Galactic Dynamics* by Binney & Tremaine for a detailed discussion]. However, under certain symmetries, solutions to the Poisson equation are fairly straightforward. In particular, under **spherical symmetry** the general solution to the **Poisson equation** is
\[
\Phi(r) = -4\pi G \left[ \frac{1}{r} \int_0^r \rho(r') r'^2 dr' + \int_r^\infty \rho(r') r' dr' \right]
\]
Note that the potential at $r$ depends on the mass distribution outside of $r$. However, if we now compute the gravitational force per unit mass
\[
\vec{F}_g(r) = -\frac{d\Phi}{dr} \hat{e}_r = -\frac{G M(r)}{r^2} \hat{e}_r
\]
where
\[
M(r) \equiv 4\pi \int_0^r \rho(r') r'^2 dr
\]
is the enclosed mass within $r$. This shows that the gravitational force does not depend on the mass distribution outside of $r$.

**Newton’s first theorem:** a body that is inside a spherical shell of matter experiences no net gravitational force from that shell. The equivalent in general relativity is called **Birkhoff’s theorem**.

This is easily understood from the fact that the solid angles that extent from a point inside a sphere to opposing directions have areas on the sphere that scale as $r^2$ (where $r$ is the distance from the point to the sphere), while the gravitational force per unit mass scales as $r^{-2}$. Hence, the gravitational forces from the two opposing areas exactly cancel.

**Circular velocity:** the velocity of a particle or fluid element on a circular orbit. For a spherical mass distribution

$$V_{\text{circ}}(r) = \sqrt{r \frac{d\Phi}{dr}} = \sqrt{\frac{GM(r)}{r}}$$

In the case of an axisymmetric mass distribution, the circular velocity in the **equatorial plane** ($z = 0$, where $z$ is one of the three cylindrical coordinates $(R, \phi, z)$) is given by

$$V_{\text{circ}}(R) = \sqrt{R \frac{d\Phi}{dR}} \neq \sqrt{\frac{GM(R)}{R}}$$

**Escape velocity:** the velocity needed for a particle or fluid element to escape to infinity. Since $E = v^2/2 + \Phi(\vec{x})$, and escape requires $E > 0$, the escape velocity is

$$V_{\text{esc}}(\vec{x}) = \sqrt{2|\Phi(\vec{x})|}$$

independent of the symmetry (or lack thereof) of the mass distribution.

Since gas cannot be on self-intersecting orbits, gas in disk galaxies generally orbits on circular orbits. The measured rotation velocities therefore reflect the circular velocities, which can be used to infer the enclosed mass as a function of radius. This
method is generally used to infer the presence of dark matter halos surrounding disk galaxies.

Consider a gravitational system consisting of $N$ particles (e.g., stars, fluid elements). The total energy of the system is $E = K + W$, where

\[
K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2
\]

\[
W = - \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}
\]

The latter follows from the fact that gravitational binding energy between a pair of masses is proportional to the product of their masses, and inversely proportional to their separation. The factor $1/2$ corrects for double counting the number of pairs.

**Potential Energy in Continuum Limit:** To infer an expression for the gravitational potential energy in the continuum limit, it is useful to rewrite the above expression as

\[
W = \frac{1}{2} \sum_{i=1}^{N} m_i \Phi_i
\]

where

\[
\Phi_i = - \sum_{j \neq i} \frac{G m_j}{r_{ij}}
\]

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. In the continuum limit this simply becomes

\[
W = \frac{1}{2} \int \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3\mathbf{x}
\]

One can show (see e.g., *Galactic Dynamics*) that this is equal to the trace of the Chandrasekhar Potential Energy Tensor

\[
W_{ij} \equiv - \int \rho(\mathbf{x}) x_i \frac{\partial \Phi}{\partial x_j} d^3\mathbf{x}
\]

In particular,
\[ W = \text{Tr}(W_{ij}) = \sum_{i=1}^{3} W_{ii} = -\int \rho(\vec{x}) \vec{x} \cdot \nabla \Phi \, d^3\vec{x} \]

which is another, equally valid, expression for the gravitational potential energy in the continuum limit.

**Virial Theorem:** A stationary, gravitational system obeys

\[
2K + W = 0
\]

Actually, the correct virial equation is \(2K + W + \Sigma = 0\), where \(\Sigma\) is the surface pressure. In many, but certainly not all, applications in astrophysics this term can be ignored. Many textbooks don’t even mention the surface pressure term.

Combining the **virial equation** with the expression for the total energy, \(E = K + W\), we see that for a system that obeys the virial theorem

\[
E = -K = W/2
\]

**Example:** Consider a cluster consisting of \(N\) galaxies. If the cluster is in virial equilibrium then

\[
2 \sum_{i=1}^{N} \frac{1}{2} m v_i^2 - \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{G m_i m_j}{r_{ij}} = 0
\]

If we assume, for simplicity, that all galaxies have equal mass then we can rewrite this as

\[
N m \frac{1}{N} \sum_{i=1}^{N} v_i^2 - \frac{G (Nm)^2}{2} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{r_{ij}} = 0
\]

Using that \(M = N m\) and \(N(N - 1) \simeq N^2\) for large \(N\), this yields
\[
M = \frac{2 \langle v^2 \rangle}{G \langle 1/r \rangle}
\]

with

\[
\langle 1/r \rangle = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{r_{ij}}
\]

It is useful to define the gravitational radius \( r_g \) such that

\[
W = -\frac{GM^2}{r_g}
\]

Using the relations above, it is clear that \( r_g = 2/\langle 1/r \rangle \). We can now rewrite the above equation for \( M \) in the form

\[
M = \frac{r_g \langle v^2 \rangle}{G}
\]

Hence, one can infer the mass of our cluster of galaxies from its velocity dispersion and its gravitation radius. In general, though, neither of these is observable, and one uses instead

\[
M = \alpha \frac{R_{\text{eff}} \langle v_{\text{los}}^2 \rangle}{G}
\]

where \( v_{\text{los}} \) is the line-of-sight velocity, \( R_{\text{eff}} \) is some measure for the ‘effective’ radius of the system in question, and \( \alpha \) is a parameter of order unity that depends on the radial distribution of the galaxies. Note that, under the assumption of isotropy, \( \langle v_{\text{los}}^2 \rangle = \langle v^2 \rangle / 3 \) and one can also infer the mean reciprocal pair separation from the projected pair separations; in other words under the assumption of isotropy one can infer \( \alpha \), and thus use the above equation to compute the total, gravitational mass of the cluster. This method was applied by Fritz Zwicky in 1933, who inferred that the total dynamical mass in the Coma cluster is much larger than the sum of the masses of its galaxies. This was the first observational evidence for dark matter, although it took the astronomical community until the late 70’s to generally accept this notion.
For a self-gravitating fluid

\[ K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \frac{1}{2} N m \langle v^2 \rangle = \frac{3}{2} N k_B T \]

where the last step follows from what we have learned in Chapter 11 about ideal gases of monoatomic particles. In fact, we can use the above equation for any fluid (including a collisionless one), if we interpret \( T \) as an effective temperature that measures the rms velocity of the constituent particles. If the system is in virial equilibrium, then

\[ E = -K = -\frac{3}{2} N k_B T \]

which, as we show next, has some important implications...

**Heat Capacity**: the amount of heat required to increase the temperature by one degree Kelvin (or Celsius). For a self-gravitating fluid this is

\[ C \equiv \frac{dE}{dT} = -\frac{3}{2} N k_B \]

which is negative! This implies that by losing energy, a gravitational system gets hotter!! This is a very counter-intuitive result, that often leads to confusion and wrong expectations. Below we give three examples of implications of the negative heat capacity of gravitating systems,

**Example 1: Drag on satellites** Consider a satellite orbiting Earth. When it experiences friction against the (outer) atmosphere, it loses energy. This causes the system to become more strongly bound, and the orbital radius to shrink. Consequently, the energy loss results in the gravitational potential energy, \( W \), becoming more negative. In order for the satellite to re-establish virial equilibrium \((2K + W = 0)\), its kinetic energy needs to increase. Hence, contrary to common intuition, friction causes the satellite to speed up, as it moves to a lower orbit (where the circular velocity is higher).

**Example 2: Stellar Evolution** A star is a gaseous, self-gravitating sphere that radiates energy from its surface at a luminosity \( L \). Unless this energy is replenished (i.e., via some energy production mechanism in the star’s interior), the star will react
by shrinking (i.e., the energy loss implies an increase in binding energy, and thus a potential energy that becomes more negative). In order for the star to remain in virial equilibrium its kinetic energy, which is proportional to temperature, has to increase; the star’s energy loss results in an increase of its temperature.

In the Sun, hydrogen burning produces energy that replenishes the energy loss from the surface. As a consequence, the system is in equilibrium, and will not contract. However, once the Sun has used up all its hydrogen, it will start to contract and heat up, because of the negative heat capacity. This continues until the temperature in the core becomes sufficiently high that helium can start to fuse into heavier elements, and the Sun settles in a new equilibrium.

Example 3: Core Collapse a system with negative heat capacity in contact with a heat bath is thermodynamically unstable. Consider a self-gravitating fluid of ‘temperature’ $T_1$, which is in contact with a heat bath of temperature $T_2$. Suppose the system is in thermal equilibrium, so that $T_1 = T_2$. If, due to some small disturbance, a small amount of heat is transferred from the system to the heat bath, the negative heat capacity implies that this results in $T_1 > T_2$. Since heat always flows from hot to cold, more heat will now flow from the system to the heat bath, further increasing the temperature difference, and $T_1$ will continue to rise without limit. This run-away instability is called the gravothermal catastrophe. An example of this instability is the core collapse of globular clusters: Suppose the formation of a gravitational system results in the system having a declining velocity dispersion profile, $\sigma^2(r)$ (i.e., $\sigma$ decreases with increasing radius). This implies that the central region is (dynamically) hotter than the outskirts. IF heat can flow from the center to those outskirts, the gravothermal catastrophe kicks in, and $\sigma$ in the central regions will grow without limits. Since $\sigma^2 = GM(r)/r$, the central mass therefore gets compressed into a smaller and smaller region, while the outer regions expand. This is called core collapse. Note that this does NOT lead to the formation of a supermassive black hole, because regions at smaller $r$ always shrink faster than regions at somewhat larger $r$. In dark matter halos, and elliptical galaxies, the velocity dispersion profile is often declining with radius. However, in those systems the two-body relaxation time is so long that there is basically no heat flow (which requires two-body interactions). However, globular clusters, which consist of $N \sim 10^4$ stars, and have a crossing time of only $t_{\text{cross}} \sim 5 \times 10^6$ yr, have a two-body relaxation time of only $\sim 5 \times 10^8$ yr. Hence, heat flow in globular clusters is not negligible, and they can (and do) undergo core collapse. The collapse does not proceed indefinitely, because of binaries (see Galactic Dynamics by Binney & Tremaine for more details).
Consider an encounter between two collisionless N-body systems (i.e., dark matter halos or galaxies): a perturber $P$ and a system $S$. Let $q$ denote a particle of $S$ and let $b$ be the impact parameter, $v_\infty$ the initial speed of the encounter, and $R_0$ the distance of closest approach (see Fig. 14).

Typically what happens in an encounter is that orbital energy (of $P$ wrt $S$) is converted into random motion energy of the constituent particles of $P$ and $S$ (i.e., $q$ gains kinetic energy wrt $S$).

The velocity impulse $\Delta \vec{v}_q = \int \vec{g}(t) \, dt$ of $q$ due to the gravitational field $\vec{g}(t)$ from $P$ decreases with increasing $v_\infty$ (simply because $\Delta t$ will be shorter). Consequently, when $v_\infty$ increases, less and less orbital energy is transferred to random motion, and there is a critical velocity, $v_{\text{crit}}$, such that

\[
\begin{align*}
  v_\infty > v_{\text{crit}} & \Rightarrow S \text{ and } P \text{ escape from each other} \\
  v_\infty < v_{\text{crit}} & \Rightarrow S \text{ and } P \text{ merge together}
\end{align*}
\]

There are only two cases in which we can calculate the outcome of the encounter analytically:

- **high speed encounter** ($v_\infty \gg v_{\text{crit}}$). In this case the encounter is said to be impulsive and one can use the impulsive approximation to compute its outcome.

- **large mass ratio** ($M_P \ll M_S$). In this case one can use the treatment of dynamical friction to describe how $P$ loses orbital energy and angular momentum to $S$.

In all other cases, one basically has to resort to numerical simulations to study the outcome of the encounter. In what follows we present treatments of first the impulse approximation and then dynamical friction.
Impulse Approximation: In the limit where the encounter velocity $v_\infty$ is much larger than the internal velocity dispersion of $S$, the change in the internal energy of $S$ can be approximated analytically. The reason is that, in a high-speed encounter, the time-scale over which the tidal forces from $P$ act on $q$ is much shorter than the dynamical time of $S$ (or $q$). Hence, we may consider $q$ to be stationary (fixed wrt $S$) during the encounter. Consequently, $q$ only experiences a change in its kinetic energy, while its potential energy remains unchanged:

$$\Delta E_q = \frac{1}{2}(\vec{v} + \Delta \vec{v})^2 - \frac{1}{2} \vec{v}^2 = \vec{v} \cdot \Delta \vec{v} + \frac{1}{2} |\Delta \vec{v}|^2$$

We are interested in $\Delta E_S = \sum_q \Delta E_q$, where the summation is over all its constituent particles:

$$\Delta E_S = \int \Delta E(\vec{r}) \rho(r) \, d^3\vec{r} \approx \frac{1}{2} \int |\Delta \vec{v}|^2 \rho(r) \, d^3\vec{r}$$

where we have used that, because of symmetry, the integral

$$\int \vec{v} \cdot \Delta \vec{v} \rho(r) \, d^3\vec{r} \approx 0$$

In the large $v_\infty$ limit, we have that the distance of closest approach $R_0 \to b$, and the velocity of $P$ wrt $S$ is $v_p(t) \simeq v_\infty \vec{e}_y \equiv v_p \vec{e}_y$. Consequently, we have that

$$\vec{R}(t) = (b, v_p t, 0)$$
Let \( \vec{r} \) be the position vector of \( q \) wrt \( S \) and adopt the distant encounter approximation, which means that \( b \gg \max[RS, RP] \), where \( R_S \) and \( R_P \) are the sizes of \( S \) and \( P \), respectively. This means that we may treat \( P \) as a point mass \( M_P \), so that

\[
\Phi_P(\vec{r}) = -\frac{GM_P}{|\vec{r} - \vec{R}|}
\]

Using geometry, and defining \( \phi \) as the angle between \( \vec{r} \) and \( \vec{R} \), we have that

\[
|\vec{r} - \vec{R}|^2 = (R - r \cos \phi)^2 + (r \sin \phi)^2
\]

so that

\[
|\vec{r} - \vec{R}| = \sqrt{R^2 - 2rR \cos \phi + r^2}
\]

Next we use the series expansion

\[
\frac{1}{\sqrt{1 + x}} = 1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{246}x^3 + \ldots
\]

to write

\[
\frac{1}{|\vec{r} - \vec{R}|} = \frac{1}{R} \left[ 1 - \frac{1}{2} \left( -\frac{2r}{R} \cos \phi + \frac{r^2}{R^2} \right) + \frac{3}{8} \left( -\frac{2r}{R} \cos \phi + \frac{r^2}{R^2} \right)^2 + \ldots \right]
\]

Substitution in the expression for the potential of \( P \) yields

\[
\Phi_P(\vec{r}) = -\frac{GM_P}{R} - \frac{GM_P r \cos \phi - GM_P r^2}{R^2} \left( \frac{3}{2} \cos^2 \phi - \frac{1}{2} \right) + \mathcal{O}[(r/R)^3]
\]

- The first term on rhs is a constant, not yielding any force (i.e., \( \nabla \Phi_P = 0 \)).
- The second term on the rhs describes how the center of mass of \( S \) changes its velocity due to the encounter with \( P \).
- The third term on the rhs corresponds to the tidal force per unit mass and is the term of interest to us.

It is useful to work in a rotating coordinate frame \((x', y', z')\) centered on \( S \) and with the \( x'\)-axis pointing towards the instantaneous location of \( P \), i.e., \( x' \) points along \( \vec{R}(t) \). Hence, we have that \( x' = r' \cos \phi \), where \( r'^2 = x'^2 + y'^2 + z'^2 \). In this new coordinate frame, we can express the third term of \( \Phi_P(\vec{r}) \) as

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\[
\Phi_3(\vec{r}) = -\frac{GM_P}{R^3} \left( \frac{3}{2} \frac{r'^2}{2} \cos^2 \phi - \frac{1}{2} r'^2 \right) \\
= -\frac{GM_P}{R^3} \left( x'^2 - \frac{1}{2} y'^2 - \frac{1}{2} z'^2 \right)
\]

Hence, the tidal force is given by

\[
\vec{F}_{\text{tid}}(\vec{r}) \equiv -\nabla \Phi_P = \frac{GM_P}{R^3} (2x', -y', -z')
\]

We can relate the components of \( \vec{F}_{\text{tid}} \) to those of the corresponding tidal force, \( \vec{F}_{\text{tid}} \) in the \((x, y, z)\)-coordinate system using

\[
\begin{align*}
    x' &= x \cos \theta - y \sin \theta \\
y' &= x \sin \theta + y \cos \theta \\
z' &= z
\end{align*}
\]

where \( \theta \) is the angle between the \( x \) and \( x' \) axes, with \( \cos \theta = b/R \) and \( \sin \theta = v_P t/R \). After some algebra one finds that

\[
\begin{align*}
    F_x &= \frac{GM_P}{R^3} \left[ x(2 - 3 \sin^2 \theta) - 3y \sin \theta \cos \theta \right] \\
    F_y &= \frac{GM_P}{R^3} \left[ y(2 - 3 \cos^2 \theta) - 3x \sin \theta \cos \theta \right] \\
    F_z &= -\frac{GM_P}{R^3} z
\end{align*}
\]

Using these, we have that

\[
\Delta \vec{v} = \left( \Delta v_x, \Delta v_y, \Delta v_z \right) = \frac{2GM_P}{v_P b^2} (x, 0, -z)
\]

Substitution in the expression for \( \Delta E_S \) yields
\[ \Delta E_S = \frac{1}{2} \int |\Delta \vec{v}|^2 \rho(r) \, d^3\vec{r} = \frac{2 G^2 M_S^2}{v_p^2 b^4} M_S \langle x^2 + z^2 \rangle \]

Under the assumption that \( S \) is spherically symmetric we have that \( \langle x^2 + z^2 \rangle = \frac{2}{3} \langle x^2 + y^2 + z^2 \rangle = \frac{2}{3} \langle r^2 \rangle \) and we obtain the final expression for the energy increase of \( S \) as a consequence of the impulsive encounter with \( P \):

\[
\Delta E_S = \frac{4}{3} G^2 M_S \left( \frac{M_P}{v_p} \right)^2 \frac{\langle r^2 \rangle}{b^4}
\]

This derivation, which is originally due to Spitzer (1958), is surprisingly accurate for encounters with \( b > 5 \max\{R_P, R_S\} \), even for relatively slow encounters with \( v_\infty \sim \sigma_S \). For smaller impact parameters one has to make a correction (see Galaxy Formation and Evolution by Mo, van den Bosch & White 2010 for details).

The impulse approximation shows that high-speed encounters can pump energy into the systems involved. This energy is tapped from the orbital energy of the two systems wrt each other. Note that \( \Delta E_S \propto b^{-4} \), so that close encounters are far more important than distant encounters.

Let \( E_b \propto GM_S/R_S \) be the binding energy of \( S \). Then, if \( \Delta E_S > E_b \) the impulsive encounter will lead to the tidal disruption of \( S \). If, on the other hand, \( \Delta E_S < E_b \), then there will be individual particles (stars) of \( S \) that get accelerated to speeds in excess of the local escape speed. Hence, \( S \) survives but experiences some mass loss (tidal stripping).

After the encounter, \( S \) has gained kinetic energy (in the amount of \( \Delta E_S \)), but its potential energy has remained unchanged (recall, this is the assumption that underlies the impulse approximation). As a consequence, after the encounter \( S \) will no longer be in virial equilibrium; \( S \) will have to readjust itself to re-establish virial equilibrium.

Let \( K_0 \) and \( E_0 \) be the initial (pre-encounter) kinetic and total energy of \( S \). The virial theorem ensures that \( E_0 = -K_0 \). The encounter causes an increase of (kinetic) energy, so that \( K_0 \to K_0 + \Delta E_S \) and \( E_0 \to E_0 + \Delta E_S \). After \( S \) has re-established virial equilibrium, we have that \( K_1 = -E_1 = -(E_0 + \Delta E_S) = K_0 - \Delta E_S \). Thus, we see that virialization after the encounter changes the kinetic energy of
$S$ from $K_0 + \Delta E_S$ to $K_0 - \Delta E_S$! The gravitational energy after the encounter is $W_1 = 2E_1 = 2E_0 + 2\Delta E_S = W_0 + 2\Delta E_S$, which is less negative than before the encounter. Using the definition of the **gravitational radius** (see Chapter 12), $r_g = GM_S^2/|W|$, from which it is clear that the (gravitational) radius of $S$ increases due to the impulsive encounter.

**Dynamical Friction:** Consider the motion of a subject mass $M_S$ through a medium of individual particles of mass $m \ll M_S$. The subject mass $M_S$ experiences a "drag force", called dynamical friction, which transfers orbital energy and angular momentum from $M_S$ to the sea of particles of mass $m$.

There are three different "views" of dynamical friction:

1. Dynamical friction arises from **two-body encounters** between the subject mass and the particles of mass $m$, which drives the system towards equipartition. i.e., towards $\frac{1}{2}M_Sv_S^2 = \frac{1}{2}m\langle v_m^2 \rangle$. Since $M_S \gg m$, the system thus evolves towards $v_S \ll v_m$ (i.e., $M_S$ slows down).

2. Due to **gravitational focussing** the subject mass $M_S$ creates an overdensity of particles behind its path (the "wake"). The gravitational back-reaction of this wake on $M_S$ is what gives rise to dynamical friction and causes the subject mass to slow down.

3. The subject mass $M_S$ causes a perturbation $\delta \Phi$ in the potential of the collection of particles of mass $m$. The gravitational interaction between the **response density** (the density distribution that corresponds to $\delta \Phi$ according to the Poisson equation) and the subject mass is what gives rise to dynamical friction (see Fig. 13).

Although these views are similar, there are some subtle differences. For example, according to the first two descriptions dynamical friction is a **local** effect. The third description, on the other hand, treats dynamical friction more as a **global** effect. As we will see, there are circumstances under which these views make different predictions, and if that is the case, the third and latter view presents itself as the better one.

Chandrasekhar derived an expression for the dynamical friction force which, although it is based on a number of questionable assumptions, yields results in reasonable
agreement with simulations. This so-called **Chandrasekhar dynamical friction** force is given by
\[
\vec{F}_{df} = M_S \frac{d\vec{v}_S}{dt} = -\frac{4\pi G^2 M_S^2}{v_S^3} \ln \Lambda \rho(< v_S) \frac{\vec{v}_S}{v_S}
\]

Here \(\rho(< v_S)\) is the density of particles of mass \(m\) that have a speed \(v_m < v_S\), and \(\ln \Lambda\) is called the **Coulomb logarithm**. It’s value is uncertain (typically \(3 \lesssim \ln \Lambda \lesssim 30\)). One often approximates it as \(\ln \Lambda \sim \ln(M_h/M_S)\), where \(M_h\) is the total mass of the system of particles of mass \(m\), but this should only be considered a very rough estimate at best. The uncertainties for the Coulomb logarithm derive from the oversimplified assumptions made by Chandrasekhar, which include that the medium through which the subject mass is moving is infinite, uniform and with an isotropic velocity distribution \(f(v_m)\) for the sea of particles.

Similar to **frictional drag** in fluid mechanics, \(\vec{F}_{df}\) is always pointing in the direction **opposite** of \(v_S\).

Contrary to frictional drag in fluid mechanics, which always increases in strength when \(v_S\) increases, dynamical friction has a more complicated behavior: In the low-\(v_S\) limit, \(F_{df} \propto v_S\) (similar to hydrodynamical drag). However, in the high-\(v_S\) limit one has that \(F_{df} \propto v_S^{-2}\) (which arises from the fact that the factor \(\rho(< v_S)\) saturates).

Note that \(\vec{F}_{df}\) is **independent** of the mass \(m\) of the constituent particles, and proportional to \(M_S^2\). The latter arises, within the second or third view depicted above, from the fact that the wake or response density has a mass that is proportional to \(M_S\), and the gravitational force between the subject mass and the wake/response density therefore scales as \(M_S^2\).

One shortcoming of Chandrasekhar’s dynamical friction description is that it treats dynamical friction as a purely **local phenomenon**; it is treated as the cumulative effect of many **uncorrelated** two-body encounters between the subject masses and the individual field particles. That this local treatment is incomplete is evident from the fact that an object \(A\) orbiting **outside** of an \(N\)-body system \(B\) still experiences dynamical friction. This can be understood with the picture sketched under view 3 above, but not in a view that treats dynamical friction as a local phenomenon.
Orbital decay: As a consequence of dynamical friction, a subject mass $M_S$ orbiting inside (or just outside) of a larger $N$-body system of mass $M_h > M_S$, will transfer its orbital energy and angular momentum to the constituent particles of the 'host' mass. As a consequence it experiences orbital decay.

Let us assume that the host mass is a **singular isothermal sphere** with density and potential given by

$$
\rho(r) = \frac{V_c^2}{4\pi Gr^2} \quad \Phi(r) = V_c^2 \ln r
$$

where $V_c^2 = GM_h/r_h$ with $r_h$ the radius of the host mass. If we further assume that this host mass has, at each point, an **isotropic** and **Maxwellian** velocity distribution, then

$$
f(v_m) = \frac{\rho(r)}{(2\pi\sigma^2)^{3/2}} \exp \left[ -\frac{v_m^2}{2\sigma^2} \right]
$$

with $\sigma = V_c/\sqrt{2}$.

**NOTE:** the assumption of a singular isothermal sphere with an isotropic, Maxwellian
velocity distribution is unrealistic, but it serves the purpose of the order-of-magnitude estimate for the orbital decay rate presented below.

Now consider a subject of mass $M_S$ moving on a circular orbit ($v_S = V_c$) through this host system of mass $M_h$. The Chandrasekhar dynamical friction that this subject mass experiences is

$$F_{df} = -\frac{4\pi \ln \Lambda G^2 M_S^2 \rho(r)}{V_c^2} \left[ \text{erf}(1) - \frac{2}{\sqrt{n}} e^{-1} \right] \approx -0.428 \ln \Lambda \frac{GM_S^2}{r^2}$$

The subject mass has specific angular momentum $L = rv_S$, which it loses due to dynamical friction at a rate

$$\frac{dL}{dt} = r \frac{dv_S}{dt} = r \frac{F_{df}}{M_S} \approx -0.428 \ln \Lambda \frac{GM_S}{r}$$

Due to this angular momentum loss, the subject mass moves to a smaller radius, while it continues to move on a circular orbit with $v_S = V_c$. Hence, the rate at which the orbital radius changes obeys

$$V_c \frac{dr}{dt} = \frac{dL}{dt} = -0.428 \ln \Lambda \frac{GM_S}{r}$$

Solving this differential equation subject to the initial condition that $r(0) = r_i$, one finds that the subject mass $M_S$ reaches the center of the host after a time

$$t_{df} = \frac{1.17 \ r_i^2 V_c}{\ln \Lambda \ GM_S} = \frac{1.17}{\ln \Lambda} \left( \frac{r_i}{r_h} \right)^2 \frac{M_h}{M_S} \frac{r_h}{V_c}$$

In the case where the host system is a virialized dark matter halo we have that

$$\frac{r_h}{V_c} \approx \frac{1}{10H(z)} = 0.1t_H$$

where $t_H$ is called the Hubble time, and is approximately equal to the age of the Universe corresponding to redshift $z$ (the above relation derives from the fact that virialized dark matter halos all have the same average density). Using that $\ln \Lambda \sim \ln(M_h/M_S)$ and assuming that the subject mass starts out from an initial radius $r_i = r_h$, we obtain a dynamical friction time

$$t_{df} = 0.12 \frac{M_h/M_S}{\ln(M_h/M_S)} t_H$$

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Hence, the time $t_{df}$ on which dynamical friction brings an object of mass $M_S$ moving in a host of mass $M_h$ from an initial radius of $r_i = r_h$ to $r = 0$ is shorter than the Hubble time as long as $M_S \gtrsim M_h/30$. Hence, dynamical friction is only effective for fairly massive objects, relative to the mass of the host. In fact, if you take into account that the subject mass experiences mass stripping as well (due to the tidal interactions with the host), the dynamical friction time increases by a factor 2 to 3, and $t_{df} < t_H$ actually requires that $M_S \gtrsim M_h/10$.

For a more detailed treatment of collisions and encounters of collisionless systems, see Chapter 12 of Galaxy Formation and Evolution by Mo, van den Bosch & White.
To understand radiative processes, and the interaction of photons with matter, it is important to realize that all photon emission mechanisms arise from accelerating electrical charge.

The interactions of light with matter can be split in two categories:

- **scattering** (photon + matter $\rightarrow$ photon + matter)
- **absorption** (photon + matter $\rightarrow$ matter)

We first discuss scattering, which gives rise to a number of astrophysical phenomena:

- reflection nebulae (similar to looking at street-light through fog)
- light echos
- polarization
- Ly-\(\alpha\) forest in quasar spectra

The scattering cross-section, \(\sigma_s\), is a hypothetical area \([\sigma_s] = \text{cm}^2\) which describes the likelihood of a photon being scattered by a target (typically an electron or atom). In general, the scattering cross-section is different from the geometrical cross-section of the particle, and it depends upon the frequency of the photon, and on the details of the interaction (see below).

Scattering interactions are categorized as either elastic (coherent), where the photon energy is unchanged by the scattering event, or inelastic (incoherent), where the photon energy changes.
Elastic scattering comes in three forms:

- **Thomson scattering** \( \gamma + e \rightarrow \gamma + e \)
- **Resonant scattering** \( \gamma + X \rightarrow X^+ \rightarrow \gamma + X \)
- **Rayleigh scattering** \( \gamma + X \rightarrow \gamma + X \)

Here \( \gamma \) indicates a photon, \( e \) a free electron, \( X \) an atom or ion, and \( X^+ \) an excited state of \( X \).

Inelastic scattering comes in two forms:

- **Compton scattering** \( \gamma + e \rightarrow \gamma' + e' \)
- **Fluorescence** \( \gamma + X \rightarrow X^{++} \rightarrow \gamma' + X^+ \rightarrow \gamma' + \gamma'' + X \)

Here accents indicate that the particle has a different energy (i.e., \( \gamma' \) is a photon with a different energy than \( \gamma \)), and \( X^{++} \) indicates a higher-excited state of \( X \) than \( X^+ \).

In what follows we discuss each of these five processes in more detail.

**Thomson scattering:** is the elastic (coherent) scattering of electromagnetic radiation by a free charged particle, as described by classical electromagnetism. It is the low-energy limit of **Compton scattering** in which the particle kinetic energy and photon frequency are the same before and after the scattering. In Thomson scattering the electric field of the incident wave (photon) accelerates the charged particle, causing it, in turn, to emit radiation at the same frequency as the incident wave, and thus the wave is scattered. The particle will move in the direction of the oscillating electric field, resulting in **electromagnetic dipole radiation** that appears **polarized** unless viewed in the forward or backward scattered directions (see Fig. 23).

The cross-section for Thomson scattering is the **Thomson cross section:**

\[
\sigma_s = \sigma_T = \frac{8\pi}{3} r_e^2 = \frac{8\pi e^4}{3m_e^2 c^4} \approx 6.65 \times 10^{-25} \text{cm}^2
\]
Figure 23: Illustration of how Thomson scattering causes polarization in the directions perpendicular to that of the incoming EM radiation. The incoming EM wave causes the electron to oscillate in the direction of the oscillation of the $\vec{E}$-field. This acceleration of the electrical charge results in the emission of dipolar EM radiation.

Note that this cross section is independent of wavelength!

In the quantum mechanical view of radiation, electromagnetic waves are made up of photons which carry both energy ($h\nu$) and momentum ($h\nu/c$). This implies that during scattering the photon exchanges momentum with the electron, causing the latter to recoil. This recoil is negligible, until the energy of the incident photon becomes comparable to the rest-mass energy of the electron, in which case Thomson scattering becomes Compton scattering.

**Compton scattering:** is an inelastic scattering of a photon by a free charged particle, usually an electron. It results in a decrease of the photon’s energy/momentum (increase in wavelength), called the Compton effect. Part of the energy/momentum of the photon is transferred to the scattering electron (‘recoil’). In the case of scattering off of electrons at rest, the Compton effect is only important for high-energy photons with $E_\gamma > m_e c^2 \sim 0.511$ MeV (X-ray and/or gamma ray photons).
Figure 24: The Klein-Nishina cross section for Compton scattering. As long as $h\nu \ll m_e c^2$ one is in the Thomson scattering regime, and $\sigma_s = \sigma_T$. However, once the photon energy becomes comparable to the rest-mass energy of the electron, Compton scattering takes over, and the cross-section (now called the Klein-Nishina cross-section), starts to drop as $\nu^{-1}$.

Because of the recoil effect, the energy of the outgoing photon is

$$E'_\gamma = \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e c^2}(1 - \cos \theta)}$$

where $\theta$ is the angle between incident and outgoing photon. This can also be written as

$$\lambda' - \lambda = \lambda_C (1 - \cos \theta)$$

which expresses that Compton scattering increases the wavelength of the photon by of order the Compton wavelength $\lambda_C = \frac{h}{m_e c} \sim 2.43 \times 10^{-10}\text{cm}$. If $\lambda \gg \lambda_C$ such a shift is negligible, and we are in the regime that is well described by Thomson scattering.

Compton scattering is a quantum-mechanical process. The quantum aspect also influences the actual cross section, which changes from the Thomson cross section, $\sigma_T$, at the low-frequency end, to the Klein-Nishina cross section, $\sigma_{KN}(\nu)$, for $h\nu > m_e c^2$ (see Fig. 24). Note how scattering becomes less efficient for more energetic photons.
So far we have considered the scattering of photons off of electrons at rest. A more realistic treatment takes into account that electrons are also moving, and may do so relativistically. This adds the possibility of the electron giving some of its kinetic energy to the photon, which results in Inverse Compton (IC) scattering.

Whether the photon loses (Compton scattering) or gains (IC scattering) energy depends on the energies of the photon and electron. Without derivation, the average energy change of the photon per Compton scattering against electrons of temperature $T_e = m_e \langle v^2 \rangle / (3 k_B)$ is

$$\left\langle \frac{\Delta E_\gamma}{E_\gamma} \right\rangle = \frac{4 k_B T_e - h \nu}{m_e c^2}$$

Hence, we have that

- $h \nu > 4 k_B T_e$: Compton effect; photon loses energy to electron
- $h \nu = 4 k_B T_e$: No energy exchange
- $h \nu < 4 k_B T_e$: Inverse Compton effect; electron loses energy to photon

As an example, consider ultra-relativistic electrons with $4 k_B T_e \gg h \nu$. In that case it can be shown that Inverse Compton (IC) scattering causes the photons to increase their frequency according to $\nu_{\text{out}} \approx \gamma^2 \nu_{\text{in}}$, where

$$\gamma = \frac{1}{\sqrt{1 - \frac{\nu^2}{c^2}}}$$

is the Lorentz factor. Hence, for ultra-relativistic electrons, which have a large Lorentz factor, the frequency boost of a single IC scattering event can be enormous. It is believed that this process, upscattering of low energy photons by the IC effect, is at work in Active Galactic Nuclei.

Another astrophysical example of IC scattering is the Sunyaev-Zel’dovic (SZ) effect in clusters; the hot (but non-relativistic) electrons of the intra-cluster gas (with a typical electron temperature of $T_e \sim 10^8$ K) upscatter Cosmic Microwave Background (CMB) photons by a small, but non-negligible amount. The result is a comptonization of the energy spectrum of the photons; while Compton scattering maintains photon numbers, it increases their energies, so that they no longer can be fit by a Planck curve. The strength of this Comptonization (typically expressed in
terms of the Compton-y parameter) is measure for the electron pressure \( P_e \propto n_e T_e \) along the line-of-sight through the cluster. Observations of the SZ effect provide a nearly redshift-independent means of detecting galaxy clusters.

**Resonant scattering:** Resonant scattering, also known as line scattering or bound-bound scattering is the scattering of photons off electrons bound to nuclei in atoms or ions. Before we present the quantum mechanical view of this process, it is useful to consider the classical one, in which the electron is viewed as being bound to the nucleus via a spring with a natural, angular frequency, \( \omega_0 = 2\pi\nu_0 \). If the electron is perturbed, it will oscillate at this natural frequency, which will result in the emission of photons of energy \( E_\gamma = h\nu_0 \). This in turn implies energy loss; hence, the bound electron is an example of a damped, harmonic oscillator. The classical damping constant is given by \( \Gamma_{cl} = \omega_0^2 \tau_e \), where \( \tau_e = 2e^2/(3m_e c^2) \sim r_e/c \sim 6.3 \times 10^{-24} \text{s} \). This damping, and the corresponding emission of EM radiation, is the classical analog of spontaneous emission.

Now consider the case of an EM wave of angular frequency, \( \omega \), interacting with the atom/ion. The result is a forced, damped, harmonic oscillator, whose effective cross section is given by

\[
\sigma_s(\omega) = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega^2 \Gamma_{cl} / 2)}
\]

(see Rybicki & Lightman 1979 for a derivation). We can distinguish three regimes:

- **[\( \omega \gg \omega_0 \)]** In this case \( \sigma_s(\omega) = \sigma_T \) and we are in the regime of regular Thomson scattering. The oscillator responds to the high-frequency forcing by adopting the forced frequency; hence, the system behaves as if the electron is free.

- **[\( \omega \simeq \omega_0 \)]** In this case

\[
\sigma_s(\omega) \simeq \sigma_T \frac{\Gamma_{cl}/2}{2\tau_e (\omega - \omega_0)^2 + (\Gamma_{cl}/2)^2}
\]

which corresponds to resonant scattering, in which the cross section is hugely boosted wrt the Thomson case. NOTE: for resonant scattering to be important, it is crucial that spontaneous de-excitation occurs before collisional excitation or de-excitation (otherwise the photon energy is lost, and we are in the realm of absorption, rather than scattering). Typically, this requires sufficiently low densities.
Figure 25: Illustration of how Rayleigh scattering causes the sky to be blue. Because of its strong ($\lambda^{-4}$) wavelength dependence, blue light is much more scattered than red light. This causes the Sun light to appear redder than it really is, an effect that strengthens when the path length through the atmosphere is larger (i.e., at sunrise and sunset). The blue light is typically scattered multiple times before hitting the observer, so that it appears to come from random directions on the sky.

In this case

$$\sigma_s(\omega) \simeq \sigma_T \left( \frac{\omega}{\omega_0} \right)^4$$

which corresponds to Rayleigh scattering, which is characterized by a strong wavelength dependence for the effective cross section of the form $\sigma_s \propto \sigma_T \lambda^{-4}$.

**Rayleigh scattering** results from the electric polarizability of the particles. The oscillating electric field of a light wave acts on the charges within a particle, causing them to move at the same frequency (recall, the forcing frequency in this case is much smaller than the natural frequency). The particle therefore becomes a small radiating dipole whose radiation we see as scattered light.

Rayleigh scattering, and its strong wavelength dependence of $\sigma_s$, is responsible for the fact that the sky appears blue during the day, and for the fact that sunsets turn the sky red (see Fig. 25).
We now turn our attention to a quantum-mechanical view of resonant scattering. The main difference between the classical view (above) and the quantum view (below), is that in the latter there is not one, but many ‘natural frequencies’, \( \nu_{ij} \), corresponding to all the possible energy-level-transitions \( \Delta E_{ij} = h\nu_{ij} \) that correspond to the atom/ion in question.

**Oscillator strength:** With each transition corresponds an oscillator strength, \( f_{ij} \), which is a dimensionless quantity that expresses the ‘strength’ of the \( i \leftrightarrow j \) transition. It expresses the quantum mechanical probability that transition \( i \rightarrow j \) occurs under the incidence of a \( \nu_{ij} \) photon given the quantum-mechanical selection rules, which state the degree to which a certain transition between degrees is allowed. You can think of \( f_{ij} \) as being proportional to the probability that the incidence of a \( \nu_{ij} \) photon results in the corresponding electronic transition.

In the quantum-mechanical view, the three regimes of bound-bound scattering have effective cross sections:

\[
\begin{align*}
\sigma_s &= \sigma_T & \nu \gg \nu_{ij} & \text{Thomson scattering} \\
\sigma_s(\nu) &= \frac{\pi e^2}{mc} f_{ij} \phi_L(\nu) & \nu \simeq \nu_{ij} & \text{Resonant scattering} \\
\sigma_s(\nu) &= \sigma_T f_{ij} \left( \frac{\nu}{\nu_{ij}} \right)^4 & \nu \ll \nu_{ij} & \text{Rayleigh scattering}
\end{align*}
\]

Here \( \phi_L(\nu) \) is the Lorentz profile, which describes the natural line broadening associated with the transition in question. The non-zero width of this Lorentz profile implies that resonant scattering is not perfectly coherent; typically the energy of the outgoing photon will be slightly different from that of the incident photon. The probability distribution for this energy shift is described by \( \phi_L(\nu) \), and originates from the Heisenberg Uncertainty Principle, according to which \( \Delta E \Delta t \geq \hbar/2 \); hence, the uncertainty related to the time it takes for the electron to spontaneously de-excite results in a related ‘uncertainty’ in energy.

Fig. 26 shows the frequency dependence of a (quantum-mechanical) atom/ion. It shows the Rayleigh regime at small \( \nu \), the resonant scattering peaks at a few transition frequencies, and the Thomson regime at large \( \nu \). Note that the height of the various peaks are set by their respective oscillator strengths.
Figure 26: Illustration of the scattering cross section of an atom or ion with at least one bound electron. At high (low) frequency, scattering is in the Thomson (Rayleigh) regime; at specific, intermediate frequencies, set by the transition energies of the atom/ion, resonant scattering dominates; the profiles are Lorentz profiles, and reflect the natural line broadening. The relative heights of the peaks are set by their oscillator strengths. NOTE: figure is not to scale; typically the cross section for resonant scattering is orders of magnitude larger than the Thomson cross section.
An important, astrophysical example of resonant scattering is the Ly-α forest in the spectra of (high-redshift) quasars. Redward of the quasar’s Ly-α emission line one typically observes a ‘forest’ of ‘absorption lines’, called the Ly-α forest (see Fig. 27). These arise from resonant scattering in the Ly-α line of neutral hydrogen in gas clouds along the line-of-sight between the quasar and observer. NOTE: although these are called ‘absorption lines’ they really are a manifestation of (resonant) scattering.

**Fluorescence:** fluorescence is an inelastic (incoherent) scattering mechanism, in which a photon excites an electron by at least two energy states, and the spontaneous de-excitation occurs to one or more of the intermediate energy levels. Consequently, the photon that is ‘scattered’ (i.e., absorbed and re-emitted) has changed its energy.
Absorption: The absorption of photons can have three effects:

- **heating of the absorbing medium** (heating of dust grains, or excitation of gas followed by collisional de-excitation)
- **acceleration of absorbing medium** (radiation pressure)
- **change of state of absorbing medium** (ionization, sublimation or dissociation)

Note that ionization (transition from neutral to ionized), sublimation (transition from solid to gas) and dissociation (transition from molecular to atomic) can also occur as a consequence of particle collisions. Therefore one often uses terms such as *photo-ionization* and *collisional ionization* to distinguish between these.

**Photoionization:** Photoionization is the process in which an atom is ionized by the absorption of a photon. For hydrogen, this is

\[ \text{HI} + \gamma \rightarrow \text{p} + \text{e}, \]

where HI denotes a neutral hydrogen atom. The photoionization rate, \( \Gamma_{\gamma, \text{H}} \), is proportional to the number density of ionizing photons and to the photoionization cross section, \( \sigma_{\pi}(\nu) \), according to:

\[
\Gamma_{\gamma, \text{H}} = \int_{\nu_t}^{\infty} c \sigma_{\pi}(\nu) N_\gamma(\nu) \, d\nu
\]

where \( \nu_t \) is the threshold frequency for ionization (corresponding to 13.6eV in the case of hydrogen). \( N_\gamma(\nu) \, d\nu \) is the number density of photons with frequencies in the range \( \nu \) to \( \nu + d\nu \), and is related to the energy flux of the radiation field, \( J(\nu) \), by
\[ N_{\gamma}(\nu) = \frac{4 \pi J(\nu)}{c \hbar \nu}. \]

The photoionization cross sections can be obtained from quantum electrodynamics by calculating the bound-free transition probability of an atom in a radiation field (see e.g., Rybicki & Lightman 1979).

**Recombination:** Recombination is the process by which an ion recombines with an electron. For hydrogen ions (i.e. protons), the process is

\[ p + e \rightarrow HI + \gamma. \]

For hydrogen (or a hydrogenic ion, i.e., an ion with a single electron), the recombination cross section to form an atom (or ion) at level \( n \), \( \sigma_{\text{rec}}(v, n) \), is related to the corresponding photoionization cross section by the Milne relation:

\[
\sigma_{\text{rec}}(v, n) = \frac{g_n}{g_{n+1}} \left( \frac{h \nu}{m_e c v} \right)^2 \sigma_{\text{pi}}(\nu, n),
\]

where \( g_n = 2n^2 \) is the statistical weight of energy level \( n \) and \( \nu \) and \( v \) are related by \( \frac{m_e v^2}{2} = h(\nu - \nu_n) \), with \( h\nu_n \) the threshold energy required to ionize an atom whose electron sits in energy state \( n \). The recombination coefficient for a given level \( n \) is the product of the capture cross section and velocity, \( \sigma_{\text{rec}}(v, n) v \), averaged over the velocity distribution \( f(v) \). For an optically thin gas where all photons produced by recombination can escape without being absorbed, the total recombination coefficient is the sum over all \( n \):

\[
\alpha_A = \sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \int \sigma_{\text{rec}}(v, n) v f(v) \, dv.
\]

This is called the Case A recombination coefficient, to distinguish it from the Case B recombination in an optically thick gas. In Case B, recombinations to the ground level generate ionizing photons that are absorbed by the gas, so that they do not contribute to the overall ionization state of the gas. It is easy to see that the Case B recombination coefficient is \( \alpha_B = \alpha_A - \alpha_1 \).
**Strömgren sphere:** A sphere of ionized hydrogen (H II) around an ionizing source (e.g., AGN, O or B star, etc.). Ionization of hydrogen (from the ground state) requires a photon energy of at least 13.6 eV, which implies UV photons. In a (partially) ionized medium, electrons and nuclei recombine to produce neutral atoms. The region around an ionizing source will ultimately establish ionization equilibrium in which the number of ionizations is equal to the number of recombinations.

Consider an ionizing source in a uniform medium of pure hydrogen. Let $\dot{N}_{\text{ion}}$ be the number of ionizing photons produced per second. The corresponding recombination rate is given by

$$\dot{N}_{\text{rec}} = n_e n_p \alpha_{\text{rec}} V = n_e^2 \alpha_B \frac{4}{3} \pi R_s^3$$

where we have used that, for a pure hydrogen gas, $n_e = n_p$, and $R_s$ is the radius of the Strömgren sphere (i.e., the radius of the sphere that is going to be ionized), which can be written as

$$R_s = \left( \frac{3 \dot{N}_{\text{ion}}}{4 \pi \alpha_B n_e^2} \right)^{1/3}$$

Using that the luminosity of the ionizing source, $L_*$, is related to its surface intensity, $I_*$, according to

$$L_* = 4 \pi R_*^2 F_* = 4 \pi^2 R_*^2 I_*$$

where $R_*$ is the radius of the ionizing source (i.e., an O-star) and we have used that $F_* = \pi I_*$ (see Appendix J). Hence, we have that

$$\dot{N}_{\text{ion}} = 4 \pi^2 R_*^2 \int_{\nu_*}^{\infty} \frac{B_\nu(T)}{h \nu} \, d\nu = \frac{\pi L_*}{\sigma_{\text{SB}} T_{\text{eff}}^4} \int_{\nu_*}^{\infty} \frac{B_\nu(T)}{h \nu} \, d\nu$$

where we have assumed that the ionizing source is a Black Body of temperature $T$, and, in the second part, that $L_* = 4\pi R_*^2 \sigma_{\text{SB}} T_{\text{eff}}^4$.

Thus, by measuring the luminosity and effective temperature of a star, and the radius of its Strömgren sphere, one can infer the (electron) density of its surroundings.
As we have seen, there are numerous processes by which a photon can interact with matter. It is useful to define the mean-free path, $l$, for a photon and the related opacity and optical depth.

**Opacity:** a measure for the impenetrability to electro-magnetic radiation due to the combined effect of scattering and absorption. If the opacity is caused by dust we call it extinction.

**Optical Depth:** the dimensionless parameter, $\tau_\nu$, describing the opacity/extinction at frequency $\nu$. In particular, the infinitesimal increase in optical depth along a line of sight, $d\tau_\nu$, is related to the infinitesimal path length $dl$ according to

$$d\tau_\nu = \sigma_\nu n\,dl = \kappa_\nu \rho\,dl = \alpha_\nu \,dl$$

Here $\sigma_\nu$ is the effective cross section ($[\sigma_\nu] = \text{cm}^2$), $\kappa_\nu$ is the mass absorption coefficient ($[\kappa_\nu] = \text{cm}^2\,g^{-1}$), $\alpha_\nu$ is the absorption coefficient ($[\alpha_\nu] = \text{cm}^{-1}$), and $n$ and $\rho$ are the number and mass densities, respectively. The optical depth to a source at distance $d$ is therefore

$$\tau_\nu = \int_0^d d\tau_\nu = \int_0^d \kappa_\nu(l)\rho(l)\,dl$$

The ISM/IGM between source and observer is said to be **optically thick (thin)** if $\tau_\nu > 1$ ($\tau_\nu < 1$).

Opacity/extinction reduces the intensity of a source according to

$$I_{\nu,\text{obs}} = I_{\nu,0} e^{-\tau_\nu}$$

where $I_{\nu,0}$ is the unextincted intensity (i.e., for $\tau_\nu = 0$).
Rosseland Mean Opacities: In the case of stars opacity is crucially important for understanding stellar structure. Opacities within stars are typically expressed in terms of the Rosseland mean opacities, $\bar{\kappa}$, which is a weighted average of $\kappa_\nu$ over frequency. Typically, one finds that $\bar{\kappa} \propto \rho T^{-3.5}$, which is known as Kramer’s opacity law, and is a consequence of the fact that the opacity is dominated by bound-free and/or free-free absorption. A larger opacity implies stronger radiation pressure, which gives rise to the concept of the Eddington luminosity.

Eddington Luminosity: the maximum luminosity a star (or, more general, emitter) can achieve before the star’s radiation pressure starts to exceed the force of gravity.

Consider an outer layer of a star with a thickness $l$ such that $\tau_\nu \simeq \kappa_\nu \rho l = 1$. Then, a photon passing this layer will be absorbed and contribute to radiation pressure. The resulting force exerted on the matter is

$$F_{\text{rad}} = \frac{L}{c}$$

This has to be compared to the gravitational force

$$F_{\text{grav}} = \frac{G M_* m_{\text{layer}}}{r^2}$$

Using that $m_{\text{layer}} = 4\pi r^2 \rho = 4\pi r^2 / \kappa_\nu$, we find that $F_{\text{rad}} = F_{\text{grav}}$ if the luminosity is equal to

$$L_{\text{Edd}} = \frac{4\pi G M_* c}{\kappa_\nu}$$

which is called the Eddington luminosity. Stars with $L > L_{\text{Edd}}$ cannot exist, as they would blow themselves apart ($F_{\text{rad}} > F_{\text{grav}}$). The most massive stars known have luminosities that are very close to their Eddington luminosity.

Since the luminosities of AGN (supermassive black holes with accretion disks) are set by their accretion rate, the same argument implies an upper limit to the accretion rate of AGN, known as the Eddington limit.
Figure 28: Empirical extinction laws, defined in terms of the ratio of color excesses, for the Milky Way (MW), the Large Magellanic Cloud (LMC) and the Small Magellanic Cloud (SMC). Note the strong feature around 2100 Å in the MW extinction law, believed to be due to graphite dust grains.

**Extinction by Dust:** Dust grains can scatter and absorb photons. Their ability to do so depends on (i) grain size, (ii) grain composition, and (iii) the presence of a magnetic field, which can cause grain alignment. Observationally, the extinction in the V-band is defined by

\[ A_V \equiv -2.5 \log \left( \frac{f_V}{f_{V,0}} \right) = -2.5 \log \left( \frac{I_V}{I_{V,0}} \right) \]

where the subscript zero refers to the unextincted flux/intensity. Using that \( I_V = I_{V,0} e^{-\tau_V} \) we have that

\[ A_V = 1.086 \tau_V \]

More generally, \( A_\lambda = 1.086 \tau_\lambda \); hence, an optical depth of unity roughly corresponds to an extinction of one magnitude.

**Reddening:** In addition to extinction, dust also causes reddening, due to the fact that dust extinction is more effective at shorter (bluer) wavelengths.
Color Excess: $E(B - V) \equiv A_B - A_V$, which can also be defined for any other wavebands.

**Extinction law:** conventionally, dust extinction is expressed in terms of an *empirical extinction law*:

$$k(\lambda) \equiv \frac{A_\lambda}{E(B - V)} \equiv R_V \frac{A_\lambda}{A_V}$$

where

$$R_V \equiv \frac{A_V}{E(B - V)}$$

is a quantity that is insensitive to the total amount of extinction; rather it expresses a property of the extinction law. Empirically, dust in the Milky Way seems to have $R_V \simeq 3.1$, while the dust in the Small Magellanic Cloud (SMC) is better characterized by $R_V \simeq 2.7$. Dust extinction law is not universal; rather, it is believed to depend on the local UV flux and the metallicity, among others (see Fig. 28).

Theoretical attempts to model the extinction curve have shown that dust comes in two varieties, graphites and silicates, while the grain-size distribution is well fit by $dN/da \propto a^{-3.5}$ and covers the range from $\sim 0.005\mu m$ to $\sim 0.25\mu m$. Note that for radiation with $\lambda > a_{\text{max}} \simeq 2500 \text{ Å}$ dust mainly causes Rayleigh scattering.
Consider an incoming signal of specific intensity $I_{\nu,0}$ passing through a cloud (i.e., any gaseous region). As the radiation transits a small path length $dr$ through the cloud, its specific intensity changes by $dI_{\nu} = dI_{\nu,\text{loss}} + dI_{\nu,\text{gain}}$. The loss-term describes the combined effect of scattering and absorption, which remove photons from the line-of-sight, while the gain-term describes all processes that add photons to the line-of-sight; these include all emission processes from the gas itself, as well as scattering of photons from any direction into the line-of-sight.

In what follows we ignore the contribution of scattering to $dI_{\nu,\text{gain}}$, as this term makes solving the equation of radiative transfer much more complicated. We will briefly comments on that below, but for now the only process that is assumed to contribute to $dI_{\nu,\text{gain}}$ are emission processes from the gas.

It is useful to define the following two coefficients:

- **Absorption coefficient**, $\alpha_{\nu} = n \sigma_{\nu} = \rho \kappa_{\nu}$, which has units $[\alpha_{\nu}] = \text{cm}^{-1}$.

- **Emission coefficient**, $j_{\nu}$, defined as the energy emitted per unit time, per unit volume, per unit frequency, per unit solid angle (i.e., $dE = j_{\nu} \, dt \, dV \, d\nu \, d\Omega$, and thus $[j_{\nu}] = \text{erg s}^{-1} \, \text{cm}^{-3} \, \text{Hz}^{-1} \, \text{sr}^{-1}$).

In terms of these two coefficients, the **equation of radiative transfer** can be written in either of the following two forms:

\[
\frac{dI_{\nu}}{dr} = -\alpha_{\nu} \, I_{\nu} + j_{\nu} \quad \text{(form I)}
\]

\[
\frac{dI_{\nu}}{d\tau_{\nu}} = -I_{\nu} + S_{\nu} \quad \text{(form II)}
\]
Here $S_\nu \equiv j_\nu / \alpha_\nu$ is called the **source function**, and has units of specific intensity (i.e., $[S_\nu] = \text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1} \text{sr}^{-1}$). In order to derive form II from form I, recall that $d\tau_\nu = \alpha_\nu \, dr$ (see Chapter 22).

**NOTE**: we use the convention of $\tau_\nu$ increasing *from* the source *towards* the observer. Some textbooks adopt the opposite convention, which results in some sign differences.

To get some insight, we now consider a number of different cases:

**Case A** | **No Cloud**
---
In this case, there is no absorption ($\alpha_\nu = 0$) or emission ($j_\nu = 0$), other than the emission from the background source. Hence, we have that

$$\frac{dI_\nu}{dr} = 0 \quad \Rightarrow \quad I_\nu = I_{\nu,0}$$

which expresses that intensity is a conserved quantity in vacuum.

**Case B** | **Absorption Only**
---
In this case, the cloud absorbs background radiation, but does not emit anything ($j_\nu = S_\nu = 0$). Hence,

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu$$

which is easily solved to yield

$$I_\nu = I_{\nu,0} e^{-\tau_\nu}$$

which is the expected result (see Chapter 22).

**Case C** | **Emission Only**
---
If the cloud does not absorb ($\alpha_\nu = 0$) but does emit we have

$$\frac{dI_\nu}{dr} = j_\nu \quad \Rightarrow \quad I_\nu = I_{\nu,0} + \int_0^l j_\nu(r) \, dr$$

where $l$ is the size of the cloud along the line-of-sight. This equation simply expresses that the *increase* of intensity is equal to the emission coefficient integrated along the line-of-sight.
Case D  Cloud in Thermal Equilibrium w/o Background Source

Consider a cloud in TE, i.e., specified by a single temperature $T$ (kinetic temperature is equal to radiation temperature). Since in a system in TE there can be no net transport of energy, we have that

$$\frac{dI_\nu}{dr} = -\alpha_\nu I_\nu + j_\nu = 0 \quad \Rightarrow \quad I_\nu = \frac{j_\nu}{\alpha_\nu} = S_\nu$$

Since the observer must see a black body of temperature $T$, we also have that $I_\nu = B_\nu(T)$ (i.e., the intensity is given by a Planck curve corresponding to the temperature of the cloud), and we thus have that

$$I_\nu = \alpha_\nu B_\nu(T)$$

The latter of these equivalent relations is sometimes called Kirchoff’s law, and simply expresses that a black body needs to establish a balance between emission and absorption (i.e., $B_\nu(T) = j_\nu/\alpha_\nu$).

Case E  Emission & Absorption (formal solution)

Consider the general case with both emission and absorption (but where we ignore the fact that scattering can scatter photons into my line of sight). Starting from form II of the equation of radiative transfer, multiplying both sides with $e^{\tau_\nu}$, we obtain that

$$\frac{d\tilde{I}_\nu}{d\tau_\nu} = -\tilde{S}_\nu$$

where $\tilde{I}_\nu \equiv I_\nu e^{\tau_\nu}$ and $\tilde{S}_\nu \equiv S_\nu e^{\tau_\nu}$. We can rewrite the above differential equation as

$$\int_{I_{\nu,0}}^{I_\nu} d\tilde{I}_\nu = \int_0^{\tau_\nu} \tilde{S}_\nu d\tau_\nu$$

Using that $\tilde{I}_{\nu,0} = I_{\nu,0} e^0 = I_{\nu,0}$ the solution to this simple integral equation is

$$I_\nu = I_{\nu,0} e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu) e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu$$

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where $\tau_\nu$ is the total optical depth along the line of sight (i.e., through the cloud). The above is the formal solution, which, under the simplifying assumption that the source function is constant along the line of sight reduces to

$$I_\nu = I_{\nu,0} e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})$$

The first term expresses the attenuation of the background signal, the second term expresses the added signal due to the emission from the cloud, while the third term describes the cloud’s self-absorption.

Using the above formal solution to the equation of radiative transfer, we have the following two extremes:

$$\tau_\nu \gg 1 \quad \Rightarrow \quad I_\nu = S_\nu$$

$$\tau_\nu \ll 1 \quad \Rightarrow \quad I_\nu = I_{\nu,0} (1 - \tau_\nu) + S_\nu \tau_\nu$$

where, for the latter case, we have used the Taylor series expansion for the exponential. In the high optical depth case, the observer just ‘sees’ the outer layers of the cloud, and therefore the observed intensity is simply the source function of the cloud (the observed signal contains no contribution from the background source). In the small optical depth limit, the contribution from the cloud is suppressed by a factor $\tau_\nu$, while that from the background source is attenuated by a factor $(1 - \tau_\nu)$. Using that $S_\nu = j_\nu / \alpha_\nu$ and $\tau_\nu = \alpha_\nu l$ (if the absorption coefficient is constant throughout the cloud), we see that $\tau_\nu S_\nu = j_\nu l$; in other words, the contribution from the cloud itself is simply its emission coefficient (assumed constant throughout the cloud) multiplied with the pathlength through the cloud.

To get some further insight into the source function and radiative transfer in general, consider form II of the radiative transfer equation. If $I_\nu > S_\nu$ then $dI_\nu/d\tau_\nu < 0$, so that the specific intensity decreases along the line of sight. If, on the other hand, $I_\nu < S_\nu$ then $dI_\nu/d\tau_\nu > 0$, indicating that the specific intensity increases along the line of sight. Hence, $I_\nu$ tends towards $S_\nu$. If the optical depth of the cloud is sufficiently large than this ‘tendency’ will succeed, and $I_\nu = S_\nu$.

An important special case of the general solution derived above is if the cloud is in local thermal equilibrium (LTE). This is very often the case, since over the mean free path of the photons, every system will tend to be in LTE, unless it was recently
disturbed and has not yet been able to equilibrate. In the case of LTE, we have that, over a patch smaller than or equal to the mean free path of the photons, we have that 

\[ S_\nu \equiv \frac{j_\nu}{\alpha_\nu} = B_\nu(T), \]

where \( T \) is the kinetic temperature (= radiation temperature) of the patch.

The solution to the equation of radiative transfer now is

\[
I_\nu = I_{\nu,0} e^{-\tau_\nu} + B_\nu(T) \left[ 1 - e^{-\tau_\nu} \right]
\]

Note that \( I_\nu \) is not constant throughout the cloud, as was the case for a cloud in TE. In the case of LTE, however, there can be a non-zero gradient \( dI_\nu/dr \).

Before we interpret this result in detail, it is important to distinguish

**Blackbody Radiation:** \( I_\nu = B_\nu(T) \)

**Thermal Radiation:** \( S_\nu = B_\nu(T) \)

**NOTE:** thermal radiation is radiation emitted by matter in thermal equilibrium.

Keeping this difference in mind, we now look at the solution to our equation of radiative transfer for a cloud in LTE at its two extremes:

\[
\tau_\nu \gg 1 \quad \Rightarrow \quad I_\nu = B_\nu(T)
\]

\[
\tau_\nu \ll 1 \quad \Rightarrow \quad I_\nu = I_{\nu,0} \left( 1 - \tau_\nu \right) + B_\nu(T) \tau_\nu
\]

The former expresses that *an optically thick cloud in LTE emits black body radiation*. This is characterized by the fact that (i) if there is a background source, you can’t see it, (ii) you can look into the source only for about one mean free path of the photons (which is much smaller than the size of the source), and (iii) the only information available to an observer is the temperature of the cloud (the observed intensity is a Planck curve of temperature \( T \)).

A good example of gas clouds in LTE are stars!

In the optically thin limit, the observed intensity depends on the background source (if present), and depends on *both* the temperature (sets source function) and density (sets optical depth) of the cloud (recall that \( \tau_\nu \propto \kappa_\nu \rho l \)).

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In the case without background source we have that

\[ I_\nu = \begin{cases} B_\nu(T) & \text{if } \tau_\nu \gg 1 \\ \tau_\nu B_\nu(T) & \text{if } \tau_\nu \ll 1 \end{cases} \]

Note that this is different from case D, in which we considered a cloud in TE without background source. In that case we obtained that \( I_\nu = B_\nu(T) \) independent of \( \tau_\nu \). In the case of LTE, however, there are radial gradients, which are responsible for diminishing the intensity by the optical depth in the case where \( \tau_\nu \ll 1 \). This may seem somewhat ‘counter-intuitive’, as it indicates that a cloud of larger optical depth is more intense!!! To understand this, consider the limit \( \tau_\nu \to 0 \). In this case all photons pass through the cloud with zero probability to be absorbed/scattered. In this situation, there is simply no way to establish an equilibrium between emission and absorption required for the establishment of a black body; or, put differently, if there is no absorption, there is no emission either (after all, we are in LTE), and thus, \( I_\nu = 0 \).

Based on the above, we have that, in the case of a cloud in LTE without background source, \( I_\nu \leq B_\nu(T) \), where \( T \) is the temperature of the cloud. If we express the intensity in terms of the brightness temperature we have that \( T_{B,\nu} \leq T \). Hence, for a cloud in LTE without background source the observed brightness temperature is a lower limit on the kinetic temperature of the cloud.

**What about scattering?** In the most general case, any element in the cloud receives radiation coming from all \( 4\pi \) steradians, and a certain fraction of that radiation will be scattered into the line-of-sight of an observer.

In general, the scattering can (will) be non-isotropic (e.g., Thomson scattering) and incoherent (e.g., Compton scattering or resonant scattering), and the final equation of radiative transfer can only be solved numerically.

In the simplified case of isotropic, coherent scattering the corresponding emission coefficient can be found by simply equating the power absorbed per unit volume to that emitted (for each frequency);

\[ j_{\nu,\text{scat}} = \alpha_{\nu,\text{scat}} J_{\nu} \]
where $\alpha_{\nu,\text{scat}}$ is the absorption coefficient of the scattering processes, while
\[
J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega
\]
is the mean intensity, averaged over all $4\pi$ steradian.

The source function due to scattering is then simply
\[
S_\nu \equiv \frac{\dot{J}_\nu}{\alpha_{\nu,\text{scat}}} = J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega
\]
Hence, the source function due to isotropic, coherent scattering is simply the mean intensity.

The radiative transfer equation for pure scattering (no background source, and no emission) is
\[
\frac{dI_\nu}{dr} = -\alpha_{\nu,\text{scat}} \left( I_\nu - J_\nu \right)
\]
Even this oversimplified case of pure isotropic, coherent scattering is not easily solved. Since $J_\nu$ involves an integration (over all $4\pi$ steradian), the above equation is an integro-differential equation, which are extremely difficult to solve in general; one typically has to resort to numerical methods (see Rybicki & Lightman 1979 for more details).

**Observability of Emission & Absorption Lines:** Consider a cloud in front of some background source. Assume the cloud is in LTE at temperature $T$. Assume that $\alpha_\nu$ is only non-zero at a specific frequency, $\nu_1$, corresponding to some electron transition. Given that resonant scattering is typically orders of magnitude more efficient than other scattering mechanisms, this is a reasonable approximation. The intensity observed is
\[
I_\nu = I_{\nu,0} e^{-\tau_\nu} + B_{\nu_1}(T) \left[ 1 - e^{-\tau_{\nu_1}} \right]
\]
At all frequencies other than $\nu_1$ we have $\tau_\nu = 0$, and thus $I_\nu = I_{\nu,0}$. Now assume that the observer sees an absorption line at $\nu = \nu_1$. This implies that
\[
I_{\nu_1} = I_{\nu_1,0} e^{-\tau_{\nu_1}} + B_{\nu_1}(T) \left[ 1 - e^{-\tau_{\nu_1}} \right] < I_{\nu_1,0}
\]
while in the case of an emission line
\[
I_{\nu_1} = I_{\nu_1,0} e^{-\tau_{\nu_1}} + B_{\nu_1}(T) \left[ 1 - e^{-\tau_{\nu_1}} \right] > I_{\nu_1,0}
\]
Rearranging, we then have that

\[
\begin{array}{l}
\text{Absorption Line: } B_\nu(T) < I_{\nu,0} \quad T < T_{B,\nu} \\
\text{Emission Line: } B_\nu(T) > I_{\nu,0} \quad T > T_{B,\nu}
\end{array}
\]

where \( T \) is the (kinetic) temperature of the cloud, and \( T_{B,\nu} \) is the \textbf{brightness temperature} of the background source, at the frequency of the line. Hence, if the cloud is colder (hotter) than the source, an absorption (emission) line will arise. In the case of no background source, we effectively have that \( T_{B,\nu} = 0 \), and the cloud will thus reveal an emission line. In the case where \( T = T_{B,\nu} \) no line will be visible, independent of the optical depth of the cloud!
Continuum radiation is any radiation that forms a continuous spectrum and is not restricted to a narrow frequency range. In what follows we briefly describe five continuum emission mechanisms:

- Thermal (Black Body) Radiation
- Bremsstrahlung (free-free emission)
- Recombination (free-bound emission)
- Two-Photon emission
- Synchrotron emission

In general, the way to proceed is to ‘derive’ the emission coefficient, $j_\nu$, the absorption coefficient, $\alpha_\nu$, and then use the equation of radiative transfer to compute the specific intensity, $I_\nu$, (i.e., the ‘spectrum’), for a cloud of gas emitting continuum radiation using any one of those mechanisms.

First some general remarks: when talking about continuum processes it is important to distinguish thermal emission, in which the radiation is generated by the thermal motion of charged particles and in which the intensity therefore depends (at least) on temperature, i.e., $I_\nu = I_\nu(T, ..)$, from non-thermal emission, which is everything else.

Examples of thermal continuum emission are black body radiation and (thermal) bremsstrahlung, while synchrotron radiation is an example of non-thermal emission. Another non-thermal continuum mechanism is inverse compton radiation. However, since this is basically an incoherent photon-scattering mechanism, rather than a photon-production mechanism, we will not discuss IC scattering any further here (see Chapter 20 instead).
Characteristics of Thermal Continuum Emission:

- **Low Brightness Temperatures:** Since one rarely encounters gases with kinetic temperatures $T > 10^7 - 10^8 K$, and since $T_B \leq T$ (see Chapter 23), if the brightness temperature of the radiation exceeds $\sim 10^8 K$ it is most likely non-thermal in origin (or has experienced IC scattering).

- **No Polarization:** Since there is no particular directionality to the thermal motion of particles, thermal emission is essentially unpolarized. In other words, if emission is found to be polarized, it is either non-thermal, or the signal became polarized after it was emitted (i.e., via Thomson scattering).

**Thermal Radiation & Black Body Radiation:** Thermal radiation is the continuum emission arising from particles colliding, which causes acceleration of charges (atoms typically have electric or magnetic dipole moments, and colliding those results in the emission of photons). This thermal radiation tries to establish thermal equilibrium with the matter that produces it via photon-matter interactions. If thermal equilibrium is established (locally), then the source function $S_\nu \equiv j_\nu / \alpha_\nu = B_\nu(T)$ (Kirchoff’s law).

As we have seen in the previous Chapter;

$$I_\nu = \begin{cases} B_\nu(T) & \text{if } \tau_\nu \gg 1 \\ \tau_\nu B_\nu(T) & \text{if } \tau_\nu \ll 1 \end{cases}$$

where $\tau_\nu = \alpha_\nu l$ is the optical depth through the cloud, which has a dimension $l$ along the line-of-sight.

**Free-free emission (Bremsstrahlung):** Bremsstrahlung (German for ‘braking radiation’) arises when a charged particle (i.e., an electron) is accelerated though the Coulomb interaction with another charged particle (i.e., an ion of charge $Ze$). Effectively what happens is that the two charges make up an electric dipole which, due to the motion of the charges, is time variable. A variable dipole is basically an antenna, and emits electromagnetic waves. The energy in these EM waves (photons) emitted is lost to the electron, which therefore loses (kinetic) energy (the electron is ‘braking’).

It is fairly straightforward to compute the amount of energy radiated by a single
electron moving with velocity \( v \) when experiencing a Coulomb interaction with a charge \( Z e \) over an impact parameter \( b \) (see Rybicki & Lightman 1979 for a detailed derivation).

The next step is to integrate over all possible impact parameters. This are all impact parameters \( b > b_{\text{min}} \), where from a classical perspective \( b_{\text{min}} \) is set by the requirement that the kinetic energy of the electron, \( E_k = \frac{1}{2} m_e v^2 \), is larger than the binding energy, \( E_b = Z e^2 / b \) (otherwise we are in the regime of recombination; see below). However, there are some quantum mechanical corrections one needs to make to this \( b_{\text{min}} \) which arise from Heisenberg’s Uncertainty Principle (\( \Delta x \Delta p \geq \hbar / 2 \)). This correction factor is called the free-free Gaunt factor, \( g_{\text{ff}}(\nu, T_e) \), which is close to unity, and has only a weak frequency dependence. The final step in obtaining the emission coefficient is the integration over the Maxwellian velocity distribution of the electrons, characterized by \( T_e \). The result (in \( \text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1} \text{sr}^{-1} \)) is:

\[
\dot{j}_\nu = 5.44 \times 10^{-39} \left( \frac{Z^2}{T_e^{1/2}} \right) n_e n_i g_{\text{ff}}(\nu, T_e) e^{-\hbar \nu / k_B T_e}
\]

In the case of a pure (ionized) hydrogen gas, \( Z = 1 \) and \( n_i = n_e \). Upon inspection, it is clear that free-free emission has a flat spectrum \( j_\nu \propto \nu^\alpha \) with \( \alpha \approx 0 \) (controlled by the weak frequency dependence of the Gaunt factor) with an exponential cut-off for \( h \nu > k_B T_e \) (the maximum photon energy is set by the temperature of the electrons). This reveals that a measurement of the exponential cut-off is a direct measure of the electron temperature.

The above emission coefficient tells us the emissive behavior of a pocket of (ionized) gas without allowance for the internal absorption. Accounting for the latter requires radiative transfer. Since Bremsstrahlung arises from collisions, we may use the LTE approximation. Hence, Kirchhoff’s law tells us that \( \alpha_\nu = j_\nu / B_\nu(T) \), which allows us to compute the absorption coefficient, and thus the optical depth \( \tau_\nu = \alpha_\nu l \). Substitution of \( B_\nu(T) \), with \( T = T_e \), yields

\[
\tau_\nu \simeq 3.7 \times 10^8 Z^2 T_e^{-1/2} \nu^{-3} [1 - e^{-h \nu / k_B T_e}] g_{\text{ff}}(\nu, T_e) \mathcal{E}
\]

where

\[
\mathcal{E} \equiv \int n_e^2 \, dl \simeq n_e^2 l
\]
is called the emission measure, and we have assumed that \( n_e = n_i \). Upon inspection, one notices that \( \tau_\nu \propto \nu^{-2} \) (for \( h\nu \ll k_B T_e \)), indicating that the opacity of the cloud increases with decreasing frequency. This opacity arises from free-free absorption, which is simply the inverse process of free-free emission; a photon is absorbed by an electron that is experiencing a Coulomb interaction.

If we now substitute our results in the equation of radiative transfer (without background source),

\[
I_\nu = B_\nu(T) \left[ 1 - e^{-\tau_\nu} \right]
\]

then we obtain that

\[
I_\nu = \begin{cases} 
B_\nu(T_e) & \text{if } \tau_\nu \gg 1 \\
\tau_\nu B_\nu(T) = j_\nu l & \text{if } \tau_\nu \ll 1
\end{cases}
\]

Fig. 29 shows an illustration of a typical free-free emission spectrum: at low frequencies the gas is optically thick, and one probes the Rayleigh-Jeans part of the Planck curve corresponding to the electron temperature (\( I_\nu \propto \nu^2 T_e \)). At intermediate frequencies, where the cloud is optically thin, the spectrum is flat (\( I_\nu \propto \mathcal{E} T_e^{-1/2} \)), and at the high-frequency end there is an exponential cut-off (\( I_\nu \propto \exp[-h\nu/k_B T_e] \)).

**Free-Bound emission (Recombination):** this involves the capture of a free electron by a nucleus into a quantized bound state. Hence, this requires the medium to be ionized, similar to free-free emission, and in general both will occur (complicating the picture). Free-bound emission is basically the same as free-free emission (they have the same emission coefficient, \( j_\nu \)), except that they involve different integration ranges for the impact parameter \( b \), and therefore different Gaunt factors; the free-bound Gaunt factor \( g_{fb}(\nu, T_e) \) has a different temperature dependence than \( g_{ff}(\nu, T_e) \), and also has more ‘structure’ in its frequency dependence; in the limit where the bound state has a large quantum number (i.e., the electron is weakly bound), we have that \( g_{fb} \sim g_{ff} \). However, for more bound states the frequency dependence of \( g_{fb} \) reveals sharp ‘edges’ associated with the discrete bound states.

When \( k_B T_e \gg h\nu \) recombination is negligible (electrons are moving too fast to become bound), and the emission is dominated by the free-free process. At lower electron temperatures (or, equivalently, higher photon frequencies), recombination becomes more and more important, and often will dominate over bremsstrahlung (see Fig. 30).
Two-Photon Emission: two photon emission occurs between bound states in an atom, but it produces continuum emission rather than line emission.

Two photon emission occurs when an electron finds itself in a quantum level for which any downward transition would violate quantum mechanical selection rules. Each transition is therefore highly forbidden. However, there is a non-zero chance that the electron decays under the emission of two, rather than one, photons. Energy conservation guarantees that $\nu_1 + \nu_2 = \nu_{tr} = \Delta E_{tr}/h$, where $\Delta E_{tr}$ is the energy difference associated with the transition. The most probable configuration is the one in which $\nu_1 = \nu_2 = \nu_{tr}/2$, but all configurations that satisfy the above energy conservation are possible; they become less likely the larger $|\nu_1 - \nu_{tr}/2|$, resulting in a ‘continuum’ emission that appears as an extremely broad ‘emission line’. In fact, whereas the number of photons with $0 < \nu < \nu_{tr}/2$ is equal to that with $\nu_{tr}/2 < \nu < \nu_{tr}$,
the latter have more energy (i.e., $E_\gamma = h\nu$). Consequently, the spectral energy distribution, $L_\nu$ (erg s$^{-1}$ Hz$^{-1}$) is skewed towards higher frequency.

For two photon emission to occur, we require that spontaneous emission happens before collisional de-excitation has a chance. Consequently, two-photon emission occurs in low density ionized gas. The strength of the two photon emission depends on the number of particles in the excited states. This in turn depends on the recombination rate; although two-photon emission is quantum-mechanical in nature, it can still be thought of as ‘thermal emission’, and the density dependence is the same as for free-free and free-bound emission (i.e., $j_\nu \propto n_e^2$).

An important example of two-photon emission is associated with the Ly$\alpha$ recombination line, which results from a de-excitation of an electron from the $n = 2$ to $n = 1$ energy level in a Hydrogen atom. As it turns out, the $n = 2$ quantum level consists of both $2s$ and $2p$ states. The transition $2p \rightarrow 1s$ is a permitted transition with $A_{2p\rightarrow1s} = 6.27 \times 10^8$ s$^{-1}$. However, the $2s \rightarrow 1s$ transition is highly forbidden, and has a two-photon-emission rate coefficient of $A_{2s\rightarrow1s} = 8.2$ s$^{-1}$. Although this is orders of magnitude lower than for the $2p \rightarrow 1s$ transition, the two photon emission can still be important in low-density nebulae ($n < \sim 10^4$ cm$^{-3}$).

**Synchrotron & Cyclotron Emission:** A free electron moving in a magnetic field experiences a Lorentz force:

$$\vec{F}_e = e \left( \frac{\vec{v}}{c} \times \vec{B} \right) = \frac{e}{c} v B \sin \phi = \frac{e}{c} v B_{\perp} = \frac{e}{c} v_{\perp} B$$

where $\phi$ is the pitch angle between $\vec{v}$ and $\vec{B}$. If $\phi = 0$ the particle moves along the magnetic field, and the Lorentz force is zero. If $\phi = 90^\circ$ the particle will move in a circle around the magnetic field line, while for $0^\circ < \phi < 90^\circ$ the electron will spiral (‘cork-screw’) around the magnetic field line. In the latter two cases, the electron is being accelerated, which causes the emission of photons. Note that this applies to both electrons and ions. However, since the cyclotron (synchrotron) emission from ions is negligible compared to that from electrons, we will focus on the latter.

If the particle is non-relativistic, then the emission is called cyclotron emission. If, on the other hand, the particles are relativistic, the emission is called synchrotron emission. We will first focus on the former.
Figure 30: Emission spectra of plasmas with solar abundances. The histogram indicates the total spectrum, including line radiation. The spectrum has been binned in order to highlight the relative importance of line radiation. The thick solid line is the total continuum emission, the thin solid line the contribution due to Bremsstrahlung, the dashed line free-bound emission and the dotted line two-photon emission. Note how recombination becomes less and less important when the gas gets hotter. [From Kaastra et al. 2008, Space Science Reviews, 134, 155]
**Cyclotron emission:** the gyrating electron emits dipolar emission that (i) has the frequency of gyration, and (ii) is highly polarized. Depending on the viewing angle the observer can see circular polarization (if line-of-sight is aligned with $\vec{B}$), linear polarization, if line of sight is perpendicular to $\vec{B}$, or elliptical polarization (for any other orientation).

The gyration frequency can be obtained by equating the Lorentz force with the centripetal force:

$$F_e = \frac{e v_\perp}{c} B = \frac{m_e v_\perp^2}{r_0}$$

where $v_\perp = v \sin \phi$, which results in

$$r_0 = \frac{m_e v_\perp c}{e B}$$

where $B = |\vec{B}|$. This is called the **gyration radius** (or gyro-radius). The period of gyration is $T = 2\pi r_0/v_\perp$, which implies a **gyration frequency** (i.e., the frequency of the emitted photons) of

$$\nu_0 = \frac{1}{T} = \frac{e B}{2\pi m_e c}$$

Note that this frequency is **independent of the velocity of the electron**! It only depends on the magnetic field strength $B$;

$$\nu_0 \text{ MHz} = 2.8 \frac{|\vec{B}|}{\text{Gauss}}$$

We thus see that **cyclotron emission** really is line emission, rather than continuum emission. The nature of this line emission is very different though, from ‘normal’ spectral lines which result from quantum transitions within atoms or molecules. Note, though, that if the ‘source’ has a varying magnetic field, then the variance in $B$ will result in a ‘broadening’ of the line, which, if sufficiently large, may appear as ‘continuum emission’.

In principle, observing cyclotron emission immediately yields the magnetic field strength. However, unless $B$ is extremely large, the frequency of the cyclotron emission is extremely low; typical magnetic field strengths in the IGM are of the order of
several $\mu$G, which implies cyclotron frequencies in the few Hz regime. The problem is that such low frequency radiation will not be able to travel through an astrophysical plasma, because the frequency is lower than the \textbf{plasma frequency}, which is the natural frequency of a plasma (see Chapter P2). In addition, the Earth’s ionosphere blocks radiation with a frequency $\nu \lesssim 10$MHz, so that we can only observe cyclotron emission from the Earth’s surface if it originates from objects with $B \gtrsim 3.5$G. For this reason, cyclotron emission is rarely observed, with the exception of the Sun, some of the planets in our Solar System, and an occasional pulsar.

\textbf{Synchrotron Emission:} this is the same as cyclotron emission, but in the limit in which the electrons are relativistic. As we demonstrate below, this has two important effects: it makes the gyration frequency dependent on the energy (velocity) of the electron, and it causes strong beaming of the electron’s dipole emission.

In the relativistic regime, the electron energy becomes $E_e = \gamma m_e c^2$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the \textbf{Lorentz factor}. This boosts the gyration radius by a factor
\( \gamma \), and reduces the gyration frequency by \( 1/\gamma \):

\[
\begin{align*}
r_0 &= \frac{\gamma m_e v_\perp c}{e B} \approx \frac{\gamma m_e c^2}{e B} \\
\nu_0 &= \frac{e B}{2\pi \gamma m_e c}
\end{align*}
\]

Note that now the gyration frequency *does* depend on the velocity (energy) of the (relativistic) electrons, which in principle implies that because the electrons will have a distribution in energies, the synchrotron emission is going to be continuum emission. However, you can also see that the gyration frequency is even lower than in the case of cyclotron emission, by a factor \( 1/\gamma \). For the record, Lorentz factors of up to \( \sim 10^{11} \) have been measured, indicating that \( \gamma \) can be extremely large! Hence, if the photon emission were to be at the gyration frequency, we would never be able to see it, because of the plasme-frequency-shielding.

However, the gyration frequency is not the only frequency in this problem. Because of the relativistic motion, the dipole emission from the electron, as seen from the observer’s frame, is highly beamed (see Fig. 31), with an opening angle \( \sim 1/\gamma \) (which can thus be tiny). Consequently, the observer does not have a continuous view of the electron, but only sees EM radiation when the beam sweeps over the line-of-sight. The width of these ‘pulses’ are a factor \( 1/\gamma^3 \) shorter than the gyration period. The corresponding frequency, called the *critical frequency*, is given by

\[
\nu_{\text{crit}} = \frac{3 e}{4 \pi m_e c} \gamma^2 B_\perp
\]

which translates into

\[
\frac{\nu_{\text{crit}}}{\text{MHz}} = 4.2 \gamma^2 \frac{B_\perp}{\text{Gauss}}
\]

So although the gyration frequency will be small, the critical frequency can be extremely large. This critical frequency corresponds to the shortest time period (the pulse duration), and therefore represents the largest frequency, above which the emission is negligible. The longest time period, which is related to the gyration period, determines the *fundamental frequency*

\[
\frac{\nu_t}{\text{MHz}} = 2.8 \frac{|\vec{B}|}{\gamma \sin^2 \phi \text{ Gauss}}
\]

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Figure 32: Specific intensity of synchrotron emission, including the effect of synchrotron self absorption at low frequencies, where the optical depth exceeds unity.

The emission spectrum due to synchrotron radiation will contain this fundamental frequency plus all its harmonics up to $\nu_{\text{crit}}$. Since these harmonics are extremely closely spaced (after all, the gyration frequency is extremely small), the synchrotron spectrum for one value of $\gamma$ looks essentially continuum. When taking the $\gamma$-distribution into account (which is related to the energy distribution of the relativistic electrons), the distribution becomes truly continuum, and the critical and fundamental frequencies no longer can be discerned (because they depend on $\gamma$).

After integrating over the energy distribution of the relativistic electrons, which typically has a power-law distribution $N(E) \propto E^{-\Gamma}$ one obtains the following emission and absorption coefficients:

$$ j_\nu \propto B_{\perp}^{(\Gamma+1)/2} \nu^{-\Gamma+1/2} $$

$$ \alpha_\nu \propto B_{\perp}^{(\Gamma+2)/2} \nu^{-\Gamma+4/2} $$

Note that $\alpha_\nu$ describes synchrotron self-absorption. The resulting source func-
tion and optical depth are

\[ S_\nu = \frac{j_\nu}{\alpha_\nu} \propto B_{\perp}^{-1/2} \nu^{5/2} \]
\[ \tau_\nu = \alpha_\nu l \propto B_{\perp}^{(\Gamma+2)/2} \nu^{-(\Gamma+4)/2} l \]

Using that typically \( \Gamma > 0 \), we have that \( \tau_\nu \propto \nu^a \) with \( a < 0 \); synchrotron self-absorption becomes more important at lower frequencies.

Application of the equation of radiative transfer, \( I_\nu = S_\nu (1 - e^{-\tau_\nu}) \), yields

\[ I_\nu = \begin{cases} 
S_\nu \propto \nu^{5/2} & \text{if } \tau_\nu \gg 1 \\
j_\nu l \propto \nu^\alpha & \text{if } \tau_\nu \ll 1
\end{cases} \]

where \( \alpha \equiv -\frac{\Gamma-1}{2} \). Fig. 32 shown an illustration of a typical synchrotron spectrum: at low frequencies, where \( \tau_\nu \gg 1 \), we have that \( I_\nu \propto \nu^{5/2} \), which transits to \( I_\nu \propto \nu^{-(\Gamma-1)/2} \) once the emitting medium becomes optically thin for synchroton self-absorption. Note that there is no cut-off related to the critical frequency, since \( \nu_{\text{crit}} = \nu_{\text{crit}}(E) \).
The following three chapters are NOT part of the course material on which the students will be examined. They are solely provided here for those students interested in learning some basics about plasma physics, which is an intriguing, and important topic, for which we unfortunately have no time in this course.

**P1:** Plasma Orbit Theory .................................................. 181
**P2:** Vlasov Equation and the Two-Fluid Model .......................... 187
**P3:** Magnetohydrodynamics .................................................. 195
Roughly speaking a **plasma** is a fluid in which the constituent particles are charged. More specifically, a plasma is a fluid for which the **plasma parameter** (defined below) $g < 1$.

**NOTE:** in the three chapters discussing plasma physics we adopt the Gaussian system of units (see textbooks on Electro-Magnetism if you are not familiar with this).

In a **neutral fluid**, the interactions (also called collisions) among the particles are **well-separated**, **short-range**, and causing **large deflections**. In between the collisions, the particles travel in straight lines.

In a **plasma**, the interactions are **long-range**, not well-separated (i.e., each particle undergoes **many interactions simultaneously**), and each individual collision typically causes only a **small deflection**. Consequently, the trajectories of particles in a plasma are very different from those in a neutral fluid (see Fig.1 in Chapter 1).

**NOTE:** As discussed in the chapters on radiative processes, accelerated charges emit photons. Hence, the charged particles in a plasma will lose energy, which impacts their motion (trajectories). As long as the timescale of this energy loss is long compared to the processes of interest, we are justified in ignoring these **radiative losses**. Throughout we assume this to be the case.

**Plasma Orbit Theory:** The study of the motion of individual particles in a plasma.

A particle of mass $m$ and charge $q$ moving in an EM field satisfies the following equation of motion:

$$ m \frac{d\vec{v}}{dt} = q \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) $$

Each particle is subjected to the EM field produced by the other particles. In addition, there may be an external magnetic field imposed on the plasma. The interior
of a plasma is usually shielded from external electrical fields. As we have already seen in the discussion of cyclotron radiation (Chapter 23), a charged particle in a uniform, magnetic field, $\vec{B}$, has a motion that can be decomposed into

1. a circular motion with **gyrofrequency** $\omega_c = |q|B/mc$ and **gyroradius** $r_0 = mv_\perp c/|q|B$ around a central point called the **guiding center**. Here $v_\perp$ is the component of velocity perpendicular to the magnetic field lines.

2. a translatory motion of the guiding center.

This combination gives rise to a **helical motion** along the magnetic field lines.

But what about the motion in a non-uniform, magnetic field, $\vec{B}(\vec{x})$?

As long as the non-uniformities in $\vec{B}(\vec{x})$ are small over the scale of the gyroradius, i.e., $|\vec{B}/(d\vec{B}/dr)| < r_0$, one can still meaningfully decompose the motion into a circular motion around the guiding center and the motion of the guiding center itself. The latter can be quite complicated, though. The aim of plasma orbit theory is to find equations describing the motion of the guiding center. Unfortunately, there is no general equation of motion for the guiding center in an arbitrary EM field. Rather, plasma orbit theory provides a ‘bag of tricks’ to roughly describe what happens under certain circumstances. In what follows we discuss four examples: three circumstances under which the guiding center experiences a **drift**, and one in which the guiding center is **reflected**.

There are three cases in which the guiding center experiences a ‘drift’:

(A) drift due to the effect of a perpendicular force
(B) drift due to a gradient in the magnetic field
(C) drift due to curvature in the magnetic field

We now discuss these in turn.

(A) **The effect of a perpendicular force:**
Consider the case of an **external force**, $\vec{F}$ (e.g., gravity), acting on a charged particle in a direction **perpendicular** to a uniform magnetic field. The equation
Figure 33: Drift of a gyrating particle in crossed gravitational and magnetic fields. The magnetic field is pointing out of the page, while the gravitational force is pointing upward. Note that the positively and negatively charged particles drift in opposite directions, giving rise to a non-zero current in the plasma.

of motion then is

\[ m \frac{d\vec{v}}{dt} = \vec{F} + \frac{q}{c} \vec{v} \times \vec{B} \]

In the limit where \( \vec{B} \) vanishes, the particle simply moves in the direction of \( \vec{F} \). In the limit where \( \vec{F} \) vanishes, the particle makes a circular motion around the magnetic field line. When \textbf{both} \( \vec{F} \) and \( \vec{B} \) are present, the guiding center of the circular motion will drift in a direction perpendicular to both \( \vec{F} \) and \( \vec{B} \), with a magnitude given by

\[ \vec{v}_{GC} = \frac{c}{q} \frac{\vec{F} \times \vec{B}}{B^2} \]

(see chapter 10 of Choudhuri for a derivation). The trajectory of the particle looks as depicted in Fig. 29. Note that positively and negatively charged particles will drift in opposite directions, thereby giving rise to a non-zero \textbf{current} in the plasma. This will \textbf{not} be the case if the external force is the electrical force, i.e., if \( \vec{F} = q\vec{E} \). In that case

\[ \vec{v}_{GC} = c \frac{\vec{E} \times \vec{B}}{B^2} \]

which does not depend on the charge; hence, all particles drift in the same direction and no current arises.

\textbf{(B) Gradient Drift:}
If the magnetic field has a gradients $\nabla \vec{B}$ in the direction perpendicular to $\vec{B}$, there will also be a drift of the guiding center given by:

$$\vec{v}_{GC} = \pm \frac{1}{2} v_\perp r_0 \frac{\vec{B} \times \nabla \vec{B}}{B^2}$$

where the $+$ and $-$ signs correspond to positive and negative charges. Hence, particles of opposite charge drift in opposite directions, once again giving rise to a non-zero current in the plasma.

(C) Curvature Drift:
If the magnetic field is curved, with a radius of curvature $R_c$, once again the guiding center experiences a drift, given by:

$$\vec{v}_{GC} = -\frac{c m v_\perp^2}{q R_c^2} \vec{R}_c \times \vec{B} \frac{B}{B^2}$$

with $v_\perp$ the velocity component parallel to $\vec{B}$, while $\vec{R}_c$ is the curvature vector directed towards the curvature center. Like the gradient drift, the curvature drift is also in opposite directions for positively and negatively charged particles, and can thus give rise to a non-zero current.

Magnetic Mirrors:
As a final example of plasma orbit theory, we consider magnetic mirrors. Consider a magnetic field as depicted in Fig. 30. Close to the coils, the magnetic field is stronger than in between. Now consider a particle gyrating along one of these field lines. One can show that the motion in the $z$-direction (the symmetry axis of the magnetic bottle) obeys $d\mu/dt = 0$, where

$$\mu = \frac{1}{2} \frac{m v_\perp^2}{B}$$

is the magnetic moment, with $v_\perp$ the motion in the direction perpendicular to the magnetic field line (see Chapter 10 of Choudhuri for a detailed derivation). Since the total kinetic energy of a charged particles moving in a static magnetic field cannot change, the sum of the kinetic energies in the direction perpendicular to and along the magnetic field lines must remain constant. As the particle moves in the direction along which the strength of $\vec{B}$ increases (i.e., towards one of the coils), $v_\perp$ must increase in order to guarantee conservation of the magnetic moment. However, the
transverse kinetic energy can never exceed the total kinetic energy. Therefore, when
the particle reaches a region of sufficiently strong $B$, where the transverse kinetic
energy equals the total kinetic energy, it is not possible for the particle to penetrate
further into regions of even stronger magnetic field: the particle will be reflected
back, and the region of increasing magnetic field thus acts as a reflector, known as a
magnetic mirror.

The contraption shown in Fig. 30 is known as magnetic bottle as it can be used to
‘store’ charged particles. Note, though, that a magnetic bottle is inherently ‘leaky’.
To see this, let $B_0$ denote the magnetic field strength in the middle of the bottle,
and $B_{\text{max}}$ the maximum magnetic field strength, which arises at the positions of the
two coils. Let $\vec{v}_0$ be the velocity of the particle at the middle of the bottle, and
let $v_{\perp,0}$ be its corresponding transverse speed: $v_{\perp,0} = v_0 \sin \alpha$. Since the transverse
velocity at the time of reflection has to equal $v_0$, we see that only those particles will
be reflected for which $\sin^2 \alpha > B_0/B_{\text{max}}$. Particles for which $\alpha$ is smaller make up a
loss cone, as these particles will leak out of the magnetic bottle.

Magnetic bottles are not only found in laboratories; the Earth’s magnetic field creates
its own magnetic bottles due to its toroidal topology. The charged particles that are
trapped give rise to what are called the Van Allen belts (electrons and protons have their own belts, as depicted in Fig. 31). As the trapped particles move back and forth between the North and South poles of the Earth’s magnetic field, they experience curvature drift (in opposite directions for the electrons and protons). The resulting currents are called the ring currents.
In the previous chapter we focussed on the trajectories of individual charged particles in a plasma. We now focus on the dynamics of many charged particles.

If the plasma is weakly ionized, then a charged particle is more likely to have a collision with a neutral particle. Such collisions take place when the particles are very close to each other and usually produce large deflections, similar to collisions between two neutral particles. Hence, a weakly ionized plasma can be described using the Boltzmann equation.

If the plasma is highly ionized, Coulomb interactions among the charged particles dominate. These are long-range interactions, and typically result in small deflections. In addition, a particle typically has interactions with multiple other particles simultaneously. Hence, the collisions are not instantaneous, well-separated, and localized (i.e., short-range). Consequently, the Boltzmann equation does not apply, and we need to derive an alternative dynamical model. Unfortunately, this is a formidable problem that is not completely solved for an arbitrary, inhomogeneous magnetized plasma.

In our discussion of neutral fluids we have seen that a system of particles can be treated like a continuum fluid if frequent collisions keep the distribution function in local regions close to a Maxwellian. Although not easy to proof, there is ample experimental evidence that shows that the collisions in a plasmas also relax to a Maxwellian. We therefore seek to develop some continuum fluid model to describe our plasma.

In a plasma, the strong electrostatic interactions among positively and negatively charged particles ensure that volumes with statistically large numbers of particles are nearly neutral. Hence, plasmas have some characteristics in common with neutral fluids. However, on small enough scales, particles feel the strong, electromagnetic forces from individual particles. Although a charged particle in a plasma in principle
produces a long-range EM field, its effect is usually screened off by particles of the opposite charge within a distance called the Debye length

\[ \lambda_d = \left( \frac{k_B T}{8\pi ne^2} \right)^{1/2} \]

with \( e \) the electrical charge of an electron.

- On scales \( \lambda \gg \lambda_d \), a plasma can be considered charge-neutral: any charge imbalance produces strong electrostatic forces which quickly restore charge-neutrality.

- On scales \( \lambda \ll \lambda_d \), particles experience strong Coulomb interactions.

The average number of particles on which a charged particle exerts an influence is roughly \( n \lambda_d^3 \), with \( n \) the average number density of particles. Associated with this is the **Plasma Parameter**

\[ g \equiv \frac{1}{n \lambda_d^3} = \frac{(8\pi)^{3/2} e^3 n^{1/2}}{(k_B T)^{3/2}} \]

- When \( g \gg 1 \), the number of particles on which a charged particle exerts an influence is small, and the system behaves like a neutral fluid.

- When \( g \ll 1 \), many particles undergo simultaneous interactions, and as a consequence, the fluid manifests **collective behavior**. This is known as the **plasma limit**.

**NOTE:** The plasma parameter \( g \propto n^{1/2} \). Hence, low density plasma’s are more ‘plasma-like’ (display more collective phenomenology). Even though the number of particles per volume is smaller, the total number of particles within a Debye volume, \( \lambda_d^3 \), is larger.

The average distance between particles is of the order \( n^{-1/3} \). Hence, the average potential energy of electrostatic interactions is of the order \( e^2 n^{1/3} \). We thus see that

\[ \frac{\langle P.E. \rangle}{\langle K.E. \rangle} \propto \frac{e^2 n^{1/3}}{k_B T} \propto g^{2/3} \]
In other words, the plasma parameter is a measure for the ratio between the average potential energy associated with collisions and the average kinetic energy of the particles.

- When $g \ll 1$, interactions among the particles are weak, but a large number of particles interact simultaneously, giving rise to plasma behavior.

- When $g \gg 1$, interactions among the particles are strong, but few particles interact collectively, and the fluid behaves like a neutral fluid.

In order to develop a dynamical equation for plasmas, it is useful to first recall how we developed our dynamical equation for neutral fluids (the Boltzmann equation). We did so by starting from the **Liouville theorem**, constructing the **BBGKY hierarchy**, and then truncating the hierarchy by assuming "molecular chaos", i.e., that

$$
f^{(2)}(\vec{x}_1, \vec{v}_1, \vec{x}_2, \vec{v}_2, t) = f^{(1)}(\vec{x}_1, \vec{v}_1, t) f^{(1)}(\vec{x}_2, \vec{v}_2, t)
$$

Note that the two-particle distribution function is only required at one and the same location (i.e., $\vec{x}_2 = \vec{x}_1$); the reason is that the collisions in our neutral fluid are localized, instantaneous, and well separated. The assumption of molecular chaos is basically an assumption that the velocities of the particles interacting are independent of each other (the two-point correlation function of velocities is independent). By making the above assumption we were able to derive a closed equation for the 1-particle distribution function, $f^{(1)}(\vec{x}, \vec{v}, t)$, which we called the **Boltzmann equation**, given by

$$
\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} = I[f]
$$

Here we use $f$ as shorthand for $f^{(1)}(\vec{x}, \vec{v}, t)$, and $I[f]$ is the **collision integral**.

For our plasma, we also start from the Liouville theorem, and use the same BBGKY hierarchy (all of this is perfectly general). The BBGKY equation for the 1-particle DF is given by

$$
\frac{\partial f^{(1)}}{\partial t} + \vec{v} \cdot \frac{\partial f^{(1)}}{\partial \vec{x}} + n \int d\vec{x}_2 d\vec{v}_2 \frac{\vec{F}_{12}}{m} \cdot \frac{\partial f^{(2)}(1, 2)}{\partial \vec{v}} = 0
$$

As is evident, this is not a closed expression, as it depends on the 2-particle DF

$$
f^{(2)}(1, 2) \equiv f^{(2)}(\vec{x}_1, \vec{v}_1, \vec{x}_2, \vec{v}_2, t) = f^{(1)}(1) f^{(1)}(2) + g(1, 2)
$$
with \( g(1, 2) \) the two-point correlation function. Using the BBGKY hierarchy, one can write down a similar expression for the evolution of \( f^{(2)}(1, 2) \), but that contains a term that depends on the 3-particle DF, \( f^{(3)}(1, 2, 3) \), etc.

In order to derive at a useful expression to describe the dynamics of a plasma one assumes that \( g(1, 2) = 0 \), in which case the above equation for the 1-particle DF becomes what is known as the **Vlasov equation**:

\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = 0
\]

with

\[
\vec{F} = n \int d\vec{x}_2 d\vec{v}_2 \vec{F}_{12} f^{(1)}(\vec{x}_2, \vec{v}_2, t)
\]

the smooth force acting on particle 1 due to the long-range interactions of all other particles. Note that by neglecting \( g(1, 2) \), which arises due to interactions between neighboring particles (within each other’s Debye length), the force \( \vec{F} \) does NOT include the forces exerted during collisions.

Upon inspection of the Vlasov equation, it is clear that it has exactly the same form as the **collisionless Boltzmann equation** (CBE); i.e., the Boltzmann equation with the collision integral \( I[f] = 0 \). Because of this, the Vlasov equation is often called the CBE. We emphasize, though, that there are some subtle differences. Among these is the fact that the CBE is used to describe systems that are truly collisionless, whereas the Vlasov equation is used to describe a collisional system (a plasma). In fact, as mentioned above, a plasma will relax towards an equilibrium with a Maxwellian distribution, and this relaxation is due to collisions. Such relaxation, however, cannot be described by the Vlasov equation. Rather, the Vlasov equation can only be used to describe plasma phenomena for which collisions can be neglected. As we will see below, an example is the study of waves excited in a plasma in thermodynamic equilibrium; if the frequency of the waves is sufficiently high, such that collisions can be neglected during an oscillation period of the wave, then the perturbations to the DF can be handled by the Vlasov equation.

An important application of the Vlasov equation is the so-called **two-fluid model of plasma physics**, in which the plasma is regarded as an inter-penetrating mixture of a negatively charged fluid of electrons, and a positively charged fluid of ions. In
that case,

\[ f(\vec{x}, \vec{v}, t) = f_e(\vec{x}, \vec{v}, t) + f_i(\vec{x}, \vec{v}, t) \]

where the subscripts ‘e’ and ‘i’ refer to electrons and ions, respectively. Since the Vlasov equation is linear, both \( f_e \) and \( f_i \) obey the Vlasov equation. If the force \( \vec{F} \) is purely electromagnetic, then we have that

\[
\frac{\partial f_a}{\partial t} + \vec{v} \cdot \frac{\partial f_a}{\partial \vec{x}} + \frac{q_a}{m} \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \cdot \frac{\partial f_a}{\partial \vec{v}} = 0
\]

where ‘a’ is either ‘e’ or ‘i’.

Rather than solving the Vlasov equation, we follow the same approach as with our neutral fluids, and our collisionless fluids, and solve instead the moment equations, by multiplying the Vlasov equation with \( \chi(\vec{v}) \) and integrating over all of velocity (momentum) space (cf. Chapter 6).

For \( \chi = 1 \) this yields the continuity equation

\[
\frac{\partial n_a}{\partial t} + \nabla \cdot (n_a \vec{u}_a) = 0
\]

while \( \chi = m_a \vec{v}_a \) yields the momentum equations

\[
m_a n_a \left[ \frac{\partial \vec{u}_a}{\partial t} + (\vec{u}_a \cdot \nabla) \vec{u}_a \right] = -\nabla P_a + q_a n_a \left( \vec{E} + \frac{\vec{v}_a}{c} \times \vec{B} \right)
\]

Note that the continuity equation is exactly the same as for a neutral fluid or a collisionless fluid, while the moment equations are the same as the Euler equations for a neutral, inviscid fluid or the Jeans equations for a collisionless fluid, except that the force is now electromagnetic, rather than gravitational.

As for neutral fluids, we need to complement these moment equations with an equation of state (EoS) in order to close the equations. Without going into detail, in most cases the EoS of a plasma can be taken to have one of the following three forms:

\[
\begin{align*}
P_a &= 0 \quad & \text{("cold plasma")} \\
P_a &= n_a k_B T_a \quad & \text{("ideal plasma")} \\
P_a &= C n_a^\gamma \quad & \text{("adiabatic processes")}
\end{align*}
\]

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A ‘cold plasma’ is a plasma in which the random motions of the particles are not important.

**NOTE:** in the presence of strong magnetic fields, the thermodynamic properties of the plasma can be very different in directions parallel and perpendicular to \(\vec{B}\); in those cases the pressure cannot be fully described by a scalar, but requires a stress-tensor-equivalent instead. We will not consider such situations here.

Since the momentum equations for our plasma contain the electric and magnetic fields, we need to complement the moment equations and EoS with the Maxwell equations

\[
\nabla \cdot \vec{E} = 4\pi (n_i - n_e) e \\
\nabla \cdot \vec{B} = 0 \\
\n\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\
\n\nabla \times \vec{B} = \frac{4\pi}{c} (n_i \vec{u}_i - n_e \vec{u}_e) e + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\]

Upon inspection, this complete set of 18 equations (8 Maxwell eqs, 2 × 3 momentum equations, 2 continuity equations, and 2 equations of state) describes the evolution of a total of 16 scalar quantities: \(\vec{E}\) (3), \(\vec{B}\) (3), \(\vec{u}_i\) (3), \(\vec{u}_e\) (3), \(n_i\), \(n_e\), \(P_i\), and \(P_e\). This set of equations constitutes the **two-fluid model** of plasma physics. Note that this model derives from the Vlasov equation, and can therefore only be used to describe plasma phenomena in which collisions can be neglected.

As an example of an application of the two-fluid model, consider **electro-magnetic oscillations** in a plasma.

Let’s assume the plasma to be ‘cold’ (i.e., \(P_e = P_i = 0\)), and consider perturbations in a uniform, homogeneous plasma. The perturbation analysis treatment is exactly analogous to that of acoustic waves in Chapter 15: First, apply small perturbations to the dynamical quantities (i.e., \(n_0 \rightarrow n_0 + n_1\), \(\vec{E}_0 \rightarrow \vec{E}_0 + \vec{E}_1\), etc, where subscripts ‘0’ refer to the unperturbed equilibrium solution. Next, linearize the equation, and combine the equations into a single equation. Finally, assume all perturbations to be of the form \(\text{EXP}[-i(\vec{k} \cdot \vec{x} - \omega t)]\), which implies that \(\partial / \partial t \rightarrow -i\omega\) and \(\nabla \rightarrow -i\vec{k}\)
and obtain the dispersion relation $\omega(\vec{k})$. In the case of our two-fluid model, this dispersion relation has the form

$$\vec{k} \times (\vec{k} \times \vec{E}_1) = -\frac{\omega^2}{c^2} \left( 1 - \frac{\omega_p^2}{\omega^2} \right) \vec{E}_1$$

Here

$$\omega_p = \left( \frac{4\pi n_0 e}{m_e} \right)^{1/2}$$

is the plasma frequency. This dispersion relation corresponds to two physically distinct types of wave modes:

**Plasma Oscillations:**
These are oscillation modes for which

$$E_{1x} = E_{1y} = 0, \quad \omega^2 = \omega_p^2$$

where the $z$-direction is taken to be along $\vec{k}$. Hence, these correspond to non-propagating, longitudinal oscillations with a frequency equal to the plasma frequency. These are called plasma waves, or Langmuir waves. Physically, they are waves in which perturbations in $\vec{E}$, cause a separation between electrons and ions, which results in an electrostatic restoring force. Note that the plasma frequency depends only on the density of the plasma. If the plasma is not cold (i.e., $P_a \neq 0$), then it follows that these Langmuir oscillations become travelling waves.

**Electromagnetic waves:**
These are oscillation modes for which

$$E_{1z} = 0, \quad \omega^2 = \omega_p^2 + k^2 c^2$$

where as before the $z$-direction is taken to be along $\vec{k}$. Hence, these are transverse waves. In fact, these are simply electromagnetic waves, but modified by the plasma.

- If $\omega \gg \omega_p$, we have that $\omega^2 = k^2 c^2$ which is the usual dispersion relation for EM waves in a vacuum. The frequency of EM waves is too high for the plasma to respond, and plasma effects are negligible.
• If $\omega \ll \omega_p$, the wave cannot penetrate the plasma, and is reflected back (see discussion on cyclotron radiation in Chapter 23). Note that in this regime, strictly speaking the Vlasov equation is no longer applicable, as the time scale for collisions becomes comparable to, or shorter than, that of the perturbation.

The above analysis is based on a perturbation analysis of the two-fluid model, which is based on moment equations of the Vlasov equation. Landau performed a more thorough analysis, by actually perturbing the Vlasov equation itself. He found that the Langmuir waves will damp, a process known as Landau damping.

This damping may come as a surprise. After all, damping is typically associated with dissipation, and hence requires either radiation, or collisions that convert wave energy into random, thermal energy. But the Vlasov equation includes neither radiation nor collisions. So where does this damping come from? Without going into detail, the answer is that gradients in the distribution function can cause a transfer of energy from the wave to the particles; Landau damping is a prime example of a wave-particle interaction. It is somewhat similar to violent relaxation (see Chapter 11) for a purely collisionless, gravitational system, in which the energy in potential fluctuations (i.e., oscillations in the gravitational system, for example due to gravitational collapse) are transferred into random motions, ultimately leading to virialization (relaxation).
As discussed in the previous chapter, when discussing phenomena in which electrons and ions respond differently (such as the response of a plasma to electromagnetic oscillations), the two-fluid model (or Vlasov equation) has to be applied.

When we consider phenomena with length scales much larger than the Debye length, and time scales much longer than the inverse of the plasma frequency, charge separation can be neglected. In that case we don’t need to treat electrons and ions separately. Rather, we treat the plasma as a single fluid.

In the one-fluid model of plasma physics, we account for collisions, the main effect of which is to cause a transfer of momentum between electrons and ions. This momentum transfer manifests as an electrical current, with current density

\[ \vec{j} = n e (\vec{u}_i - \vec{u}_e) \]

Here \( n = n_e = n_i \), and, for simplicity, we assume all ions to have a charge +e. One can show that, under a number of assumptions (see Choudhuri, § 13.4 for details), this current density can be related to the EM fields according to

\[ \vec{j} = \sigma (\vec{E} + \frac{\vec{u}}{c} \times \vec{B}) \]

This is known as Ohm’s law. Here

\[ \vec{u} = \frac{m_i \vec{u}_i + m_e \vec{u}_e}{m_i + m_e} \]

and \( \sigma \) is called the electrical conductivity, which is the inverse of the electrical resistivity

\[ \eta \equiv \frac{1}{\sigma} \propto \frac{m_e^{1/2} e^2}{(k_B T)^3/2} \]

The one-fluid model for a plasma is known as magnetohydrodynamics, or MHD. If we are in the regime where charge separation is negligible (a prerequisite for MHD),
then it can be shown that the electrical field, $\vec{E}(\vec{x})$, is not an independent variable. Rather, it follows from $\vec{B}(\vec{x})$ and $\vec{v}(\vec{x})$ according to

$$\vec{E} = \frac{c}{4\pi \sigma} \left( \nabla \times \vec{B} \right) - \frac{\vec{v}}{c} \times \vec{B}$$

This follows from the above expression for the current density, $\vec{j}$, and the fact that neglecting $\partial \vec{E}/\partial t$ in the Maxwell equations implies that

$$\nabla \times \vec{B} = \frac{4\pi \gamma}{c} \vec{j}$$

Without going through any derivation, we now present the MHD equations. These are:

(1) **The continuity equation**
This equation is identical to that for a neutral fluid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

with $\rho = n(m_i + m_e)$.

(2) **The momentum equations**
These are simply the Navier-Stokes equations complemented with a term that describes the magnetic body forces:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = \vec{F} - \frac{\nabla P}{\rho} + \nu \nabla^2 \vec{u} + \frac{1}{\rho c} \gamma \vec{j} \times \vec{B}$$

where we have assumed an incompressible plasma, so that the $\nabla(\vec{u} \cdot \vec{u})$-term in the Navier-Stokes equations vanishes. If we now use that $\vec{j} = (c/4\pi) \nabla \times \vec{B}$, these momentum equations can be written in Lagrangian index-form as

$$\rho \frac{d\vec{u}_i}{dt} = \rho \vec{F}_i - \frac{\partial}{\partial x_j} (\sigma_{ij} + M_{ij})$$

Here $\sigma_{ij}$ is the stress tensor and $M_{ij}$ is the magnetic stress tensor given by

$$M_{ij} = \frac{B_i^2}{8\pi} \delta_{ij} - \frac{B_i B_j}{4\pi}$$
As with the stress tensor, the diagonal terms describe a pressure. In this case, the **magnetic pressure** $B^2/8\pi$. The off-diagonal terms of the stress tensor correspond to the **viscous shear**. In the case of $M_{ij}$, the off-diagonal terms reflect the **magnetic tension** along the magnetic field lines.

(3) **The energy equation**
This is the same as for a neutral fluid, except that you need to add an Ohmic heating term, $j^2/\sigma$. Here $j^2 = \vec{j} \cdot \vec{j}$ and $\sigma$ is the electric conductivity. The Ohmic heating term describes how collisions convert magnetic energy into thermal energy.

(4) **The induction equation**
Finally, in order to close the set of MHD equations, we need to complement it with an equation that describes the evolution of the magnetic field. Combining the Maxwell equation $\partial \vec{B}/\partial t = -c\nabla \times \vec{E}$, with the expression for $\vec{E}$ in terms of $\vec{B}$ and $\vec{u}$, yields the induction equation:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \lambda \nabla^2 \vec{B}$$

Here $\lambda = c/(4\pi\sigma)$ is the **magnetic diffusivity**.

The above set of 8 equations (continuity, momentum, energy and induction) for a total of 8 unknowns ($\rho, \vec{u}, T, \vec{B}$) makes up the MHD model. A fluid that obeys these MHD equations is called a **magnetofluid**.

Note that the induction equation is very similar to the vorticity equation

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{u} \times \vec{\omega}) - \nabla \times \left(\frac{\nabla P}{\rho}\right) + \nu \nabla^2 \vec{\omega}$$

(see Chapter 7). Here $\vec{\omega}$ is the **vorticity** and $\nu$ the **kinetic viscosity**. Except for the baroclinic term, which is absent in the induction equation, vorticity and magnetic field (in the MHD approximation) behave very similar.

Motivated by this similarity, we now define the **magnetic Reynolds number**

$$R_m = \frac{UL}{\lambda}$$

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with $U$ and $L$ the characteristic velocity and length scales of the plasma flow. Recall (from Chapter 9), the definition of the Reynolds number $\mathcal{R} = UL/\nu$. We thus merely replaced the kinetic viscosity with the magnetic diffusivity, which is proportional to the electric resistivity.

- When $\mathcal{R}_m \ll 1$, the second term in the induction equation dominates, which therefore becomes
  \[ \frac{\partial B}{\partial t} \simeq \lambda \nabla^2 \vec{B} \]
  This is the situation one typically encounters in laboratory plasmas, where $U$ and $L$ are small. The implication is that, the magnetic field in laboratory plasmas, when left to itself, decays away due to magnetic diffusion. This can be understood from the fact that magnetic fields are directly related to currents, which die away due to Ohmic dissipation unless one applies a source of voltage.

- When $\mathcal{R}_m \gg 1$, the first term in the induction equation dominates, which therefore becomes
  \[ \frac{\partial B}{\partial t} \simeq \nabla \times (\vec{u} \times \vec{B}) \]
  This is the situation we typically encounter in astrophysics, where $U$ and $L$ are large. In the limit of infinite conductivity (i.e., zero electrical resistivity, and thus zero magnetic diffusivity), the above equation is exact, and we talk of ideal MHD. An implication of ideal MHD is that
  \[ \frac{d}{dt} \int_S \vec{B} \cdot d^2 s = 0 \]
  This expresses that the magnetic flux is conserved as it moves with the fluid. This is known as Alfvén’s theorem of flux freezing. It is the equivalent of Helmholtz’ theorem that $d\Gamma/dt = 0$ for an inviscid fluid (with $\Gamma$ the circularity). An implication is that, in the case of ideal MHD, two fluid elements that are connected by a magnetic flux line, will remain connected by that same magnetic flux line.

(Ideal) MHD is used to describe many astrophysical processes, from the magnetic field topology of the Sun, to angular momentum transfer in accretion disks, and from the formation of jets in accretion disks, to the magnetic breaking during star formation. One can also apply perturbation theory to the MHD equations, to examine what happens to a magnetofluid if it is perturbed. If one ignores viscosity, heat conduction, and electric resistivity (i.e., we are in the ideal MHD regime), then the
resulting dispersion relation reveals a characteristic velocity, called the Alfvén velocity, given by

\[ \vec{u}_A = \frac{\vec{B}_0}{\sqrt{4\pi \rho_0}} \]

Any wave is driven by some restoring force. In the case of acoustic waves these are pressure gradients, while the restoring force in the case of the plasma oscillations discussed in the previous chapter arise from the electrical field that results from a separation of electrons and ions. In the case of perturbations to a magnetofluid, there are two restoring forces that play a role; pressure gradients and magnetic tension. One wave solution for the dispersion relation is based solely on magnetic tension as the restoring force. These waves are called Alfvén waves, which are transverse waves moving in the direction of the magnetic field line with velocity \( u_A \).

These Alfvén waves are similar to the waves in a rope or string, which are also transverse waves. The group velocity of these waves is proportional to \( \sqrt{\text{tension/density}} \). Since the magnetic tension is given by \( B^2/4\pi \), we see that the Alfvén velocity has exactly the same form.

If the resistivity of the magnetofluid is non-zero, Alfvén waves will experience damping, thereby transferring the energy stored in the magnetic wave to random, thermal energy.
The following appendices provide supplemental material, including some calculus and radiation essentials that is relevant for this course. Students are assumed to be familiar with this. An exception are Appendices F, G, H and I which provide details that are NOT considered part of this course’s curriculum. They are included to provide background information for those readers that want to know a bit more.

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Appendix A

Vector Calculus

Vector: $\vec{A} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Amplitude of vector: $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit vector: $|\vec{A}| = 1$

Basis: In the above example, the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a vector basis. Any 3 vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ can form a vector basis as long as $\det(\vec{A}, \vec{B}, \vec{C}) \neq 0$.

Determinant: $\det(\vec{A}, \vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

$\det(\vec{A}, \vec{B}, \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

Geometrically: $\det(\vec{A}, \vec{B}) = \pm$ area of parallelogram
$\det(\vec{A}, \vec{B}, \vec{C}) = \pm$ volume of parallelepiped

Multiplication by scalar: $\alpha \vec{A} = (\alpha a_1, \alpha a_2, \alpha a_3)$

$|\alpha \vec{A}| = |\alpha| |\vec{A}|$

Summation of vectors: $\vec{A} + \vec{B} = \vec{B} + \vec{A} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
Einstein Summation Convention: \( a_i b_i = \sum_i a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a} \cdot \vec{b} \)
\( \partial A_i / \partial x_i = \partial A_1 / \partial x_1 + \partial A_2 / \partial x_2 + \partial A_3 / \partial x_3 = \nabla \cdot \vec{A} \)
\( A_{ii} = A_{11} + A_{22} + A_{33} = \text{Tr} \vec{A} \) (trace of \( \vec{A} \))

Dot product (aka scalar product): \( \vec{A} \cdot \vec{B} = a_i b_i = |\vec{A}| |\vec{B}| \cos \theta \)
\( \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \)

Useful for:
- computing angle between two vectors: \( \cos \theta = \vec{A} \cdot \vec{B} / (|\vec{A}| |\vec{B}|) \)
- check orthogonality: two vectors are orthogonal if \( \vec{A} \cdot \vec{B} = 0 \)
- compute projection of \( \vec{B} \) in direction of \( \vec{A} \), which is given by \( \vec{A} \cdot \vec{B} / |\vec{A}| \)

Cross Product (aka vector product): \( \vec{A} \times \vec{B} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix} = \varepsilon_{ijk} a_i b_j \hat{e}_k \)
\( |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta = \text{det}(\vec{A}, \vec{B}) \)

Levi-Civita tensor: \( \varepsilon_{ijk} = \begin{cases} 
0 & \text{any two of } i, j, k \text{ are the same} \\
+1 & i, j, k \text{ is even permutation of } 1, 2, 3 \\
-1 & i, j, k \text{ is odd permutation of } 1, 2, 3 
\end{cases} \)

\[
\begin{array}{c}
\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \\
(\alpha \vec{A}) \cdot \vec{B} = \alpha (\vec{A} \cdot \vec{B}) = \vec{A} \cdot (\alpha \vec{B}) \\
\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \\
\vec{A} \cdot \vec{B} = 0 \rightarrow \vec{A} \perp \vec{B} \\
\vec{A} \cdot \vec{A} = |\vec{A}|^2 \\
\end{array}
\]

\[
\begin{array}{c}
\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \\
(\alpha \vec{A}) \times \vec{B} = \alpha (\vec{A} \times \vec{B}) = \vec{A} \times (\alpha \vec{B}) \\
\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \\
\vec{A} \times \vec{B} = 0 \rightarrow \vec{A} \parallel \vec{B} \\
\vec{A} \times \vec{A} = 0 \\
\end{array}
\]

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Triple Scalar Product:  
\[ \vec{A} \cdot (\vec{B} \times \vec{C}) = \det(\vec{A}, \vec{B}, \vec{C}) = \varepsilon_{ijk} a_i b_j c_k \]
\[ \vec{A} \cdot (\vec{B} \times \vec{C}) = 0 \rightarrow \vec{A}, \vec{B}, \vec{C} \text{ are coplanar} \]
\[ \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \]

Triple Vector Product:  
\[ \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C} \]
as is clear from above, \( \vec{A} \times (\vec{B} \times \vec{C}) \) lies in plane of \( \vec{B} \) and \( \vec{C} \).

Useful to remember:  
\[ (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C}) (\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D}) (\vec{B} \cdot \vec{C}) \]
\[ (\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \left[ \vec{A} \cdot (\vec{B} \times \vec{D}) \right] \vec{C} - \left[ \vec{A} \cdot (\vec{B} \times \vec{C}) \right] \vec{D} \]

Gradient Operator:  
\[ \nabla = \vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \]
This vector operator is sometimes called the nabla or del operator.

Laplacian operator:  
\[ \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]
This is a scalar operator.

Differential:  
\[ f = f(x, y, z) \rightarrow df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy + \frac{\partial f}{\partial z} \, dz \]

Chain Rule:  
If \( x = x(t), y = y(t) \) and \( z = z(t) \) then \( \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \)
If \( x = x(s, t), y = y(s, t) \) and \( z = z(s, t) \) then \[ \frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} + \frac{\partial f}{\partial z} \frac{dz}{ds} \]

Gradient Vector:  
\[ \nabla f = \text{grad} \, f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \]
the gradient vector at \((x, y, z)\) is normal to the level surface through the point \((x, y, z)\).

Directional Derivative:  
The derivative of \( f = f(x, y, z) \) in direction of \( \vec{u} \) is  
\[ D_\vec{u} f = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|} = |\nabla f| \cos \theta \]

Vector Field:  
\[ \vec{F}(\vec{x}) = (F_x, F_y, F_z) = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \]
where \( F_x = F_x(x, y, z), F_y = F_y(x, y, z) \), and \( F_z = F_z(x, y, z) \).
Divergence of Vector Field: \( \text{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \)

A vector field for which \( \nabla \cdot \vec{F} = 0 \) is called solenoidal or divergence-free.

Curl of Vector Field: \( \text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \)

A vector field for which \( \nabla \times \vec{F} = 0 \) is called irrotational or curl-free.

Laplacian of Vector Field: \( \nabla^2 \vec{F} = (\nabla \cdot \nabla) \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla \times (\nabla \times \vec{F}) \)

Note that \( \nabla^2 \vec{F} \neq \nabla(\nabla \cdot \vec{F}) \): do not make this mistake.

Let \( S(x) \) and \( T(x) \) be scalar fields, and let \( \vec{A}(x) \) and \( \vec{B}(x) \) be vector fields:

<table>
<thead>
<tr>
<th>( \nabla S = \text{grad} S = \text{vector} )</th>
<th>( \nabla^2 S = \nabla \cdot (\nabla S) = \text{scalar} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla \cdot \vec{A} = \text{div} \vec{A} = \text{scalar} )</td>
<td>( \nabla^2 \vec{A} = (\nabla \cdot \nabla) \vec{A} = \text{vector} )</td>
</tr>
<tr>
<td>( \nabla \times \vec{A} = \text{curl} \vec{A} = \text{vector} )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\nabla \times (\nabla S) &= 0 & \text{curl grad } S &= 0 \\
\nabla \cdot (\nabla \times \vec{A}) &= 0 & \text{div curl } \vec{A} &= 0
\end{align*}
\]
\[ \nabla(ST) = S \nabla T + T \nabla S \]
\[ \nabla \cdot (S \vec{A}) = S(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla S \]
\[ \nabla \times (S \vec{A}) = (\nabla S) \times \vec{A} + S(\nabla \times \vec{A}) \]
\[ \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \]

\[ \nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \]
\[ \nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) \]
\[ \vec{A} \times (\nabla \times \vec{A}) = \frac{1}{2} \nabla(\vec{A} \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{A} \]
\[ \nabla \times (\nabla^2 \vec{A}) = \nabla^2(\nabla \times \vec{A}) \]
Appendix B

Conservative Vector Fields

Line Integral of a Conservative Vector Field: Consider a curve $\gamma$ running from location $\vec{x}_0$ to $\vec{x}_1$. Let $d\vec{l}$ be the directional element of length along $\gamma$ (i.e., with direction equal to that of the tangent vector to $\gamma$), then, for any scalar field $\Phi(\vec{x})$,

$$\int_{\vec{x}_0}^{\vec{x}_1} \nabla \Phi \cdot d\vec{l} = \int_{\vec{x}_0}^{\vec{x}_1} d\Phi = \Phi(\vec{x}_1) - \Phi(\vec{x}_0)$$

This implies that the line integral is independent of $\gamma$, and hence

$$\oint_c \nabla \Phi \cdot d\vec{l} = 0$$

where $c$ is a closed curve, and the integral is to be performed in the counter-clockwise direction.

Conservative Vector Fields:
A conservative vector field $\vec{F}$ has the following properties:

- $\vec{F}(\vec{x})$ is a gradient field, which means that there is a scalar field $\Phi(\vec{x})$ so that $\vec{F} = \nabla \Phi$
- Path independence: $\oint_c \vec{F} \cdot d\vec{l} = 0$
- Irrotational = curl-free: $\nabla \times \vec{F} = 0$
Appendix C

Integral Theorems

**Green’s Theorem:** Consider a 2D vector field $\vec{F} = F_x \hat{i} + F_y \hat{j}$

$$\oint \vec{F} \cdot d\vec{l} = \int \int_A \nabla \times \vec{F} \cdot \hat{n} \, dA = \int \int_A |\nabla \times \vec{F}| \, dA$$

$$\oint \vec{F} \cdot \hat{n} \, dl = \int \int_A \nabla \cdot \vec{F} \, dA$$

NOTE: in the first equation we have used that $\nabla \times \vec{F}$ is always pointing in the direction of the normal $\hat{n}$.

**Gauss’ Divergence Theorem:** Consider a 3D vector field $\vec{F} = (F_x, F_y, F_z)$

If $S$ is a closed surface bounding a region $D$ with normal pointing outwards, and $\vec{F}$ is a vector field defined and differentiable over all of $D$, then

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int \int_D \nabla \cdot \vec{F} \, dV$$

**Stokes’ Curl Theorem:** Consider a 3D vector field $\vec{F} = (F_x, F_y, F_z)$

If $C$ is a closed curve, and $S$ is any surface bounded by $C$, then

$$\oint_c \vec{F} \cdot d\vec{l} = \int \int_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

NOTE: The curve of the line integral must have positive orientation, meaning that $d\vec{l}$ points counterclockwise when the normal of the surface points towards the viewer.
Curvi-Linear Coordinate Systems

In astrophysics, one often works in curvi-linear, rather than Cartesian coordinate systems. The two most often encountered examples are the cylindrical \((R, \phi, z)\) and spherical \((r, \theta, \phi)\) coordinate systems.

In this chapter we describe how to handle vector calculus in non-Cartesian coordinate systems (Euclidean spaces only). After giving the ‘rules’ for arbitrary coordinate systems, we apply them to cylindrical and spherical coordinate systems, respectively.

**Vector Calculus in an Arbitrary Coordinate System:**

Consider a vector \(\vec{x} = (x, y, z)\) in Cartesian coordinates. This means that we can write

\[
\vec{x} = x \vec{e}_x + y \vec{e}_y + z \vec{e}_z
\]

where \(\vec{e}_x, \vec{e}_y\) and \(\vec{e}_z\) are the unit directional vectors. Now consider the same vector \(\vec{x}\), but expressed in another general (arbitrary) coordinate system; \(\vec{x} = (q_1, q_2, q_3)\). It is tempting, but terribly wrong, to write that

\[
\vec{x} = q_1 \vec{e}_1 + q_2 \vec{e}_2 + q_3 \vec{e}_3
\]

where \(\vec{e}_1, \vec{e}_2\) and \(\vec{e}_3\) are the unit directional vectors in the new \((q_1, q_2, q_3)\)-coordinate system. In what follows we show how to properly treat such generalized coordinate systems.

In general, one expresses the distance between \((q_1, q_2, q_3)\) and \((q_1 + dq_1, q_2 + dq_2, q_3 + dq_3)\) in an arbitrary coordinate system as

\[
ds = \sqrt{h_{ij} dq_i dq_j}
\]

Here \(h_{ij}\) is called the **metric tensor**. In what follows, we will only consider orthogonal coordinate systems for which \(h_{ij} = 0\) if \(i \neq j\), so that \(ds^2 = h_{ii} dq_i^2\) (Einstein summation convention) with \(h_i = \sqrt{h_{ii}}\).

An example of an orthogonal coordinate system are the Cartesian coordinates, for which \(h_{ij} = \delta_{ij}\). After all, the distance between two points separated by the infinitesimal displacement vector \(d\vec{x} = (dx, dy, dz)\) is \(ds^2 = |d\vec{x}|^2 = dx^2 + dy^2 + dz^2\).
The coordinates \((x, y, z)\) and \((q_1, q_2, q_3)\) are related to each other via the **transformation relations**

\[
\begin{align*}
x &= x(q_1, q_2, q_3) \\
y &= y(q_1, q_2, q_3) \\
z &= z(q_1, q_2, q_3)
\end{align*}
\]

and the corresponding inverse relations

\[
\begin{align*}
q_1 &= q_1(x, y, z) \\
q_2 &= q_2(x, y, z) \\
q_3 &= q_3(x, y, z)
\end{align*}
\]

Hence, we have that the **differential vector** is:

\[
d\vec{x} = \frac{\partial \vec{x}}{\partial q_1} dq_1 + \frac{\partial \vec{x}}{\partial q_2} dq_2 + \frac{\partial \vec{x}}{\partial q_3} dq_3
\]

where

\[
\frac{\partial \vec{x}}{\partial q_i} = \frac{\partial}{\partial q_i}(x, y, z)
\]

The **unit directional vectors** are:

\[
\vec{e}_i = \frac{\partial \vec{x}/\partial q_i}{|\partial \vec{x}/\partial q_i|}
\]

which allows us to rewrite the expression for the **differential vector** as

\[
d\vec{x} = \left| \frac{\partial \vec{x}}{\partial q_1} \right| dq_1 \vec{e}_1 + \left| \frac{\partial \vec{x}}{\partial q_2} \right| dq_2 \vec{e}_2 + \left| \frac{\partial \vec{x}}{\partial q_3} \right| dq_3 \vec{e}_3
\]

and thus

\[
|d\vec{x}|^2 = \left| \frac{\partial \vec{x}}{\partial q_i} \right|^2 dq_i^2
\]

(Einstein summation convention). Using the definition of the metric, according to which \(|d\vec{x}|^2 = h_i^2 dq_i^2\) we thus infer that

\[
h_i = \left| \frac{\partial \vec{x}}{\partial q_i} \right|
\]
Using this expression for the metric allows us to write the unit directional vectors as

$$\vec{e}_i = \frac{1}{h_i} \frac{\partial \vec{x}}{\partial q_i}$$

and the differential vector in the compact form as

$$d\vec{x} = h_i dq_i \vec{e}_i$$

From the latter we also have that the infinitesimal volume element for a general coordinate system is given by

$$d^3 \vec{x} = |h_1 h_2 h_3| dq_1 dq_2 dq_3$$

Note that the absolute values are needed to assure that $d^3 \vec{x}$ is positive.
Now consider a vector $\vec{A}$. In the Cartesian basis $\mathcal{C} = \{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ we have that

$$[\vec{A}]_\mathcal{C} = A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$$

In the basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, corresponding to our generalized coordinate system, we instead have that

$$[\vec{A}]_\mathcal{B} = A_1 \vec{e}_1 + A_2 \vec{e}_2 + A_3 \vec{e}_3$$

We can rewrite the above as

$$[\vec{A}]_\mathcal{B} = A_1 \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} + A_2 \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} + A_3 \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} = \begin{pmatrix} A_1 e_{11} + A_2 e_{21} + A_3 e_{31} \\ A_1 e_{12} + A_2 e_{22} + A_3 e_{32} \\ A_1 e_{13} + A_2 e_{23} + A_3 e_{33} \end{pmatrix}$$

and thus

$$[\vec{A}]_\mathcal{B} = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \equiv \mathbf{T} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Using similar logic, one can write

$$[\vec{A}]_\mathcal{C} = \begin{pmatrix} e_{x1} & e_{y1} & e_{z1} \\ e_{x2} & e_{y2} & e_{z2} \\ e_{x3} & e_{y3} & e_{z3} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \mathbf{I} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

and since $\vec{A}$ is the same object independent of its basis we have that

$$\mathbf{I} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \mathbf{T} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

and thus, we see that the relation between $[\vec{A}]_\mathcal{B}$ and $[\vec{A}]_\mathcal{C}$ is given by

$$[\vec{A}]_\mathcal{C} = \mathbf{T} [\vec{A}]_\mathcal{B}, \quad \quad \quad \quad [\vec{A}]_\mathcal{B} = \mathbf{T}^{-1} [\vec{A}]_\mathcal{C}$$

For this reason, $\mathbf{T}$ is called the transformation of basis matrix. Note that the columns of $\mathbf{T}$ are the unit-direction vectors $\vec{e}_i$, i.e., $T_{ij} = e_{ij}$. Since these are orthogonal to each other, the matrix $\mathbf{T}$ is said to be orthogonal, which implies that $\mathbf{T}^{-1} = \mathbf{T}^\top$ (the inverse is equal to the transpose), and det$(\mathbf{T}) = \pm 1$.

Now we are finally ready to determine how to write our position vector $\vec{x}$ in the new basis $\mathcal{B}$ of our generalized coordinate system. Let’s write $\vec{x} = a_i \vec{e}_i$, i.e.

$$[\vec{x}]_\mathcal{B} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$
We started this chapter by pointing out that it is tempting, but wrong, to set \( a_i = q_i \) (as for the Cartesian basis). To see this, recall that \( |\vec{x}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2} \), from which it is immediately clear that each \( a_i \) needs to have the dimension of length. Hence, when \( q_i \) is an angle, clearly \( a_i \neq q_i \). To compute the actual \( a_i \) you need to use the transformation of basis matrix as follows:

\[
[\vec{x}]_B = T^{-1} [\vec{x}]_C = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e_{11}x + e_{12}y + e_{13}z \\ e_{21}x + e_{22}y + e_{23}z \\ e_{31}x + e_{32}y + e_{33}z \end{pmatrix}
\]

Hence, using our expression for the unit direction vectors, we see that

\[
a_i = \frac{1}{h_i} \left( \frac{\partial x_j}{\partial q_i} x_j \right) = \frac{1}{h_i} \left( \frac{\partial \vec{x}}{\partial q_i} \cdot \vec{x} \right)
\]

Hence, the position vector in the generalized basis \( B \) is given by

\[
[\vec{x}]_B = \sum_i \frac{1}{h_i} \left( \frac{\partial \vec{x}}{\partial q_i} \cdot \vec{x} \right) \vec{e}_i
\]

and by operating \( d/dt \) on \( [\vec{x}]_B \) we find that the corresponding velocity vector in the \( B \) basis is given by

\[
[\vec{v}]_B = \sum_i h_i \dot{q}_i \vec{e}_i
\]

with \( \dot{q}_i = dq_i/dt \). Note that the latter can also be inferred more directly by simply dividing the expression for the differential vector \((d\vec{x} = h_i q_i \vec{e}_i)\) by \( dt \).
Next we write out the gradient, the divergence, the curl and the Laplacian for our generalized coordinate system:

**The gradient:**

\[
\nabla \psi = \frac{1}{h_i \partial q_i} \vec{e}_i
\]

**The divergence:**

\[
\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (h_2 h_3 A_1) + \frac{\partial}{\partial q_2} (h_3 h_1 A_2) + \frac{\partial}{\partial q_3} (h_1 h_2 A_3) \right]
\]

**The curl** (only one component shown):

\[
(\nabla \times \vec{A})_3 = \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial q_1} (h_2 A_2) - \frac{\partial}{\partial q_2} (h_1 A_1) \right]
\]

**The Laplacian:**

\[
\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right]
\]
Vector Calculus in Cylindrical Coordinates:

For cylindrical coordinates \((R, \phi, z)\) we have that

\[
x = R \cos \phi \quad y = R \sin \phi \quad z = z
\]

The scale factors of the metric therefore are:

\[
h_R = 1 \quad h_\phi = R \quad h_z = 1
\]

and the position vector is \(\vec{x} = R\vec{e}_R + z\vec{e}_z\).

Let \(\vec{A} = A_R\vec{e}_R + A_\phi\vec{e}_\phi + A_z\vec{e}_z\) an arbitrary vector, then

\[
A_R = A_x \cos \phi - A_y \sin \phi \\
A_\phi = -A_x \sin \phi + A_y \cos \phi \\
A_z = A_z
\]

In cylindrical coordinates the velocity vector becomes:

\[
\vec{v} = \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z
\]

The Gradient:

\[
\nabla \cdot \vec{A} = \frac{1}{R} \frac{\partial}{\partial R} (RA_R) + \frac{1}{R} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}
\]

The Laplacian:

\[
\nabla^2 \psi = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \psi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}
\]
Vector Calculus in Spherical Coordinates:

For spherical coordinates \((r, \theta, \phi)\) we have that

\[
x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta
\]

The scale factors of the metric therefore are:

\[
h_r = 1 \quad h_{\theta} = r \quad h_{\phi} = r \sin \theta
\]

and the position vector is \(\vec{x} = r \vec{e}_r\).

Let \(\vec{A} = A_r \vec{e}_r + A_\theta \vec{e}_\theta + A_\phi \vec{e}_\phi\) an arbitrary vector, then

\[
A_r = A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta \\
A_\theta = A_x \cos \theta \cos \phi + A_y \cos \theta \sin \phi - A_z \sin \theta \\
A_\phi = -A_x \sin \phi + A_y \cos \phi
\]

In spherical coordinates the velocity vector becomes:

\[
\vec{v} = \dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta + r \sin \theta \dot{\phi} \vec{e}_\phi
\]

The Gradient:

\[
\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}
\]

The Laplacian:

\[
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}
\]
Appendix E

The Levi-Civita Symbol

The Levi-Civita symbol, also known as the permutation symbol or the antisymmetric symbol, is a collection of numbers, defined from the sign of a permutation of the natural numbers 1, 2, 3, ..., n. It is often encountered in linear algebra, vector and tensor calculus, and differential geometry.

The n-dimensional Levi-Civita symbol is indicated by $\varepsilon_{i_1 i_2 \ldots i_n}$, where each index $i_1, i_2, \ldots, i_n$ takes values 1, 2, ..., n, and has the defining property that the symbol is total antisymmetric in all its indices: when any two indices are interchanged, the symbol is negated:

$$\varepsilon_{i_1 \ldots i_p \ldots i_q \ldots} = -\varepsilon_{i_1 \ldots i_q \ldots i_p \ldots}$$

If any two indices are equal, the symbol is zero, and when all indices are unequal, we have that

$$\varepsilon_{i_1 i_2 \ldots i_n} = (-1)^p \varepsilon_{1,2,\ldots,n}$$

where $p$ is called the parity of the permutation. It is the number of pairwise interchanges necessary to unscramble $i_1, i_2, \ldots, i_n$ into the order 1, 2, ..., n. A permutation is said to be even (odd) if its parity is an even (odd) number.

Example: what is the parity of \{3, 4, 5, 2, 1\}?

- \{1, 2, 3, 4, 5\}
- \{3, 2, 1, 4, 5\}
- \{3, 4, 1, 2, 5\}
- \{3, 4, 5, 2, 1\}

Answer: $p = 3$, since three pairwise interchanges are required.

In three dimensions the Levi-Civita symbol is defined by

$$\varepsilon_{ijk} = \begin{cases} 
+1 & \text{if } (i, j, k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2) \\
-1 & \text{if } (i, j, k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3) \\
0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i
\end{cases}$$
Appendix F

The Viscous Stress Tensor

As discussed in Chapter 4, the deviatoric stress tensor, \( \tau_{ij} \), is only non-zero in the presence of shear in the fluid flow. This suggests that

\[
\tau_{ij} = T_{ijkl} \frac{\partial u_k}{\partial x_l}
\]

where \( T_{ijkl} \) is a proportionality tensor of \textbf{rank} four. In what follows we derive an expression for \( T_{ijkl} \). We start by noting that since \( \sigma_{ij} \) is symmetric, we also have that \( \tau_{ij} \) will be symmetric. Hence, we expect that the above dependence can only involve the \textbf{symmetric} component of the deformation tensor, \( T_{kl} = \frac{\partial u_k}{\partial x_l} \). Hence, it is useful to split the deformation tensor in its \textbf{symmetric} and \textbf{anti-symmetric} components:

\[
\frac{\partial u_i}{\partial x_j} = e_{ij} + \xi_{ij}
\]

where

\[
e_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right], \quad \xi_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right]
\]

The symmetric part of the deformation tensor, \( e_{ij} \), is called the \textbf{rate of strain tensor}, while the anti-symmetric part, \( \xi_{ij} \), expresses the \textbf{vorticity} \( \vec{\omega} \equiv \nabla \times \vec{u} \) in the velocity field, i.e., \( \xi_{ij} = -\frac{1}{2} \varepsilon_{ijk} w_k \). Note that one can always find a coordinate system for which \( e_{ij} \) is diagonal. The axes of that coordinate frame indicate the \textbf{eigendirections} of the strain (compression or stretching) on the fluid element.

In terms of the relation between the viscous stress tensor, \( \tau_{ij} \), and the deformation tensor, \( T_{kl} \), there are a number of properties that are important.
- **Locality**: The $\tau_{ij} - T_{kl}$-relation is said to be **local** if the stress tensor is only a function of the deformation tensor and thermodynamic state functions like temperature.

- **Homogeneity**: The $\tau_{ij} - T_{kl}$-relation is said to be **homogeneous** if it is everywhere the same. The viscous stress tensor may depend on location $\vec{x}$ only insofar as $T_{ij}$ or the thermodynamic state functions depend on $\vec{x}$. This distinguishes a fluid from a solid, in which the stress tensor depends on the stress itself.

- **Isotropy**: The $\tau_{ij} - T_{kl}$-relation is said to be **isotropic** if it has no preferred direction.

- **Linearity**: The $\tau_{ij} - T_{kl}$-relation is said to be **linear** if the relation between the stress and rate-of-strain is linear. This is equivalent to saying that $\tau_{ij}$ does not depend on $\nabla^2 \vec{u}$ or higher-order derivatives.

A fluid that is local, homogeneous and isotropic is called a **Stokesian fluid**. A Stokesian fluid that is linear is called a **Newtonian fluid**. Experiments have shown that most (astrophysical) fluids are Newtonian to good approximation. Hence, in what follows we will assume that our fluids are Newtonian, unless specifically stated otherwise. For a Newtonian fluid, it can be shown (using linear algebra) that the most general form of our proportionality tensor is given by

$$T_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

Hence, for a **Newtonian fluid** the **viscous stress tensor** is

$$\tau_{ij} = 2\mu e_{ij} + \lambda e_{kk} \delta_{ij}$$

where $\mu$ is the **coefficient of shear viscosity**, $\lambda$ is a scalar, $\delta_{ij}$ is the Kronecker delta function, and $e_{kk} = \text{Tr}(e) = \partial u_k / \partial x_k = \nabla \cdot \vec{u}$ (summation convention).

Note that (in a Newtonian fluid) the viscous stress tensor depends **only** on the symmetric component of the deformation tensor (the rate-of-strain tensor $e_{ij}$), but **not** on the antisymmetric component which describes **vorticity**. You can understand the fact that viscosity and vorticity are unrelated by considering a fluid disk in solid body rotation (i.e., $\nabla \cdot \vec{u} = 0$ and $\nabla \times \vec{u} = \vec{w} \neq 0$). In such a fluid there is no "slippage", hence no shear, and therefore no manifestation of viscosity.
Thus far we have derived that the stress tensor, $\sigma_{ij}$, which in principle has 6 unknowns, can be reduced to a function of three unknowns only ($P$, $\mu$, $\lambda$) as long as the fluid is Newtonian. Note that these three scalars, in general, are functions of temperature and density. We now focus on these three scalars in more detail, starting with the pressure $P$. To be exact, $P$ is the thermodynamic equilibrium pressure, and is normally computed thermodynamically from some equation of state, $P = P(\rho, T)$. It is related to the translational kinetic energy of the particles when the fluid, in equilibrium, has reached equipartition of energy among all its degrees of freedom, including (in the case of molecules) rotational and vibrations degrees of freedom.

In addition to the thermodynamic equilibrium pressure, $P$, we can also define a mechanical pressure, $P_m$, which is purely related to the translational motion of the particles, independent of whether the system has reached full equipartition of energy. The mechanical pressure is simply the average normal stress and therefore follows from the stress tensor according to

$$P_m = -\frac{1}{3} \text{Tr}(\sigma_{ij}) = -\frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})$$

Using that

$$\sigma_{ij} = -P \delta_{ij} + 2\mu e_{ij} + \lambda e_{kk} \delta_{ij}$$

we thus obtain the following relation between the two pressures:

$$P_m = P - \eta \nabla \cdot \mathbf{u}$$

where

$$\eta = \frac{2}{3} \mu + \lambda = \frac{P - P_m}{\nabla \cdot \mathbf{u}}$$

is the coefficient of bulk viscosity. We can now write the stress tensor as

$$\sigma_{ij} = -P \delta_{ij} + \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] + \eta \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

This is the full expression for the stress tensor in terms of the coefficients of shear viscosity, $\mu$, and bulk viscosity, $\eta$. 

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Appendix G

Poisson Brackets

Canonical Coordinates: in classical mechanics, canonical coordinates are coordinates \( q_i \) and \( p_i \) in phase-space that are used in the Hamiltonian formalism and that satisfy the canonical commutation relations:

\[
\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}
\]

Here the curly brackets correspond to Poisson brackets, defined below. Often \( q_i \) are Cartesian coordinates in configuration space and \( p_i \) is the corresponding linear momentum. However, when using curvi-linear coordinates and \( q_i \) is an angle, then the corresponding \( p_i \) is an angular momentum. Hence, \( p_i \) is therefore not always equal to \( m\dot{q}_i \). To avoid confusion, \( p_i \) is called the conjugate momentum.

Poisson Brackets: Given two functions \( A(q_i, p_i) \) and \( B(q_i, p_i) \) of the phase-space coordinates \( q_i \) and \( p_i \), the Poisson bracket of \( A \) and \( B \) is defined as

\[
\{A, B\} = \sum_{i=1}^{3N} \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)
\]

In vector notation,

\[
\{A, B\} = \sum_{i=1}^{N} \left( \frac{\partial A}{\partial \vec{q}_i} \cdot \frac{\partial B}{\partial \vec{p}_i} - \frac{\partial A}{\partial \vec{p}_i} \cdot \frac{\partial B}{\partial \vec{q}_i} \right)
\]

where \( \vec{q}_i = (q_{i1}, q_{i2}, q_{i3}) \) and \( \vec{p}_i = (p_{i1}, p_{i2}, p_{i3}) \) and \( i \) now indicates a particular particle \( (i = 1, 2, ..., N) \).

Note that Poisson brackets are anti-symmetric, i.e., \( \{A, B\} = -\{B, A\} \), and that \( \{A, A\} = 0 \).

Poisson brackets are an extremely powerful construct in classical mechanics (they are the equivalent of commutators in quantum mechanics). To see this, consider a function \( A(q_i, p_i, t) \). If the system is Hamiltonian (i.e., \( q_i \) and \( p_i \) obey the Hamiltonian equations of motion), then we have the following correspondance:
\[ \frac{dA}{dt} = 0 \quad \iff \quad \frac{\partial A}{\partial t} + \{ A, H \} = 0 \]

To see this, simply write out the Poisson brackets, and substitute the Hamiltonian equations of motion:

\[
\{ A, H \} = \sum_{i=1}^{N} \left( \frac{\partial A}{\partial \vec{q}_i} \cdot \frac{\partial H}{\partial \vec{p}_i} - \frac{\partial A}{\partial \vec{p}_i} \cdot \frac{\partial H}{\partial \vec{q}_i} \right) = \sum_{i=1}^{N} \left( \frac{\partial A}{\partial \vec{q}_i} \cdot \dot{\vec{q}}_i + \frac{\partial A}{\partial \vec{p}_i} \cdot \dot{\vec{p}}_i \right)
\]

Hence, a function of the phase-space coordinates whose Poisson bracket with the Hamiltonian vanishes is conserved along its orbit/path in a static system (static means that all partial time derivates are equal to zero). Such a quantity is called an integral of motion.

As we have seen in Chapter 6, the Liouville equation states that the Lagrangian time-derivative of the \( N \)-point DF vanishes, i.e., \( df^{(N)}/dt = 0 \). Using the above, we can write this as

\[
\frac{\partial f^{(N)}}{\partial t} + \{ f^{(N)}, H \} = 0
\]

which is yet another way of writing the Liouville equation. We can also use the Poisson brackets to rewrite the Collisionless Boltzmann equation as

\[
\frac{\partial f^{(1)}}{\partial t} + \{ f^{(1)}, H \} = 0
\]

Note that in the latter the Hamiltonian is a function of 6 phase-space coordinates \((\vec{q}, \vec{p})\), whereas in the former the Hamiltonian depends on the \( 6N \) phase-space coordinates of all \( N \) particles, i.e., on \((\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N)\).

The Poisson brackets are useful for writing equations in more compact form, and we will use them for that reason in Appendix H below when discussing the BBGKY hierarchy.
Appendix H

The BBGKY Hierarchy

This Appendix makes use of the canonical variables and Poisson brackets. Readers not familiar with these should first read Appendix G above.

As derived in Chapter 6, the Liouville equation

$$\frac{d}{dt} f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N) = 0$$

expresses the incompressibility of Hamiltonian flow in $\Gamma$-space. Here $\vec{w}_i \equiv (\vec{q}_i, \vec{p}_i)$ is the 6D phase-space vector of particle $i$. In what follows we assume that all particles are (statistically) identical. This implies that $f^{(N)}$ is a symmetric function of $\vec{w}_i$, such that

$$f^{(N)}(..., \vec{w}_i, ..., \vec{w}_j, ...) = f^{(N)}(..., \vec{w}_j, ..., \vec{w}_i, ...) \quad \forall (i, j)$$

In words; if you flip the indices of any two particles, nothing changes. This allows us to derive an equation describing the evolution of the 1-point distribution function $f^{(1)}(\vec{w})$, as follows.

We first define the reduced or K-body DF, which is obtained by integrating the $N$-body DF, $f^{(N)}$, over $N - K$ six-vectors $\vec{w}_i$. Since $f^{(N)}$ is symmetric in $\vec{w}_i$, without loss of generality we may choose the integration variables to be $\vec{w}_{K+1}, \vec{w}_{K+2}, ..., \vec{w}_N$:

$$f^{(K)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_K, t) \equiv \frac{N!}{(N - K)!} \int \prod_{i=K+1}^N d^6 \vec{w}_i f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N, t)$$

where the choice of the prefactor will become clear in what follows.

In particular, the 1-particle distribution function is

$$f^{(1)}(\vec{w}_1, t) \equiv N \int \prod_{i=2}^N d^6 \vec{w}_i f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N, t)$$

Because of the prefactor, we now have that
\[ \int d^6 \vec{w} \, f^{(1)}(\vec{w}, t) = \int d^3 \vec{q} \int d^3 \vec{p} \, f^{(1)}(\vec{q}, \vec{p}, t) = N \]

Hence, \( f^{(1)}(\vec{q}, \vec{p}, t) = dN/d^3 \vec{q} d^3 \vec{p} \) is the number of particles in the phase-space volume \( d^3 \vec{q} d^3 \vec{p} \) centered on \( (\vec{q}, \vec{p}) \).

That \( f^{(1)}(\vec{w}, t) \) is an important, relevant DF is evident from the following. Consider an observable \( Q(\vec{w}) \) that involves only quantities that depend additively on the phase-space coordinates of single, individual particles \( \text{i.e., } Q_{\text{ensemble}} = Q(\vec{w}_1) + Q(\vec{w}_2) + ... + Q(\vec{w}_N) \). Examples are velocity, kinetic energy, or any other velocity moment \( v^k \). The expectation value, \( \langle Q \rangle \), can be written as

\[
\langle Q \rangle = \int d^6 \vec{w}_1 ... d^6 \vec{w}_N \, f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N) \sum_{i=1}^{N} Q_i
\]

Since all particles are statistically identical, we can rewrite this as

\[
\langle Q \rangle = \int d^6 \vec{w}_1 \, Q(\vec{w}_1) \, f^{(1)}(\vec{w}_1)
\]

Hence, computing the expectation value for any observable \( Q(\vec{w}) \) only requires knowledge of the one-particle DF.

For the time evolution of each reduced DF we can write

\[
\frac{\partial f^{(K)}}{\partial t} = \frac{N!}{(N-K)!} \int \prod_{i=K+1}^{N} d^6 \vec{w}_i \, \frac{\partial f^{(N)}}{\partial t}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N) \]

\[
= \frac{N!}{(N-K)!} \int \prod_{i=K+1}^{N} d^6 \vec{w}_i \, \{H, f^{(N)}\}
\]

Now we substitute the Hamiltonian. To do so, we use that \( \vec{w}_i = (\vec{q}_i, \vec{p}_i) \), is related to the Cartesian position and velocity coordinates, \( \vec{r}_i \) and \( \vec{v}_i \), respectively. This allows us to write

\[
\mathcal{H}(\vec{r}_i, \vec{p}_i, t) = \mathcal{H}(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N)
\]

\[
= \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m} + \sum_{i=1}^{N} V(\vec{r}_i) + \sum_{i<j} U(\vec{r}_i - \vec{r}_j)
\]
Note that the Hamiltonian contains three terms; a kinetic energy term, a term describing the potential energy due to an external force $\vec{F} = -\nabla V$ that acts equally on all particles, and the potential energy $U(\vec{r}_i - \vec{r}_j)$ related to two-body interactions between particles $i$ and $j$. Examples of the latter are the VanderWaals force in the case of a liquid, the Coulomb force in the case of a plasma, and the gravitational force in the case of dark matter halo. Without derivation (see any good textbook on kinetic theory), substituting this Hamiltonian into the evolution equation for the one-particle DF yields

$$\frac{\partial f^{(1)}}{\partial t} = N \int \prod_{i=2}^{N} d^6 \vec{w}_i \left\{ \mathcal{H}, f^{(N)} \right\} = \left\{ \mathcal{H}^{(1)}, f^{(1)} \right\} + \left( \frac{\partial f^{(1)}}{\partial t} \right)_{\text{coll}}$$

where $\mathcal{H}^{(1)} = \vec{p}_i^2/(2m) + V(\vec{r})$ is the one-particle Hamiltonian, and

$$\left( \frac{\partial f^{(1)}}{\partial t} \right)_{\text{coll}} = \int d^3 \vec{r}_2 d^3 \vec{p}_2 \frac{\partial U(\vec{r} - \vec{r}_2)}{\partial \vec{r}} \cdot \frac{\partial f^{(2)}}{\partial \vec{p}}$$

Using the convective derivative, the above can be written in concise form as

$$\frac{df^{(1)}}{dt} = \left( \frac{\partial f^{(1)}}{\partial t} \right)_{\text{coll}}$$

which expresses that the 1-particle DF at the location of any particle only changes in time due to collisions which scatter particles into or out of phase-space. Note that the collision term depends on the two-particle DF, $f^{(2)}$. In fact, you can write a similar equation for $\partial f^{(2)}/\partial t$, but will find that it depends on $f^{(3)}$, etc. The resulting set of $N$ coupled equations is known as the BBGKY hierarchy (after Bogoliubov, Born, Green, Kirkwood and Yvon, who discovered the equations independently between 1935 and 1946).

The resulting set of $N$ coupled equations is known as the BBGKY hierarchy (after Bogoliubov, Born, Green, Kirkwood and Yvon, who discovered the equations independently between 1935 and 1946).

We started with the Liouville equation, governing a complicated function of $N$ variable, and it looks like all we have achieved is to replace it with a set of $N$ coupled equations. However, the BBKGY hierarchy is useful since it allows us to make some simplifying assumptions (which will be sufficiently accurate under certain conditions), that truncates the series. The simplest and most useful of these truncations is the Boltzmann equation, which is a closed equation for $f^{(1)}$ alone given by
\[
\frac{df^{(1)}}{dt} = \left( \frac{\partial f^{(1)}}{\partial t} \right)_{\text{coll}} = I[f^{(1)}]
\]

where \( I[f^{(1)}] \) is called the collision integral.

A more heuristic derivation of the Boltzmann equation is given in Chapter 6, while an excellent textbook for a more detailed discussion and derivation of the BBGKY hierarchy is *Introduction to Plasma Theory* by Dwight R. Nicholson.
Consider a system which can exchange energy and particles with a reservoir, and the volume of which can change. There are three ways for this system to increase its internal energy; heating, changing the system’s volume (i.e., doing work on the system), or adding particles. Hence,

\[ dU = T \, dS - P \, dV + \mu \, dN \]

Note that this is the first law of thermodynamics, but now with the added possibility of changing the number of particles of the system. The scalar quantity \( \mu \) is called the chemical potential, and is defined by

\[ \mu = \left( \frac{\partial U}{\partial N} \right)_{S,V} \]

Hence, the chemical potential quantifies how the internal energy of the system changes if particles are added or removed, while keeping the entropy and volume of the system fixed. The chemical potential appears in the Fermi-Dirac distribution describing the momentum distribution of a gas of fermions or bosons, and therefore enters whenever one tries to derive an equation of state of the system in question (see Chapter 10 for details).

Consider an ideal gas, of volume \( V \), entropy \( S \) and with internal energy \( U \). Now imagine adding a particle of zero energy \( (\epsilon = 0) \), while keeping the volume fixed. Since \( \epsilon = 0 \), we also have that \( dU = 0 \). But what about the entropy? Well, we have increased the number of ways in which we can redistribute the energy \( U \) (a macrostate quantity) over the different particles (different microstates). Hence, by adding this particle we have increased the system’s entropy. If we want to add a particle while keeping \( S \) fixed, we need to decrease \( U \) to offset the increase in the number of ‘degrees of freedom’ over which to distribute this energy. Hence, keeping \( S \) (and \( V \)) fixed, requires that the particle has negative energy, and we thus see that \( \mu < 0 \).
For a fully degenerate Fermi gas, we have that \( T = 0 \), and thus \( S = 0 \) (i.e., there is only one micro-state associated with this macrostate, and that is the fully degenerate one). If we now add a particle, and demand that we keep \( S = 0 \), then that particle must have the Fermi energy (see Chapter 13); \( \epsilon = E_f \). Hence, for a fully degenerate gas, \( \mu = E_f \).

Finally, consider a photon gas in thermal equilibrium inside a container. Contrary to an ideal gas, in a photon gas the number of particles (photons) cannot be arbitrary. The number of photons at given temperature, \( T \), and thus at given \( U \), is given by the Planck distribution and is constantly adjusted (through absorption and emission against the wall of the container) so that the photon gas remains in thermal equilibrium. In other words, \( N_\gamma \) is not a degree of freedom for the system, but it set by the volume and the temperature of the gas. Since we can’t change \( N_\gamma \) while maintaining \( S \) (or \( T \)) and \( V \), we have that \( \mu = 0 \) for photons.

To end this discussion of the chemical potential, we address the origin of its name, which may, at first, seem weird. Let’s start with the ‘potential’ part. The origin of this name is clear from the following. According to its definition (see above), the chemical potential is the ‘internal energy’ per unit amount (moles). Now consider the following correspondences:

Gravitational potential is the gravitational energy per unit mass:

\[
W = \frac{G m_1 m_2}{r} \quad \Rightarrow \quad \phi = \frac{G m}{r} \quad \Rightarrow \quad \phi = \frac{\partial W}{\partial m}
\]

Similarly, electrical potential is the electrical energy per unit charge

\[
V = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r} \quad \Rightarrow \quad \phi = \frac{1}{4\pi \varepsilon_0} \frac{q}{r} \quad \Rightarrow \quad \phi = \frac{\partial V}{\partial q}
\]

These examples make it clear why \( \mu \) is considered a ‘potential’. Finally, the word chemical arises from the fact that the \( \mu \) plays an important role in chemistry (i.e., when considering systems in which chemical reactions take place, which change the particles). In this respect, it is important to be aware of the fact that \( \mu \) is an additive quantity that is conserved in a chemical reaction. Hence, for a chemical
reaction \( i + j \rightarrow k + l \) one has that \( \mu_i + \mu_j = \mu_k + \mu_l \). As an example, consider the annihilation of an electron and a positron into two photons. Using that \( \mu = 0 \) for photons, we see that the chemical potential of elementary particles (i.e., electrons) must be opposite to that of their anti-particles (i.e., positrons).

Because of the additive nature of the chemical potential, we also have that the above equation for \( dU \) changes slightly whenever the gas consists of different particle species; it becomes

\[
dU = T \, dS - P \, dV + \sum_i \mu_i \, dN_i
\]

where the summation is over all species \( i \). If the gas consists of equal numbers of elementary particles and anti-particles, then the total chemical potential of the system will be equal to zero. In fact, in many treatments of fluid dynamics it may be assumed that \( \sum_i \mu_i \, dN_i = 0 \); in particular when the relevant reactions are ‘frozen’ (i.e., occur on a timescales \( \tau_{\text{react}} \) that are much longer than the dynamical timescales \( \tau_{\text{dyn}} \) of interest), so that \( dN_i = 0 \), or if the reactions go so fast (\( \tau_{\text{react}} \ll \tau_{\text{dyn}} \)) that each reaction and its inverse are in local thermodynamic equilibrium, in which case \( \sum_i \mu_i \, dN_i = 0 \) for those species involved in the reaction. Only in the rare, intermediate case when \( \tau_{\text{react}} \sim \tau_{\text{dyn}} \) is it important to keep track of the relative abundances of the various chemical and/or nuclear species.
Appendix J

Radiation Essentials

**Spectral Energy Distribution:** The radiation from a source may be characterized by its spectral energy distribution (SED), $L_\nu \, d\nu$, or, equivalently, $L_\lambda \, d\lambda$. Some texts refer to the SEDs as the *spectral luminosity* or the *spectral power*. The SED is the total energy emitted by photons in the frequency interval $[\nu, \nu + d\nu]$, and is related to the total *luminosity*, $L \equiv dE/dt$, according to

$$ L = \int L_\nu \, d\nu = \int L_\lambda \, d\lambda $$

Note that $[L_\nu] = \text{erg s}^{-1} \text{Hz}^{-1}$, while $[L] = \text{erg s}^{-1}$.

**Flux:** The flux, $f$, of a source is the radiation energy per unit time passing through a unit area

$$ dL = f \, dA \quad [f] = \text{erg s}^{-1} \text{cm}^{-2} $$

where $A$ is the area. Similarly, we can also define the *spectral flux density* (or simply ‘flux density’), as the flux per unit spectral bandwidth:

$$ dL_\nu = f_\nu \, dA \quad [f_\nu] = \text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1} $$

In radio astronomy, one typically expresses $f_\nu$ in **Jansky**, where $1 \text{Jy} = 10^{-23} \text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1}$. As with the SEDs, one may also express spectral flux densities as $f_\lambda$. Using that $\lambda = c/\nu$, and using that $f_\nu \, d\nu = f_\lambda \, d\lambda$ one has that

$$ f_\nu = \frac{\lambda^2}{c} f_\lambda, \quad f_\lambda = \frac{\nu^2}{c} f_\nu $$

**Luminosity** and **flux** are related according to

$$ L = 4 \pi r^2 f $$

where $r$ is the distance from the source.
Figure 36: Diagrams showing intensity and its dependence on direction and solid angle. Fig. (a) depicts the ‘observational view’, where \( dA \) represents an element of a detector. The arrows show incoming rays from the center of the source. Fig. (b) depicts the ‘emission view’, where \( dA \) represents the surface of a star. At each point on the surface, photons leave in all directions away from the surface.

**Intensity:** The intensity, \( I \), also called surface brightness is the flux emitted in, or observed from, a solid angle \( d\Omega \). The intensity is related to the flux via

\[
df = I \cos \theta \, d\Omega
\]

where \( \theta \) is the angle between the normal of the surface area through which the flux is measured and the direction of the solid angle. The unit of intensity is \([I] = \text{erg} \, \text{s}^{-1} \, \text{cm}^{-2} \, \text{sr}^{-1}\). Here ‘sr’ is a steradian, which is the unit of solid angle measure (there are \( 4\pi \) steradians in a complete sphere). As with the flux and luminosity, one can also define a specific intensity, \( I_\nu \), which is the intensity per unit spectral bandwidth (\([I_\nu] = \text{erg} \, \text{s}^{-1} \, \text{cm}^{-2} \, \text{Hz}^{-1} \, \text{sr}^{-1}\)).

The flux emerging from the surface of a star with luminosity \( L \) and radius \( R_\star \) is

\[
F \equiv \frac{L}{4\pi R_\star^2} = \int_{\text{half sphere}} I \cos \theta \, d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \ I \cos \theta \sin \theta = \pi \ I
\]

where we have used that \( d\Omega = \sin \theta \, d\theta \, d\phi \), and the fact that the integration over the solid angle \( \Omega \) is only to be performed over half a sphere. Note that an observer can only measure the surface brightness of resolved objects; if unresolved, the observer can only measure the objects flux.
Consider a resolved object (i.e., a galaxy), whose surface brightness distribution on the sky is given by $I(\Omega)$. If the object’s extent is a solid angle $\Omega_S$ on the sky, its flux is given by

$$f = \int_{\Omega_S} I(\Omega) \cos \theta \, d\Omega \simeq \int I(\Omega) \, d\Omega \equiv \langle I \rangle \Omega_S$$

where we have assumed that $\Omega_S$ is small, so that variations of $\cos \theta$ across the object can be neglected. Since both $f \propto r^{-2}$ and $\Omega_S \propto r^{-2}$, where $r$ is the object’s distance, we see that the average surface brightness $\langle I \rangle$ is independent of distance.

**Energy density:** the energy density, $u$, is a measure of the radiative energy per unit volume (i.e., $[u] = \text{erg cm}^{-3}$). If the radiation intensity as seen from some specific location in space is given by $I(\Omega)$, then the energy density at that location is

$$u = \frac{1}{c} \int I \, d\Omega \equiv \frac{4\pi}{c} J$$

where

$$J = \frac{1}{4\pi} \int I \, d\Omega$$

is the **mean intensity** (i.e., average over $4\pi$ steradian). If the radiation is isotropic (i.e., the center of a star, or, to good approximation, a random location in the early Universe), then $J = I$. If the radiation intensity is due to the summed intensity from a number of individual sources, then $u = \frac{1}{c} \sum_i f_i$, where $f_i$ is the flux due to source $i$.

Recall from Chapter 11 that the number density of photons emerging from a **Black Body** of temperature $T$ is given by

$$n_\gamma(\nu, T) \, d\nu = \frac{8\pi \nu^2}{c^3} \frac{d\nu}{e^{h\nu/k_B T} - 1}$$

Hence, we have that

$$u(\nu, T) \, d\nu = n_\gamma(\nu, T) \, h\nu \, d\nu = \frac{8\pi h \nu^3}{c^3} \frac{d\nu}{e^{h\nu/k_B T} - 1}$$

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Figure 37: Various Planck curves for different temperatures, illustrating Wien’s displacement law. Note how the Planck curve for a black body with the temperature of the Sun peaks at the visible wavelengths, where the sensitivity of our eyes is maximal.

Using that $u(\nu, T) = (4\pi/c)J(\nu(T))$ we have that the mean specific intensity from a black body [for which one typically uses the symbol $B(\nu(T))$] is given by

$$B(\nu(T)) \ d\nu = \frac{2h \nu^3}{c^2} \frac{\mathrm{d}\nu}{e^{h\nu/k_B T} - 1}$$

which is called the Planck curve (or ‘formula’). Integrating over frequency yields the total, mean intensity emitted from the surface of a Black Body

$$J = J(T) = \int_0^\infty B(\nu(T)) \ d\nu = \frac{\sigma_{\text{SB}}}{\pi} T^4$$

where $\sigma_{\text{SB}}$ is the Stefan-Boltzmann constant. This implies an energy density

$$u = u(T) = \frac{4\pi}{c} J = \frac{4\sigma_{\text{SB}}}{c} T^4 \equiv a_r T^4$$

where $a_r \approx 7.6 \times 10^{-15} \text{erg cm}^{-3} \text{K}^{-4}$ is called the radiation constant (see also Chapter 11).

**Wien’s Displacement Law:** When the temperature of a Black Body emitter increases, the overall radiated energy increases and the peak of the radiation curve
moves to shorter wavelengths. It is straightforward to show that the product of the temperature and the wavelength at which the Planck curve peaks is a constant, given by

$$\lambda_{\text{max}} T = 0.29$$

where \( T \) is the \textit{absolute temperature}, expressed in degrees Kelvin, and \( \lambda_{\text{max}} \) is expressed in cm. This relation is called \textbf{Wien’s Displacement Law}.

\textbf{Stefan-Boltzmann Law:} The flux emitted by a Black Body is

$$F_{\text{BB}} = \pi I(T) = \sigma_{\text{SB}} T^4$$

which is known as the \textbf{Stefan-Boltzmann law}. This law is used to define the effective temperature of an emitter.

\textbf{Effective Temperature:} The temperature an emitter of flux \( F \) would have if it where a Black Body; using the Stefan-Boltzmann law we have that \( T_{\text{eff}} = (F/\sigma_{\text{SB}})^{1/4} \). We can also use the effective temperature to express the emitter’s luminosity;

$$L = 4 \pi R^2 \sigma_{\text{SB}} T_{\text{eff}}^4$$

where \( R \) is the emitter’s radius. The effective temperature is sometimes also called the \textbf{radiation temperature}, as a measure for the temperature associated with the radiation field.

\textbf{Brightness Temperature:} the brightness temperature, \( T_B(\nu) \), of a source at frequency \( \nu \) is defined as the temperature which, when put into the Planck formula, yields the specific intensity actually measured at that frequency. Hence, for a Black Body \( T_B(\nu) \) is simply equal to the temperature of the Black Body. If \( T_B(\nu) \) depends on frequency, then the emitter is not a Black Body. The brightness temperature is a frequency-dependent version of the effective, or radiation, temperature.

\textbf{Wavebands:} Astronomers typically measure an object’s flux through some filter (waveband). The measured flux in ‘band’ \( X \) is, \( f_X \), is related to the spectral flux density, \( f_\lambda \), of the object according to

$$f_X = \int f_\lambda F_X(\lambda) R(\lambda) T(\lambda) \, d\lambda$$
Here \( F_X(\lambda) \) describes the transmission of the filter that defines waveband \( X \), \( R(\lambda) \) is the transmission efficiency of the telescope + instrument, and \( T(\lambda) \) describes the transmission of the atmosphere. The combined effect of \( F_X \), \( R \), and \( T \) is typically ‘calibrated’ using standard stars with known \( f_\lambda \).

**Magnitudes:** For historical reasons, the flux of an astronomical object in waveband \( X \) is usually quoted in terms of **apparent magnitude**:

\[
m_X = -2.5 \log \left( \frac{f_X}{f_{X,0}} \right)
\]

where the flux zero-point \( f_{X,0} \) has traditionally been taken as the flux in the \( X \) band of the bright star Vega. In recent years it has become more common to use ‘AB-magnitudes’, for which

\[
f_{X,0} = 3.6308 \times 10^{-20} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1} \int F_X(c/\nu) \, d\nu
\]

Similarly, the luminosities of objects (in waveband \( X \)) are often quoted as an **absolute magnitude**:

\[
M_X = -2.5 \log(L_X) + C_X
\]

where \( C_X \) is a zero point. It is usually convenient to write \( L_X \) in units of the solar luminosity in the same band, \( L_{\odot,X} \), so that

\[
M_X = -2.5 \log \left( \frac{L_X}{L_{\odot,X}} \right) + M_{\odot,X},
\]

where \( M_{\odot,X} \) is the absolute magnitude of the Sun in the waveband in consideration. Using the relation between luminosity and flux we have that

\[
m_X - M_X = 5 \log(r/r_0)
\]

where \( r_0 \) is a fiducial distance at which \( m_X \) and \( M_X \) are defined to have the same value. Conventionally, \( r_0 \) is chosen to be 10 pc.

**Distance modulus:** the distance modulus of an object is defined as \( m_X - M_X \).