Dynamics of Collisionless Systems

Summer Semester 2005, ETH Zürich

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Useful Information

TEXTBOOK: Galactic Dynamics, **Binney & Tremaine**
Princeton University Press *Highly Recommended*

WEBPAGE: http://www.exp-astro.phys.ethz.ch/ vdbosch/galdyn.html

LECTURES: Wed, 14.45-16.30, HPP H2. Lectures will be in English

EXERSIZE CLASSES: to be determined

HOMEWORK ASSIGNMENTS: $\pm$ every other week

EXAM: Verbal (German possible), July/August 2005

GRADING: exam $(2/3)$ plus homework assignments $(1/3)$

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Outline

Lecture 1: Introduction & General Overview

Lecture 2: Cancelled

Lecture 3: Potential Theory

Lecture 4: Orbits I (Introduction to Orbit Theory)

Lecture 5: Orbits II (Resonances)

Lecture 6: Orbits III (Phase-Space Structure of Orbits)

Lecture 7: Equilibrium Systems I (Jeans Equations)

Lecture 8: Equilibrium Systems II (Jeans Theorem in Spherical Systems)

Lecture 9: Equilibrium Systems III (Jeans Theorem in Spheroidal Systems)

Lecture 10: Relaxation & Virialization (Violent Relaxation & Phase Mixing)

Lecture 11: Wave Mechanics of Disks (Spiral Structure & Bars)

Lecture 12: Collisions between Collisionless Systems (Dynamical Friction)

Lecture 13: Kinetic Theory (Fokker-Planck Eq. & Core Collapse)

Lecture 14: Cancelled

Summary of Vector Calculus I

$\vec{A} \cdot \vec{B} = \text{scalar} = |\vec{A}|\,|\vec{B}| \cos\psi = A_i B_i$ **(summation convention)**

$\vec{A} \times \vec{B} = \text{vector} = \epsilon_{ijk} \vec{e}_i A_j B_k$ **(with ϵ_{ijk} the Levi-Civita Tensor)**

Useful to Remember

$\vec{A} \times \vec{A} = 0$

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

$\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$

$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$

$\vec{\nabla}$ =**vector operator**= $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

$$\begin{array}{lcll} \vec{\nabla} S & = & \text{grad} S & = \text{vector} \\ \vec{\nabla} \cdot \vec{A} & = & \text{div} \vec{A} & = \text{scalar} \\ \vec{\nabla} \times \vec{A} & = & \text{curl} \vec{A} & = \text{vector} \end{array}$$

Summary of Vector Calculus II

Laplacian: $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} =$ **scalar operator** $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$
\begin{array}{lcll}
\nabla^2 S & = & \vec{\nabla} \cdot (\vec{\nabla} S) & = \text{scalar} \\
\nabla^2 \vec{A} & = & (\vec{\nabla} \cdot \vec{\nabla}) \vec{A} & = \text{vector} \\
\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) & \neq & \nabla^2 \vec{A} & = \text{vector}
\end{array}
$$

$$
\begin{array}{lcll}
\vec{\nabla} \times (\vec{\nabla} S) & = & 0 & \text{curl}(\text{grad} S) = 0 \\
\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) & = & 0 & \text{div}(\text{curl}\, \vec{A}) = 0 \\
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) & = & \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} &
\end{array}
$$

$$
\begin{array}{lcl}
\vec{\nabla}(ST) & = & S\vec{\nabla}T + T\vec{\nabla}S \\
\vec{\nabla} \cdot (S\vec{A}) & = & S(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot \vec{\nabla} S \\
\vec{\nabla} \times (S\vec{A}) & = & S(\vec{\nabla} \times \vec{A}) - \vec{A} \times \vec{\nabla} S \\
\vec{\nabla} \cdot (\vec{A} \times \vec{B}) & = & \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})
\end{array}
$$

Integral Theorems I

Gradient Theorem: **Let γ be a curve running from $\vec{x}_0$ to $\vec{x}_1$, $\mathrm{d}\vec{l}$ is the directed element of length along γ, and $\phi(\vec{x})$ is a scalar field then:**

$$\int_{\vec{x}_0}^{\vec{x}_1} \vec{\nabla}\phi \cdot \mathrm{d}\vec{l} = \int_{\vec{x}_0}^{\vec{x}_1} \mathrm{d}\phi = \phi(\vec{x}_1) - \phi(\vec{x}_0)$$

It follows that

$$\oint \vec{\nabla}\phi \cdot \mathrm{d}\vec{l} = 0$$

Divergence Theorem (Gauss' Theorem): **Let V be a 3D volume bounded by a 2D surface S, and let $\vec{A}(\vec{x})$ be a vector field, then:**

$$\int_V \vec{\nabla} \cdot \vec{A}\, \mathrm{d}^3\vec{x} = \int_S \vec{A} \cdot \mathrm{d}^2\vec{S}$$

Curl Theorem (Stokes' Theorem): **Let S be a 2D surface bounded by a 1D curve γ, and let $\vec{A}(\vec{x})$ be a vector field, then:**

$$\int_S (\vec{\nabla} \times \vec{A})\, \mathrm{d}^2\vec{S} = \oint_\gamma \vec{A} \cdot \mathrm{d}\vec{l}$$

Integral Theorems II

NOTE: Since a **conservative** force $\vec{F}$ can always be written as the gradient of a **scalar field** ϕ, we have from the **gradient theorem** that

$$\oint \vec{F} \cdot \mathrm{d}\vec{l} = 0$$

From the **curl theorem** we immediately see that

$$\vec{\nabla} \times \vec{F} = 0$$

We immediately infer that a **conservative** force is **curl free**, and that the amount of work done ($\mathrm{d}W = \vec{F} \cdot \mathrm{d}\vec{r}$) is independent of the path taken.

From the **divergence theorem** we infer that

$$\int_V \phi \vec{\nabla} \cdot \vec{A}\, \mathrm{d}^3\vec{x} = \int_S \phi \vec{A} \cdot \mathrm{d}^2\vec{S} - \int_V \vec{A} \cdot \vec{\nabla}\phi\, \mathrm{d}^3\vec{x}$$

which is the three-dimensional analog of **integration by parts**

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = \int \mathrm{d}(uv) - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x$$

Curvi-Linear Coordinate Systems I

In addition to the Cartesian coordinate system (x, y, z), we will often work with cylindrical (R, ϕ, z) or spherical (r, θ, ϕ) coordinate systems

Let (q_1, q_2, q_3) denote the coordinates of a point in an arbitrary coordinate system, defined by the metric tensor h_{ij}. The distance between (q_1, q_2, q_3) and $(q_1 + \mathrm{d}q_1, q_2 + \mathrm{d}q_2, q_3 + \mathrm{d}q_3)$ is

$$\mathrm{d}s^2 = h_{ij}\ \mathrm{d}q_i\, \mathrm{d}q_j \qquad \text{(summation convention)}$$

We will only consider orthogonal systems for which $h_{ij} = 0$ if $i \neq j$, so that $\mathrm{d}s^2 = h_i^2\ \mathrm{d}q_i^2$ with

$$h_i \equiv h_{ii} = \left|\frac{\partial \vec{x}}{\partial q_i}\right|$$

The differential vector is

$$\mathrm{d}\vec{x} = \frac{\partial \vec{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial \vec{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial \vec{x}}{\partial q_3}\mathrm{d}q_3$$

The unit directional vectors are

$$\vec{e}_i = \frac{\partial \vec{x}}{\partial q_i} \Big/ \left|\frac{\partial \vec{x}}{\partial q_i}\right| = \frac{1}{h_i}\frac{\partial \vec{x}}{\partial q_i}$$

so that $\mathrm{d}\vec{x} = \sum_i h_i\, \mathrm{d}q_i\, \vec{e}_i$ and $\mathrm{d}^3\vec{x} = h_1 h_2 h_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3$.

Curvi-Linear Coordinate Systems II

The gradient:

$$\vec{\nabla}\psi = \frac{1}{h_i}\frac{\partial \psi}{\partial q_i}\vec{e}_i$$

The divergence:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_3 h_1 A_2) + \frac{\partial}{\partial q_3}(h_1 h_2 A_3)\right]$$

The curl (only one component shown)**:**

$$(\vec{\nabla}\times\vec{A})_3 = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(h_2 A_2) - \frac{\partial}{\partial q_2}(h_1 A_1)\right]$$

The Laplacian:

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\right]$$

Cylindrical Coordinates

For cylindrical coordinates (R, ϕ, z) we have that

$$x = R\cos\phi \qquad y = R\sin\phi \qquad z = z$$

The scale factors of the metric are:

$$h_R = 1 \qquad h_\phi = R \qquad h_z = 1$$

and the position vector is $\vec{x} = R\vec{e}_R + z\vec{e}_z$

Let $\vec{A} = A_R\vec{e}_R + A_\phi\vec{e}_\phi + A_z\vec{e}_z$ an arbitrary vector, then

$$\begin{aligned} A_R &= A_x\cos\phi - A_y\sin\phi \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \\ A_z &= A_z \end{aligned}$$

Velocity: $\vec{v} = \dot{R}\vec{e}_R + R\dot{\vec{e}}_R + \dot{z}\vec{e}_z = \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z$

Gradient & Laplacian:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{R}\frac{\partial}{\partial R}(RA_R) + \frac{1}{R}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2\psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical Coordinates

For spherical coordinates (r, θ, ϕ) we have that

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

The scale factors of the metric are:

$$h_r = 1 \qquad h_\theta = r \qquad h_\phi = r\sin\theta$$

and the position vector is $\vec{x} = r\vec{e}_r$

Let $\vec{A} = A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\phi\vec{e}_\phi$ an arbitrary vector, then

$$\begin{aligned} A_r &= A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta \\ A_\theta &= A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \end{aligned}$$

Velocity: $\vec{v} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\dot{\phi}\vec{e}_\phi$

Gradient & Laplacian:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\psi^2}$$

Introduction

COLLISIONLESS DYNAMICS: **The study of the motion of large numbers of point particles orbiting under the influence of their mutual self-gravity**

EAMPLES OF COLLISIONLESS SYSTEMS

- **Galaxies (ellipticals & disk galaxies)** $N \sim 10^6 - 10^{11}$
- **Globular clusters** $N \sim 10^4 - 10^6$
- **Galaxy clusters** $N \sim 10^2 - 10^3$
- **Cold Dark Matter haloes** $N \gg 10^{50}$

MAIN GOALS

- **Infer mass distribution from observed kinematics. Comparison with light distribution $\Rightarrow$ learn about dark matter and black holes**
- **Understand observed structure of galaxies:**
  1. **Galaxies formed this way $\Rightarrow$ learn about Galaxy Formation**
  2. **Galaxies evolved this way $\Rightarrow$ learn about Stability of galaxies**

Newtonian Gravity

Newton's First Law: A body acted on by no forces moves with uniform velocity in a straight line

Newton's Second Law: $\vec{F} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$ (equation of motion)

Newton's Third Law: $\vec{F}_{ij} = -\vec{F}_{ji}$ (action = reaction)

Newton's Law of Gravity: $\vec{F}_{ij} = -G\frac{m_i m_j}{|\vec{x}_i - \vec{x}_j|^3}(\vec{x}_i - \vec{x}_j)$

$$G = 6.67 \times 10^{-11} Nm^2\,\mathrm{kg}^{-2} = 4.3 \times 10^{-9} (\,\mathrm{km\,s}^{-1})^2\,\mathrm{M}_\odot{}^{-1}\,\mathrm{Mpc}$$

Gravity is a conservative Force. This implies that:

- $\exists$ scalar field $V(\vec{x})$ (potential energy), so that $\vec{F} = -\vec{\nabla}V(\vec{x})$
- The total energy $E = \frac{1}{2}mv^2 + V(\vec{x})$ is conserved
- Gravity is a curl-free field: $\vec{\nabla} \times \vec{F} = 0$

Gravity is a central Force. This implies that:

- The moment about the center vanishes: $\vec{r} \times \vec{F} = 0$
- Angular momentum $\vec{J} = m\vec{r} \times \vec{v}$ is conserved: $(\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = \vec{r} \times \vec{F} = 0)$

The Gravitational Potential

Potential Energy: $\vec{F}(\vec{x}) = -\vec{\nabla} V(\vec{x})$

Gravitational Potential: $\Phi(\vec{x}) = \frac{V(\vec{x})}{m}$

Gravitational Field: $\vec{g}(\vec{x}) = \frac{\vec{F}(\vec{x})}{m} = -\vec{\nabla}\Phi(\vec{x})$

From now on $\vec{F}$ is the force per unit mass so that $\vec{F}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x})$

For a point mass M at $\vec{x}_0$: $\Phi(\vec{x}) = -\frac{GM}{|\vec{x}-\vec{x}_0|}$

For a density distribution $\rho(\vec{x})$: $\Phi(\vec{x}) = -G\int \frac{\rho(\vec{x}')}{|\vec{x}'-\vec{x}|}\mathrm{d}^3\vec{x}'$

The density distribution $\rho(\vec{x})$ and gravitational potential $\Phi(\vec{x})$ are related to each other by the Poisson Equation

$$\boxed{\nabla^2\Phi = 4\pi G\rho}$$

For $\rho = 0$ this reduces to the Laplace equation: $\nabla^2\Phi = 0$.

see B&T p.31 for derivation of Poisson Equation

Gauss's Theorem & Potential Theorem

If we integrate the Poisson Equation, we obtain

$$4\pi G \int \rho \, \mathrm{d}^3\vec{x} = 4\pi G M = \int_V \nabla^2 \Phi \, \mathrm{d}^3\vec{x} = \int_S \vec{\nabla}\Phi \mathrm{d}^2\vec{s}$$

Gauss's Theorem: $\boxed{\int_S \vec{\nabla}\Phi \, \mathrm{d}^2\vec{s} = 4\pi G M}$

Gauss's Theorem states that the integral of the normal component of the gravitational field [$\vec{g}(\vec{r}) = \vec{\nabla}\Phi$] over any closed surface S is equal to $4\pi G$ times the total mass enclosed by S.

cf. Electrostatics: $\boxed{\int_S \vec{E} \cdot \vec{n} \, \mathrm{d}^2\vec{s} = \frac{Q_{\mathrm{int}}}{\epsilon_0}}$

For a continuous density distribution $\rho(\vec{x})$ the total potential energy is:

$$\boxed{W = \tfrac{1}{2} \int \rho(\vec{x}) \, \Phi(\vec{x}) \, \mathrm{d}^3\vec{x}}$$

(see B&T p.33 for derivation)

NOTE: Here we follow B&T and use the symbol W instead of V.

The Discrete N-body Problem

The gravitational force on particle i due to particle j is:

$$\vec{F}_{i,j} = \frac{G m_i m_j}{|\vec{x}_i - \vec{x}_j|^3}(\vec{x}_i - \vec{x}_j)$$

(Newton's Inverse Square Law)

Equations of Motion: $\vec{F} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$

For particle i, the equations of motion are:

$$\begin{aligned} \frac{\mathrm{d}v_{k,i}}{\mathrm{d}t} &= G \sum_{j=1, j\neq i}^{N} \frac{m_j}{(x_{k,i} - x_{k,j})^2} \\ \frac{\mathrm{d}x_{k,i}}{\mathrm{d}t} &= v_{k,i} \qquad (k = 1, 3) \end{aligned}$$

This corresponds to a closed set of $6N$ equations, for a total of $6N$ unknowns (x, y, z, v_x, v_y, v_z)

Since N is typically very, very large, we can't make progress studying the dynamics of these systems by solving the $6N$ equations of motion.

Even with the most powerful computers to date, we can only run N-body simulations with $N \lesssim 10^6$

From Discrete to Smooth

The density distribution and gravitational potential of N-body system are:

$$\rho_N(\vec{x}) = \sum_{i=1}^{N} m_i \, \delta(\vec{x} - \vec{x}_i)$$

with $\delta(\vec{x})$ the Dirac delta function (B&T p.652), and

$$\Phi_N(\vec{x}) = -\sum_{i=1}^{N} \frac{G m_i}{|\vec{x} - \vec{x}_i|}$$

$$\begin{aligned}
\vec{F}_i &= G \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\vec{x}_j - \vec{x}_i|^3} (\vec{x}_j - \vec{x}_i) \\
&= G \sum_{j=1, j\neq i}^{N} \int \frac{(\vec{x}_j - \vec{x}_i)}{|\vec{x}_j - \vec{x}_i|^3} m_j \delta(\vec{x}_j - \vec{x}) \mathrm{d}^3\vec{x} \\
&= G \int \frac{(\vec{x}_j - \vec{x}_i)}{|\vec{x}_j - \vec{x}_i|^3} \rho_N(\vec{x}) \mathrm{d}^3\vec{x}
\end{aligned}$$

We will replace $\rho_N(\vec{x})$ and $\Phi_N(\vec{x})$ with smooth and continuous functions $\rho(\vec{x})$ and $\Phi(\vec{x})$

From Discrete to Smooth

For systems with large N, it is useful to try to use statistical descriptions of the system (cf. Thermodynamics)

Replacing a discrete density distribution by a continuous density distribution is familiar to us from fluid dynamics and plasma physics

However, there is one important difference:

$$\begin{aligned} \text{Plasma \& Fluid} &\iff \text{short range forces} \\ \text{Gravitational System} &\iff \text{long range forces} \end{aligned}$$

Plasma: electrostatic forces are long-range forces, but because of Debye schielding the total charge $\longrightarrow 0$ at large r: short-range forces dominate. Plasma may be collisionless.

Fluid: collisional system dominated by short-range van der Waals forces between dipoles of molecules. Always attractive, but for large r dipoles vanish. For very small r force becomes strongly repulsive.

For both plasma and fluid energy is an extensive variable: total energy is sum of energies of subsystems.

For gravitational systems, energy is a non-extensive variable: sub-systems influence each other by long-range gravitational interaction.

From Discrete to Smooth

FLUID

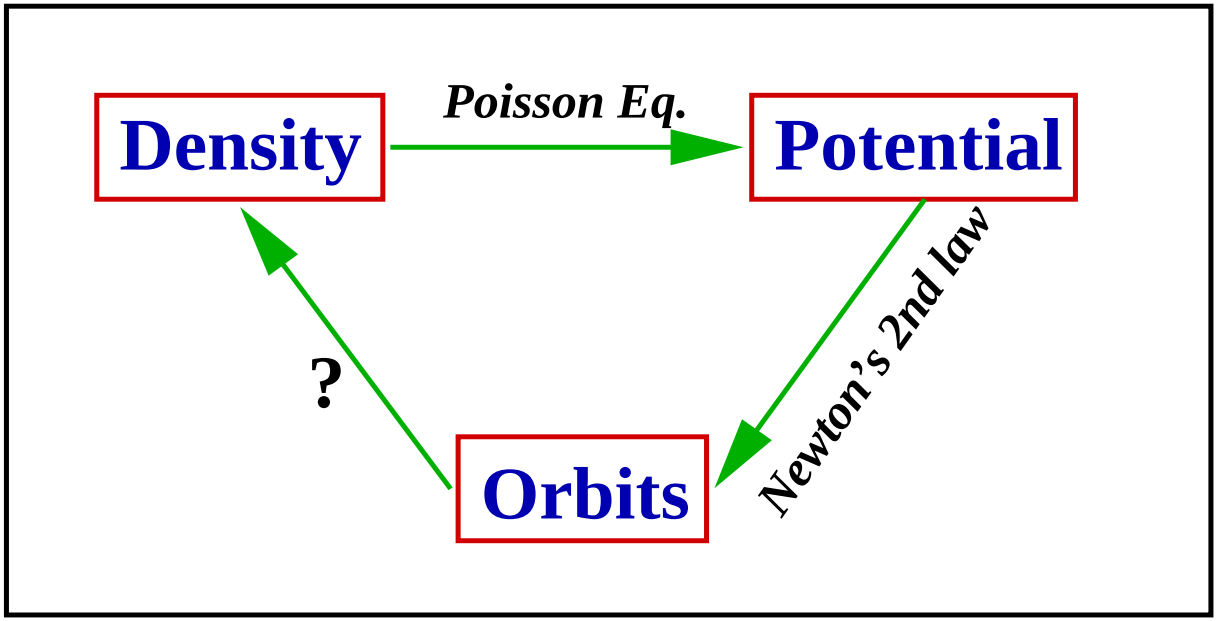
- **mean-free path of molecules $\ll$ size of system**
- **molecules collide frequently, giving rise to a well defined collisional pressure. This pressure balances gravity in hydrostatic equilibrium.**
- **Pressure related to density by equation of state. Ie, the EOS determines the (hydrostatic) equilibrium.**

GRAVITATIONAL SYSTEM

- **mean-free path of particles $\gg$ size of system**
- **No collisional pressure, although kinetic energy of particles act as a source of 'pressure', balancing the potential energy in virial equilibrium.**
- **No equivalent of equation of state. Pressure follows from kinetic energy, but kinetic energy follows from the actual orbits within gravitational potential, which in turn follows from the spatial distribution of the particles (Self-Consistency Problem)**

The Self-Consistency Problem

Given a density distribution $\rho(\vec{x})$, the Poisson equation yields the gravitational potential $\Phi(\vec{x})$. In this potential I can integrate orbits using Newton's equations of motion. The self-consistency problem is the problem of finding that combination of orbits that reproduces $\rho(\vec{x})$.



Think of self-consistency problem as follows: Given $\Phi(\vec{x})$, integrate all possible orbits $\mathcal{O}_i(\vec{x})$, and find the orbital weights w_i such that $\rho(\vec{x}) = \sum w_i \mathcal{O}_i(\vec{x})$. Here $\mathcal{O}_i(\vec{x})$ is the density contributed to $\vec{x}$ by orbit i.

Timescales for Collisions

Following fluid dynamics and plasma physics, we replace our discrete $\rho_N(\vec{x})$ with a smooth, continuous $\rho(\vec{x})$. Orbits are then integrated in the corresponding smooth potential $\Phi(\vec{x})$.

In reality, the true orbits will differ from these orbits, because the true potential is not smooth.

In addition to direct collisions ('touching' particles), we also have long-range collisions, in which the long-range gravitational force of the granularity of the potential causes small deflections.

Over time, these deflections accumulate to make the description based on the smooth potential inadequate.

It is important to distinguish between long-range interactions, which only cause a small deflection per interaction, and short-range interactions, which cause a relatively large deflection per interaction.

Direct Collisions

Consider a system of size R consisting of N identical bodies of radius r

The **cross section** for a direct collision is $\sigma = 4\pi r^2$

The **mean free path** of a particle is $\lambda = \frac{1}{n\sigma}$, with $n = \frac{3N}{4\pi R^3}$ the **number density** of bodies

$$\boxed{\frac{\lambda}{R} = \frac{4\pi R^3}{3\,N\,4\pi r^2 R} \simeq \left(\frac{R}{r}\right)^2 \frac{1}{N}}$$

It takes a **crossing time** $t_{\rm cross} \sim R/v$ to cross the system, so that the time scale for direct collisions is

$$\boxed{t_{\rm coll} = \left(\frac{R}{r}\right)^2 \frac{1}{N} t_{\rm cross}}$$

Example: A Milky Way like galaxy has $R = 10\ {\rm kpc} = 3.1 \times 10^{17}\ {\rm km}$, $v \simeq 200\ {\rm km\,s^{-1}}$, $N \simeq 10^{10}$, and r is roughly the radius of the Sun $(r = 6.9 \times 10^5\ {\rm km})$. This yields $\lambda = 2 \times 10^{13} R$. In other words, a direct collision occurs on average only once per 2000 billion crossings! The crossing time is $t_{\rm cross} = R/v = 5 \times 10^7\ {\rm yr}$, so that $t_{\rm coll} \simeq 10^{21}\ {\rm yr}$. This is about 10^{11} times the age of the Universe!!!

Relaxation Time I

Now that we have seen that direct collisions are completely negligble, let's focus on encounters

Consider once again a system of size R consisting of N identical particles of mass m. Consider one such particle crossing the system with velocity v. As we will see later, a typical value for the velocity is

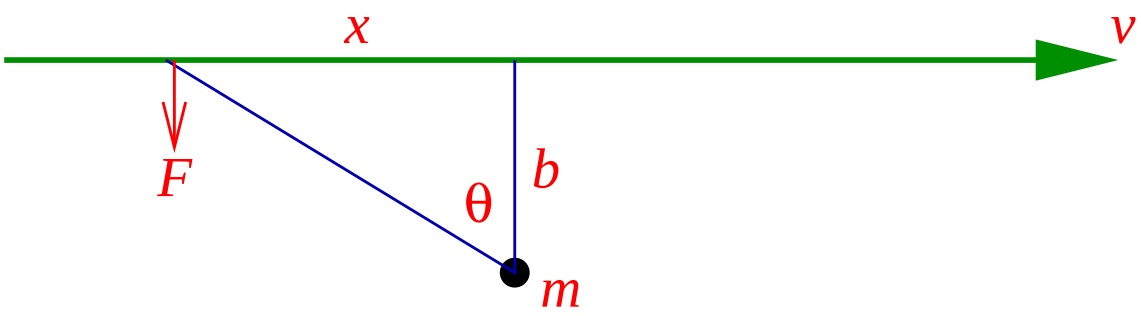
$$v = \sqrt{\frac{G\,M}{R}} = \sqrt{\frac{G\,N\,m}{R}}$$

We want to calculate how long it takes before the cumulative effect of many encounters has given our particle a kinetic energy $E_{\mathrm{kin}} \propto v^2$ in the direction perpendicular to its original motion of the order of its its initial kinetic energy.

Note that for a sufficiently close encounter, this may occur in a single encounter. We will treat this case seperately, and call such an encounter a close encounter.

Relaxation Time II

First consider a single encounter



Here b is the impact parameter, $x = v\,t$, with $t = 0$ at closest approach,

and $\cos\theta = \frac{b}{\sqrt{x^2+b^2}} = \left[1 + \left(\frac{vt}{b}\right)^2\right]^{-1/2}$

At any given time, the gravitational force in the direction perpendicular to the direction of the particle is

$$F_\perp = G\frac{m^2}{x^2+b^2}\cos\theta = \frac{Gm^2}{b^2}\left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2}$$

This force $F_\perp$ causes an acceleration in the $\perp$-direction: $F_\perp = m\frac{dv_\perp}{dt}$

We now compute the total $\Delta v_\perp$ integrated over the entire encounter, where we make the simplifying assumption that the particle moves in a straight line. This assumption is OK as long as $\Delta v_\perp \ll v$

Relaxation Time III

$$\begin{aligned}\Delta v_\perp &= 2\int_0^\infty \frac{Gm}{b^2}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-3/2}\mathrm{d}t \\ &= \frac{2Gm}{b^2}\frac{b}{v}\int_0^\infty (1+s^2)^{-3/2}\mathrm{d}s \\ &= \frac{2Gm}{bv}\end{aligned}$$

As discussed above, this is only valid as long as $\Delta v_\perp \ll v$. We define the minimum impact parameter $b_{\min}$, which borders long- and short-range interactions as: $\Delta v_\perp(b_{\min}) = v$

$$\boxed{b_{\min} = \frac{2Gm}{v^2} \simeq R/N}$$

For a MW-type galaxy, with $R = 10\,\mathrm{kpc}$ and $N = 10^{10}$ we have that $b_{\min} \simeq 3\times 10^7\,\mathrm{km} \simeq 50\,\mathrm{R}_\odot$

In a single, close encounter $\Delta E_{\mathrm{kin}} \sim E_{\mathrm{kin}}$. The time scale for such a close encounter to occur can be obtained from the time scale for direct collisions, by simply replacing r by $b_{\min}$.

$$\boxed{t_{\mathrm{short}} = \left(\frac{R}{b_{\min}}\right)^2 \frac{t_{\mathrm{cross}}}{N} = N\,t_{\mathrm{cross}}}$$

Relaxation Time IV

Now we compute the number of long-range encounters per crossing. Here we use that $(\Delta v_\perp)^2$ adds linearly with the number of encounters. (Note: this is not the case for $\Delta v_\perp$ because of the random directions).

When the particle crosses the system once, it has $n(<b)$ encounters with an impact parameter less than b, where

$$n(<b) = N\frac{\pi b^2}{\pi R^2} = N\left(\frac{b}{R}\right)^2$$

Differentiating with respect to b yields

$$n(b)\mathrm{d}b = \frac{2Nb}{R^2}\mathrm{d}b$$

Thus the total $(\Delta v_\perp)^2$ per crossing due to encounters with impact parameter $b, b+\mathrm{d}b$ is

$$(\Delta v_\perp)^2(b)\mathrm{d}b = \left(\frac{2Gm}{bv}\right)^2\frac{2Nb}{R^2}\mathrm{d}b = 8N\left(\frac{Gm}{Rv}\right)^2\frac{\mathrm{d}b}{b}$$

Integrating over the impact parameter yields

$$(\Delta v_\perp)^2 = 8N\left(\frac{Gm}{Rv}\right)^2\int_{b_{\min}}^{R}\frac{\mathrm{d}b}{b} \equiv 8N\left(\frac{Gm}{Rv}\right)^2\ln\Lambda$$

with $\ln\Lambda = \ln\left(\frac{R}{b_{\min}}\right) = \ln N$ the Coulomb logarithm

Relaxation Time V

We thus have that

$$(\Delta v_\perp)^2 = \left(\frac{GNm}{R}\right)^2 \frac{1}{v^2}\frac{8\ln N}{N}$$

Substituting the characteristic value for v then yields that

$$\boxed{\frac{(\Delta v_\perp)^2}{v^2} \simeq \frac{10\ln N}{N}}$$

Thus it takes of the order of $N/(10\ln N)$ crossings for $(\Delta v_\perp)^2$ to become comparable to v^2. This defines the relaxation time

$$\boxed{t_{\rm relax} = \frac{N}{10\ln N} t_{\rm cross}}$$

Summary of Time Scales

Let R be the size of the system, r the size of a particle (e.g., star), v the typical velocity of the particles, and N the number of particles in the system.

Hubble time: The age of the Universe. $t_H \simeq 1/H_0 \simeq 10^{10}\ \mathrm{yr}$

Formation time: The time it takes the system to form. $t_{\rm form} = \frac{\dot{M}}{M} \simeq t_H$

Crossing time: The typical time needed to cross the system. $t_{\rm cross} = R/v$

Collision time: The typical time between two direct collisions.

$$t_{\rm coll} = \left(\frac{R}{r}\right)^2 \frac{t_{\rm cross}}{N}$$

Relaxation time: The time over which the change in kinetic energy due to the long-range collisions has accumulated to a value that is comparable to the intrinsic kinetic energy of the particle.

$$t_{\rm relax} = \frac{N}{10 \ln N} t_{\rm cross}$$

Interaction time: The typical time between two short-range interactions that cause a change in kinetic energy comparable to the intrinsic kinetic energy of the particle.

$$t_{\rm short} = N t_{\rm cross}$$

For Trully Collisionless systems:

$$t_{\rm cross} \ll t_H \simeq t_{\rm form} \ll t_{\rm relax} \ll t_{\rm short} \ll t_{\rm coll}$$

Some other useful Time Scales

NOTE: Using that $v = \sqrt{\frac{GM}{R}}$ and $\bar{\rho} = \frac{3M}{4\pi R^3}$ we can write

$$t_{\rm cross} = \sqrt{\frac{3}{4\pi G\bar{\rho}}}$$

Dynamical time: the time required to travel halfway across the system.

$$t_{\rm dyn} = \sqrt{\frac{3\pi}{16G\rho}} = \frac{\pi}{2} t_{\rm cross}$$

Free-fall time: the time it takes a sphere with zero pressure to collapse to a point.

$$t_{\rm ff} = \sqrt{\frac{3\pi}{32G\rho}} = t_{\rm dyn}/\sqrt{2}$$

Orbital time: the time it takes to complete a (circular) orbit.

$$t_{\rm orb} = \sqrt{\frac{3\pi}{G\rho}} = 2\pi t_{\rm cross}$$

NOTE: All these timescales are the same as the crossing time, except for some pre-factors

$$t_{\rm cross} \lesssim t_{\rm ff} \lesssim t_{\rm dyn} \lesssim t_{\rm orb}$$

Example of Time Scales

System	Mass $M_\odot$	Radius kpc	Velocity $\mathrm{km\,s^{-1}}$	N	$t_{\rm cross}$ yr	$t_{\rm relax}$ yr
Galaxy	10^{10}	10	100	10^{10}	10^8	$> 10^{15}$
DM Halo	10^{12}	200	200	$> 10^{50}$	10^9	$> 10^{60}$
Cluster	10^{14}	1000	1000	10^3	10^9	$\sim 10^{10}$
Globular	10^4	0.01	2	10^4	5×10^6	5×10^8

- **Dark Matter Haloes and Galaxies are collisionless**
- **Collisions may or may not be important in clusters of galaxies**
- **Relaxation is expected to have occured in (some) globular clusters**

NOTE: **For a self-gravitating system, the typical velocities are** $v \simeq \sqrt{\frac{GM}{R}}$

For the crossing time this implies: $t_{\rm cross} = \frac{R}{v} = \sqrt{\frac{R^3}{GM}} = \sqrt{\frac{3}{4\pi G\rho}}$

Useful to remember: $1\ \mathrm{km/\,s} \simeq 1\ \mathrm{kpc/\,Gyr}$

$1\ \mathrm{yr} \simeq \pi \times 10^7\ \mathrm{s}$

$1\ \mathrm{M_\odot} \simeq 2 \times 10^{30}\ \mathrm{kg}$

$1\ \mathrm{pc} \simeq 3.1 \times 10^{13}\ \mathrm{km}$

The Distribution Function I

We have seen that the dynamics of our discrete system of N point masses is given by $6N$ equations of motion, which allow us to compute $6N$ unknowns $(\vec{x}, \vec{v})$ as function of time t.

The system is completely specified by $6N$ initial conditions $(\vec{x}_0, \vec{v}_0)$

We can specify these initial conditions by defining the distribution function (DF), also called the phase-space density

$$f(\vec{x}, \vec{v}, t_0) = \sum_{i=1}^{N} \delta(\vec{x} - \vec{x}_{i,0}) \sum_{i=1}^{N} \delta(\vec{v} - \vec{v}_{i,0})$$

Once $f(\vec{x}, \vec{v}, t)$ is specified at any time t, we can infer $f(\vec{x}, \vec{v}, t')$ at any other time t'

The DF $f(\vec{x}, \vec{v}, t)$ completely specifies a collisionless system

In the case of our smooth density distribution we define the 6 dimensional phase-space density :

$$f(\vec{x}, \vec{v}, t)\mathrm{d}^3\vec{x}\mathrm{d}^3\vec{v}$$

NOTE: A necessary, physical condition is that $f \geq 0$

The Distribution Function II

The density $\rho(\vec{x})$ follows from $f(\vec{x},\vec{v})$ by integrating over velocity space:

$$\rho(\vec{x},t) = \int\int\int f(\vec{x},\vec{v},t)\mathrm{d}^3\vec{v}$$

while the total mass follows from

$$M(t) = \int\int\int \rho(\vec{x},t)\mathrm{d}^3\vec{x} = \int \mathrm{d}^3\vec{x}\int \mathrm{d}^3\vec{v} f(\vec{x},\vec{v},t)$$

It is useful to think about the DF as a probability function (once normalized by M), which expresses the probability of finding a star in a phase-space volume $\mathrm{d}^3\vec{x}\mathrm{d}^3\vec{v}$. This means we can compute the expectation value for any quantity Q as follows:

$$\begin{aligned}\langle Q(\vec{x},t)\rangle &= \frac{1}{\rho(\vec{x})}\int \mathrm{d}^3\vec{v}\, Q(\vec{x},\vec{v})\, f(\vec{x},\vec{v},t)\\ \langle Q(t)\rangle &= \frac{1}{M}\int \mathrm{d}^3\vec{x}\int \mathrm{d}^3\vec{v}\, Q(\vec{x},\vec{v})\, f(\vec{x},\vec{v},t)\end{aligned}$$

EXAMPLES:

RMS velocity: $\langle v_i^2\rangle = \frac{1}{M}\int \mathrm{d}^3\vec{x}\int \mathrm{d}^3\vec{v}\, v_i^2\, f(\vec{x},\vec{v},t)$

Velocity Profile: $\mathcal{L}(x,y,v_z) = \int\int\int f(\vec{x},\vec{v},t)\,\mathrm{d}z\,\mathrm{d}v_x\,\mathrm{d}v_y$

Surface Brightness: $\Sigma(x,y) = \int\int\int\int f(\vec{x},\vec{v},t)\,\mathrm{d}z\,\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z$

The Distribution Function III

Each particle (star) follows a trajectory in the 6D phase-space $(\vec{x}, \vec{v})$, which is completely governed by Newtonian Dynamics (for a collisionless system).

This trajectory projected in the 3D space $\vec{x}$ is called the orbit of the particle.

As we will see later, the Lagrangian time-derivative of the DF, i.e. the time-derivative of $f(\vec{x}, \vec{v}, t)$ as seen when travelling through phase-space along the particle's trajectory, is

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = 0}$$

This simple equation is the single most important equation for collisionless dynamics. It completely specifies the evolution of a collisionless system, and is called the Collisionless Boltzmann Equation (C.B.E.) or Vlasov equation.

The flow in phase-space is incompressible.

NOTE: Don't confuse this with $\frac{\partial f}{\partial t} = 0$!!!!

This is the Eulerian time-derivative as seen from a fixed phase-space location, which is only equal to zero for a system in steady-state equilibrium.

Collisionless Dynamics in a Nutshell

$$\begin{aligned} \rho(\vec{x}) &= \int f(\vec{x},\vec{v})\,\mathrm{d}^3\vec{v} \\ \nabla^2\Phi(\vec{x}) &= 4\pi G\rho(\vec{x}) \\ \frac{\mathrm{d}f}{\mathrm{d}t} &= 0 \end{aligned}$$

The self-consistency problem of finding the orbits that reproduce $\rho(\vec{x})$ is equivalent to finding the DF $f(\vec{x},\vec{v})$ which yields $\rho(\vec{x})$.

Problem: **For most systems we only have constraints on a 3D projection of the 6D distribution function.**

Recall: $\mathcal{L}(x,y,v_z) = \int\int\int f(\vec{x},\vec{v},t)\,\mathrm{d}z\,\mathrm{d}v_x\,\mathrm{d}v_y$

Circular & Escape Velocities I

Consider a spherical density distribution $\rho(r)$ for which the Poisson Equation reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) = 4\pi G\rho(r)$$

from which we obtain that

$$r^2\frac{\partial \Phi}{\partial r} = 4\pi G\int_0^r \rho(r)\, r^2\, \mathrm{d}r = GM(r)$$

with $M(r)$ the enclosed mass. This allows us to write

$$\vec{F}_{\mathrm{grav}}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\vec{e}_r = -\frac{GM(r)}{r^2}\vec{e}_r$$

Because gravity is a central, conservative force, both the energy and angular momentum are conserved, and a particle's orbit is confined to a plane. Introducing the polar coordinates (r, θ) we write

$$\begin{aligned} E &= \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \Phi(r) \\ J &= r^2\dot{\theta} \end{aligned}$$

Eliminating $\dot{\theta}$ we obtain the Radial Energy Equation:

$$\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} + \Phi(r) = E$$

Circular & Escape Velocities II

In the co-rotating frame, the equation of motion reduces to a one-dimensional radial motion under influence of the effective potential $U(r) = \frac{J^2}{2r^2} + \Phi(r)$. The 'extra' term arises due to the non-inertial nature of the reference frame, and corresponds to the centrifugal force

$$\vec{F}_{\rm cen} = -\frac{\rm d}{{\rm d}r}\left(\frac{J^2}{2r^2}\right)\vec{e}_r = \frac{J^2}{r^3}\vec{e}_r = \frac{v_\theta^2}{r}\vec{e}_r$$

For a circular orbit we have that $\vec{F}_{\rm cen} = -\vec{F}_{\rm grav}$, so that we obtain the circular speed.

$$\boxed{v_c(r) = \sqrt{r\frac{{\rm d}\Phi}{{\rm d}r}} = \sqrt{\frac{GM(r)}{r}}}$$

Thus, $rv_c^2(r)$ measures the mass enclosed within radius r (in spherical symmetry). Note that for a point mass $v_c(r) \propto r^{-1/2}$, which is called a Keplerian rotation curve

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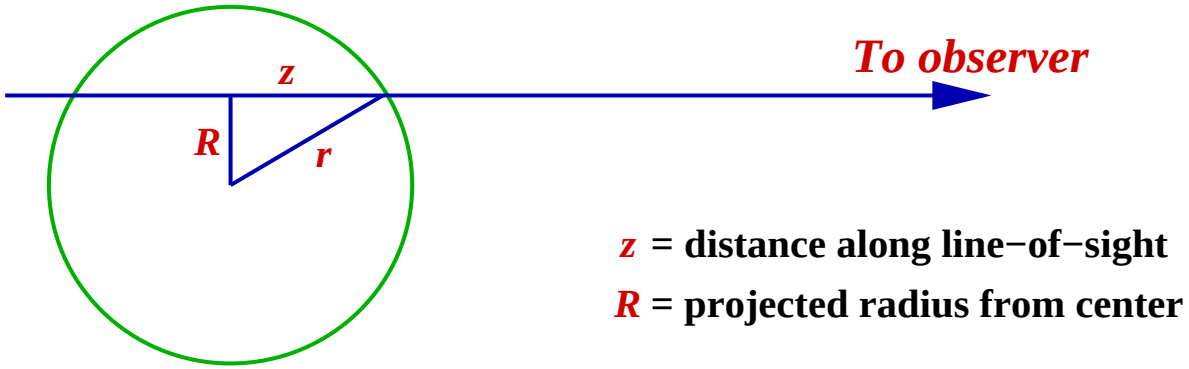
Escape Speed: The speed a particle needs in order to 'escape' to infinity

$$\boxed{v_{\rm esc}(r) = \sqrt{2|\Phi(r)|}}$$

Recall: The energy per unit mass is $E = \frac{1}{2}v^2 + \Phi(r)$. In order to escape to infinity we need $E \geq 0$, which translates into $v^2 \geq 2|\Phi(r)|$

Projected Surface Density

Consider a spherical system with intrinsic, 3D luminosity distribution $\nu(r)$.
An observer, at large distance, observes the projected, 2D surface brightness distribution $\Sigma(R)$



$$\Sigma(R) = 2\int_0^\infty \nu(r)\mathrm{d}z = 2\int_R^\infty \nu(r)\frac{r\,\mathrm{d}r}{\sqrt{r^2-R^2}}$$

This is a so-called Abel Integral, for which the inverse is:

$$\nu(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2-r^2}}$$

Thus, an observed surface brightness distribution $\Sigma(R)$ of a spherical system can be deprojected to obtain the 3D light distribution $\nu(r)$. However, because it requires the determination of a derivative, it can be fairly noisy.

Spherical Potential-Density Pairs

To compute the potential of a spherical density distribution $\rho(r)$ we can make use of Newton's Theorems

First Theorem **A body inside a spherical shell of matter experiences no net gravitational force from that shell.**

Second Theorem **The gravitational force on a body that lies outside a closed spherical shell of mass M is the same as that of a point mass M at the center of the shell.**

Based on these two Theorems, we can compute $\Phi(r)$ by splitting $\rho(r)$ in spherical shells, and adding the potentials of all these shells:

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r')\, r'^2\, \mathrm{d}r' + \int_r^\infty \rho(r')\, r'\, \mathrm{d}r' \right]$$

Using the definition of the enclosed mass $M(r) = 4\pi \int_0^r \rho(r')\, r'^2\, \mathrm{d}r'$ this can be rewritten as

$$\Phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^\infty \rho(r')\, r'\, \mathrm{d}r'$$

Power-law Density Profiles I

Consider a spherical system with a simple power-law density distribution

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-\alpha}$$

$$\Sigma(R) = 2 \int_R^\infty \rho(r) \frac{r\,\mathrm{d}r}{\sqrt{r^2 - R^2}} = \rho_0\, r_0^\alpha\, B(\tfrac{\alpha}{2} - \tfrac{1}{2}, \tfrac{1}{2})\, R^{1-\alpha}$$

$$M(<r) = 4\pi \int_0^r \rho(r')\, r'^2\, \mathrm{d}r' = \frac{4\pi\rho_0 r_0^\alpha}{3-\alpha} r^{3-\alpha} \qquad (\alpha < 3)$$

$$M(>r) = 4\pi \int_r^\infty \rho(r')\, r'^2\, \mathrm{d}r' = \frac{4\pi\rho_0 r_0^\alpha}{\alpha-3} r^{3-\alpha} \qquad (\alpha > 3)$$

NOTE: For $\alpha \geq 3$ the enclosed mass is infinite, while for $\alpha \leq 3$ the total mass ($r \rightarrow \infty$) is infinite: A pure power-law system can not exist in nature!

A more realistic density distribution consists of a double power-law:

At small radii: $\rho \propto r^{-\alpha}$ with $\alpha < 3$

At large radii: $\rho \propto r^{-\beta}$ with $\beta > 3$

$B(x,y)$ is the so-called Beta-Function, which is related to the Gamma Function $\Gamma(x)$

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = B(y,x)$$

Power-law Density Profiles II

The potential of a power-law density distribution is:

$$\Phi(r) = \begin{cases} \frac{4\pi G\rho_0 r_0^\alpha}{(\alpha-3)(\alpha-2)} r^{2-\alpha} & \text{if } 2 < \alpha < 3 \\ \infty & \text{otherwise} \end{cases}$$

The circular and escape velocities of a power-law density distribution are:

$$v_c^2(r) = r\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G\,M(r)}{r} = \frac{4\pi G\rho_0 r_0^\alpha}{3-\alpha} r^{2-\alpha}$$

$$v_{\rm esc}^2(r) = \frac{2}{\alpha-2} v_c^2(r)$$

$\alpha = 2$: **Singular Isothermal Sphere** $v_c = \text{constant}$ **(flat rotation curve)**

$\alpha = 0$: **Homogeneous Sphere** $v_c \propto r$ **(solid body rotation)**

NOTE: For $\alpha > 3$ you find that $v_c(r)$ falls off more rapidly than Keplerian. How can this be? After all, a Keplerian RC corresponds to a delta-function density distribution (point mass), which is the most concentrated mass distribution possible....

Power-law Density Profiles II

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---

answer: the circular velocity is defined via the gradient of the potential. As shown above, Φ is only defined for $2 < \alpha < 3$, and therefore so does v_c

Power-law Density Profiles: Summary

It is very useful to remember the following scaling relations:

$$
\begin{array}{lcll}
\rho(r) & \propto & r^{-\alpha} & \\
\Sigma(R) & \propto & R^{1-\alpha} & \\
\Phi(r) & \propto & r^{2-\alpha} & (2 < \alpha < 3) \\
v_c^2(r) & \propto & r^{2-\alpha} & (2 < \alpha < 3) \\
M(< r) & \propto & r^{3-\alpha} & (\alpha < 3) \\
M(> r) & \propto & r^{3-\alpha} & (\alpha > 3)
\end{array}
$$

Double Power-law Density Profiles

As we have seen, no realistic system can have a density distribution that is described by a single power-law. However, many often used density distributions have a double power-law.

$$\rho(r) = \frac{C}{r^{\gamma}(1+r^{1/\alpha})^{(\beta-\gamma)\alpha}}$$

At small radii, $\rho \propto r^{-\gamma}$, while at large radii $\rho \propto r^{-\beta}$. The parameter α determines the 'sharpness' of the break.

NOTE: In order for the mass to be finite, $\gamma < 3$ and $\beta > 3$

(α, β, γ)	Name	Reference
$(1, 3, 1)$	NFW Profile	Navarro, Frenk & White, 1997, ApJ, 490, 493
$(1, 4, 1)$	Hernquist Profile	Hernquist, 1990, ApJ, 356, 359
$(1, 4, 2)$	Jaffe Profile	Jaffe, 1983, MNRAS, 202, 995
$(1, 4, \frac{3}{2})$	Moore Profile	Moore et al., 1999, MNRAS, 310, 1147
$(\frac{1}{2}, 2, 0)$	Modified Isothermal Sphere	Sacket & Sparke, 1990, ApJ, 361, 409
$(\frac{1}{2}, 3, 0)$	Modified Hubble Profile	Binney & Tremaine, p. 39
$(\frac{1}{2}, 4, 0)$	Perfect Sphere	de Zeeuw, 1985, MNRAS, 216, 273
$(\frac{1}{2}, 5, 0)$	Plummer Model	Plummer, 1911, MNRAS, 71, 460

Ellipsoids I

Thus far we have only considered spherical systems. However, only very few systems in nature are trully spherical. A more general, though still not fully general, form to consider is the ellipsoid.

Without loosing generality, we will use the following definition of the ellipsoidal radius

$$m^2 = a_1^2 \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} \qquad a_1 \geq a_2 \geq a_3$$

Note that we have taken the three principal axes to be aligned with our Cartesian coordinate system (x, y, z). If $a_1 > a_2 > a_3$ then the ellipsoid is said to be triaxial

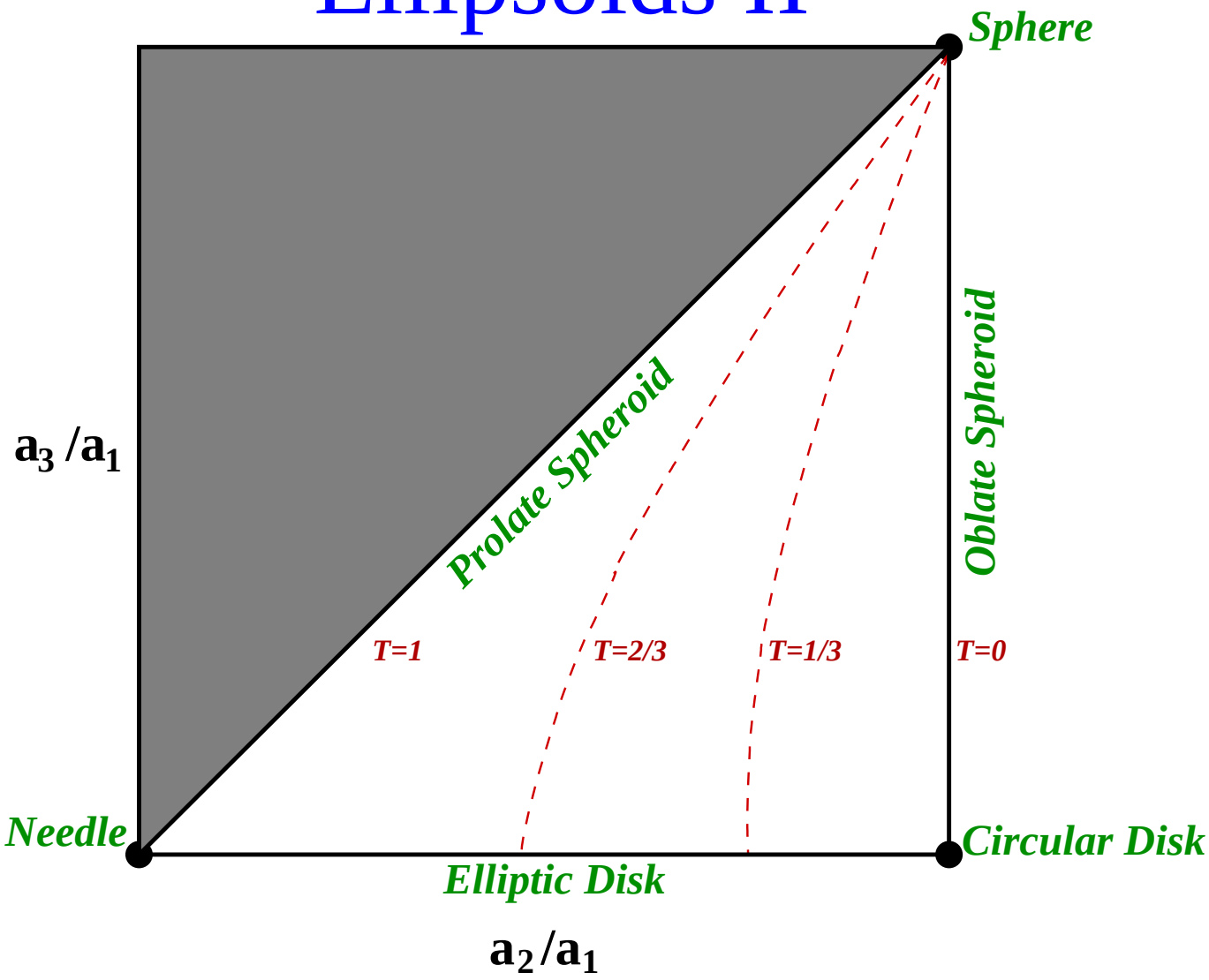
A body whose isodensity surfaces are concentric ellipsoids is called an ellipsoidal body.

Triaxiality Parameter: $$T \equiv \frac{1-(a_2/a_1)^2}{1-(a_3/a_1)^2}$$

A spheroid is an axisymmetric ellipsoid with two equal principal axes:

- **Oblate Spheroid:** $a_1 = a_2 > a_3 \quad (T = 0)$ **(i.e. Earth)**
- **Prolate Spheroid:** $a_1 > a_2 = a_3 \quad (T = 1)$ **(i.e. Cigar)**

Ellipsoids II



For an oblate spheroid with axis ratio $q = a_3/a_1$, we define:

Ellipticity: $\varepsilon = 1 - q$

Eccentricity: $e = \sqrt{1 - q^2}$

Ellipsoids III

A shell of similar, concentric ellipsoids is called a homoeoid. Note that the perpendicular distance d between the two ellipsoids is a function of the angular position.

In what follows we consider the family of ellipsoidal bodies whose density distribution is the sum of thin homoeoids.

Homoeoid Theorem: **The exterior isopotential surface of a homoeoidal shell of negligble thickness are the spheroids that are confocal with the shell itself. Insider the shell the potential is constant.**

This implies that:

- **The equipotentials of a homoeoid become spherical at large radii.**
- **The equipotential of a thin homoeoid has the same shape as the homoeoid at the location of the homoeoid.**

NOTE: the Homoeoid Theorem applies only to thin homoeoids. However, for any homoeoid of any thickness we have:

Newton's Third Theorem: **A mass that is inside a homoeoid experiences no net gravitational force from the homoeoid.** $\Phi_{\rm inside} = \text{constant}$

Ellipsoids IV

Consider a spheroidal density distribution $\rho(R,z)=\rho(m^2)$ with $m^2=R^2+z^2/(1-e^2)$, then the potential is:

$$\Phi(R,z)=-2\pi G\frac{\sqrt{1-e^2}}{e}\left[\psi(\infty)\mathrm{arcsine}-\frac{a_0 e}{2}\int_0^\infty\frac{\psi(m)\,\mathrm{d}\tau}{(\tau+a_0^2)\sqrt{\tau+b_0^2}}\right]$$

Here

$$\frac{m^2}{a_0^2}=\frac{R^2}{\tau+a_0^2}+\frac{z^2}{\tau+b_0^2}$$

with a_0 any constant and $b_0=\sqrt{1-e^2}a_0$, and

$$\psi(m)\equiv\int_0^{m^2}\rho(m^2)\,\mathrm{d}m^2$$

The corresponding circular velocity in the equatorial plane $z=0$ is

$$v_c^2(R)=R\frac{\partial\Phi}{\partial R}=4\pi G\sqrt{1-e^2}\int_0^R\frac{\rho(m^2)\,m^2\,\mathrm{d}m}{\sqrt{R^2-m^2e^2}}$$

Ellipsoids IV

Consider a spheroidal density distribution $\rho(R,z)=\rho(m^2)$ with $m^2=R^2+z^2/q^2$, then the potential is:

$$\Phi(R,z)=-2\pi Gq\frac{\arcsin e}{e}\psi(\infty)+\pi Gq\int_0^\infty\frac{\psi(m)\,\mathrm{d}\tau}{(\tau+1)\sqrt{\tau+q^2}}$$

Here $e=\sqrt{1-q^2}$ is the eccentricity,

$$m^2=\frac{R^2}{\tau+1}+\frac{z^2}{\tau+q^2}$$

and

$$\psi(m)\equiv\int_0^{m^2}\rho(m'^2)\,\mathrm{d}m'^2$$

The corresponding circular velocity in the equatorial plane $z=0$ is

$$v_c^2(R)=R\frac{\partial\Phi}{\partial R}=4\pi Gq\int_0^R\frac{\rho(m^2)\,m^2\,\mathrm{d}m}{\sqrt{R^2-m^2e^2}}$$

Ellipsoids V

- In general one finds that $v_c(R)$ increases with larger flattening q: Flatter systems with the same spheroidal, enclosed mass have larger circular speeds at given R.
- Let $\varepsilon_\rho = 1 - q$ the ellipticity of the density distribution. One always has that $\varepsilon_\Phi \leq \varepsilon_\rho$. At a few characteristic radii, a reasonable rule of thumb is that $\varepsilon_\Phi \sim \frac{1}{3}\varepsilon_\rho$

We can generalize the equations on the previous page for a triaxial, ellipsoidal density distribution $\rho(\vec{x}) = \rho(m^2)$ with

$$m^2 = a_1^2 \sum_{i=1}^{3} \frac{x_i^2}{a_i^2}$$

The corresponding potential is

$$\Phi(\vec{x}) = -\pi G \left(\frac{a_2\, a_3}{a_1}\right) \int_0^\infty \frac{[\psi(\infty) - \psi(m)]\, \mathrm{d}\tau}{\sqrt{(\tau + a_1^2)(\tau + a_2^2)(\tau + a_3^2)}}$$

with

$$\frac{m^2}{a_1^2} = \sum_{i=1}^{3} \frac{x_i^2}{\tau + a_i^2}$$

Multipole Expansion I

In order to calculate the potential of an arbitrary density distribution, it is useful to consider a Multipole expansion.

Using separation of variables, $\Phi(r,\theta,\phi) = R(r)\,P(\theta)\,Q(\phi)$, one can write

$$\Phi(r,\theta,\phi) = -4\pi G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_l^m(\theta,\phi)}{2l+1} \left[\frac{1}{r^{(l+1)}} \int_0^r \rho_{lm}(r') r'^{(l+2)} \mathrm{d}r' + r^l \int_r^{\infty} \rho_{lm}(r') \frac{\mathrm{d}r'}{r'^{(l-1)}} \right]$$

Here

$$\rho_{lm}(r) = \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \hat{Y}_l^m(\theta,\phi) \rho(r,\theta,\phi)$$

and

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}}\, P_l^{|m|}(\cos\theta) \mathrm{e}^{im\phi}$$

with $P_l(x)$ the associated Legendre functions, and $\hat{Y}_l^m(\theta,\phi)$ the complex conjugate of $Y_l^m(\theta,\phi)$

Multipole Expansion II

Monopole	$l = 0$	1 term
Dipole	$l = 1$	3 terms
Quadrupole	$l = 2$	5 terms
Octopole	$l = 3$	7 terms
Hexadecapole	$l = 4$	9 terms

The monopole term describes the potential of a spherical system with $\rho(r, \theta, \phi) = \rho(r)$. Since $Y_0^0(\theta, \phi) = 1/\sqrt{4\pi}$ and $\rho_{00} = \sqrt{4\pi}\rho(r)$, the $(l = m = 0)$-term of the multipole expansion is simply the equation for the potential of a spherical system:

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r')\, r'^2\, \mathrm{d}r' + \int_r^\infty \rho(r')\, r'\, \mathrm{d}r' \right]$$

In electrostatics you have both positive and negative charges. Consequently, the monopole term of the electrostatic potential often vanishes at large radii, while the dipole terms comes to dominate.

In gravity we have only positive charges (mass). Consequently, the monopole term always dominates at large radii, while the dipole term vanishes. The quadrupole term depends on the flattening of the density distribution.

Potentials of Disks

Since many galaxies have a dominant, thin disk component, it is useful to consider the potentials of infinitessimally thin disks.

There are three methods to compute the potential of an infinitesimally thin disk:

- **Use the formalism for ellipsoids, and apply the limit $q \longrightarrow 0$. Cumbersome! involving complicated double integrals...this method is seldomly used.**
- **Use the general definition of the potential, which results in an expression in terms of Elliptic Integrals.**
- **Use the Laplace equation subject to appropriate boundary conditions on the disk and at infinity.**

Disk Potentials via Elliptic Integrals

The potential of a thin disk with surface density $\Sigma(R)$ can be written as

$$\Phi(\vec{x}) = -G \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \mathrm{d}^3\vec{x}' = -G \int_0^\infty \Sigma(R')\, R' \mathrm{d}R' \int_0^{2\pi} \frac{\mathrm{d}\phi'}{|\vec{x}-\vec{x}'|}$$

Expressing $|\vec{x} - \vec{x}'|$ in $(R, \phi = 0, z)$ and $(R', \phi', z' = 0)$ yields

$$\Phi(R, z) = -\frac{2G}{\sqrt{R}} \int_0^\infty K(k)\, k\, \Sigma(R')\, \sqrt{R'}\, \mathrm{d}R'$$

with $k^2 \equiv 4\, R\, R' / [(R + R')^2 + z^2]$. The corresponding circular velocity can be obtained from

$$R \frac{\partial \phi}{\partial R}(R, z) = \frac{G}{\sqrt{R}} \int_0^\infty \mathrm{d}R'\, k\, \Sigma(R')\, \sqrt{R'} \times \left[K(k) - \frac{1}{4} \left(\frac{k^2}{1-k^2} \right) \left(\frac{R'}{R} - \frac{R}{R'} + \frac{z^2}{RR'} \right) E(k) \right]$$

with $K(k)$ and $E(k)$ so called complete elliptic integrals. In principle the evaluation at $z = 0$ is complicated (contains integrable singularity); in practice it often suffices to approximate the above at small z

Disk Potentials via Bessel Functions

The potential of a thin disk with surface density $\Sigma(R)$ can be written as

$$\Phi(R,z) = \int_0^\infty S(k)\, J_0(kR)\, \mathrm{e}^{-k|z|}\mathrm{d}k$$

with

$$S(k) = -2\pi G \int_0^\infty J_0(kR)\, \Sigma(R)\, R\, \mathrm{d}R$$

Here $J_0(x)$ is the cylindrical Bessel function of order zero.

The corresponding circular velocity is given by

$$v_c^2(R) = R\left(\frac{\partial\Phi}{\partial R}\right)_{z=0} = -R\int_0^\infty S(k)\, J_1(kR)\, k\, \mathrm{d}k$$

This method is simple, and most of the time well behaved. For an exponential disk with $\Sigma(R) = \Sigma_0 \mathrm{e}^{-R/R_d}$ one finds

$$v_c^2(R) = 4\pi G\Sigma_0 R_d y^2\left[I_0(y)K_0(y) - I_1(y)K_1(y)\right]$$

with $y = \frac{R}{2R_d}$ and $I_n(x)$ and $K_n(x)$ modified Bessel functions of the first and second kinds

Orbits in Central Force Fields I

Consider the central force field $F(r)$ associated with a spherical density distribution $\rho(r)$.

As we have seen before, the orbits are planar, so that we consider the polar coordinates (r, θ)

The equations of motion are: $\boxed{\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = F(r)\vec{e}_r}$

Solving these requires a careful treatment of the unit vectors in polar coordinates:

$$\boxed{\begin{aligned} \vec{e}_r &= \cos\theta\vec{e}_x + \sin\theta\vec{e}_y \\ \vec{e}_\theta &= -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y \end{aligned}}$$

$$\begin{aligned} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(r\cos\theta\vec{e}_x + r\sin\theta\vec{e}_y) \\ &= \dot{r}\cos\theta\vec{e}_x - r\dot{\theta}\sin\theta\vec{e}_x + \dot{r}\sin\theta\vec{e}_y + r\dot{\theta}\cos\theta\vec{e}_y \\ &= \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \end{aligned}$$

and similarly one obtains that

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

Orbits in Central Force Fields II

We thus obtain the following set of equations of motions:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= F(r) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0 \end{aligned}$$

Multiplying the second of these equations with r yields, after integration, that $\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0$. This simply expresses the conservation of the orbit's angular momentum $L = r^2\dot{\theta}$, i.e., the equations of motion can be written as

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{\mathrm{d}\Phi}{\mathrm{d}r} \\ r^2\dot{\theta} &= L = \text{constant} \end{aligned}$$

In general these equations have to be solved numerically. Despite the very simple, highly symmetric system, the equations of motion don't provide much insight. As we'll see later, more direct insight is obtained by focussing on the conserved quantities. Note also that the equations of motion are different in different coordinate systems: in Cartesian coordinates (x, y):

$$\begin{aligned} \ddot{x} &= F_x = -\frac{\partial\Phi}{\partial x} \\ \ddot{y} &= F_y = -\frac{\partial\Phi}{\partial y} \end{aligned}$$

Orbits in Central Force Fields III

As shown before, one can use the second equation of motion (in polar coordinates) to eliminate $\dot{\theta}$ in the first, which yields the radial energy equation

$$\boxed{\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} + \Phi(r) = E}$$

which can be rewritten as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{2[E - \Phi(r)] - \frac{J^2}{2r^2}}$$

where the $\pm$ sign is required because r can both increase and decrease. Solving for the turn-around points, where $\mathrm{d}r/\mathrm{d}t = 0$, yields

$$\boxed{\frac{1}{r^2} = \frac{2[E-\Phi(r)]}{-J^2}}$$

which has two solutions: the apocenter r_+ and the pericenter $r_- \leq r_+$. These radii reflect the maximum and minimum radial extent of the orbit.

It is customary to define the orbital eccentricity as

$$\boxed{e = \frac{r_+ - r_-}{r_+ + r_-}}$$

where $e = 0$ and $e = 1$ correspond to circular and radial orbits, resp.

The Lagrangian

The equations of motion as given by Newton's second law depend on the choice of coordinate system

Their derivation involves painful vector calculus when curvi-linear coordinates are involved

In the Lagrangian formulation of dynamics, the equations of motion are valid for any set of so-called generalized coordinates $(q_1, q_2, .., q_n)$, with n the number of degrees of freedom

Generalized coordinates are any set of coordinates that are used to describe the motion of a physical system, and for which the position of every particle in the system is a function of these coordinates and perhaps also time: $\vec{r} = \vec{r}(q_i, t)$. If $\vec{r} = \vec{r}(q_i)$ the system is said to be natural.

Define the Lagrangian function: $\boxed{\mathcal{L} = T - V}$

with T and V the kinetic and potential energy, respectively.

In Cartesian coordinates, and setting the mass $m = 1$, we have

$$\mathcal{L} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \Phi(x, y, z)$$

In Generalized coordinates we have that $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$.

Actions and Hamilton's Principle

Define the action integral (also just called the action)

$$I = \int_{t_0}^{t_1} \mathcal{L}\, \mathrm{d}t$$

which is the integral of the Lagrangian along a particle's trajectory as it moves from time t_0 to t_1.

Hamilton's Principle, also called Principle of least action: The equations of motion are such that the action integral is stationary (i.e., $\delta I = 0$) under arbitrary variations δq_i which vanish at the limits of integration t_0 to t_1.

Note that these stationary points are not necessarily minima. They may also be maxima or sadle points.

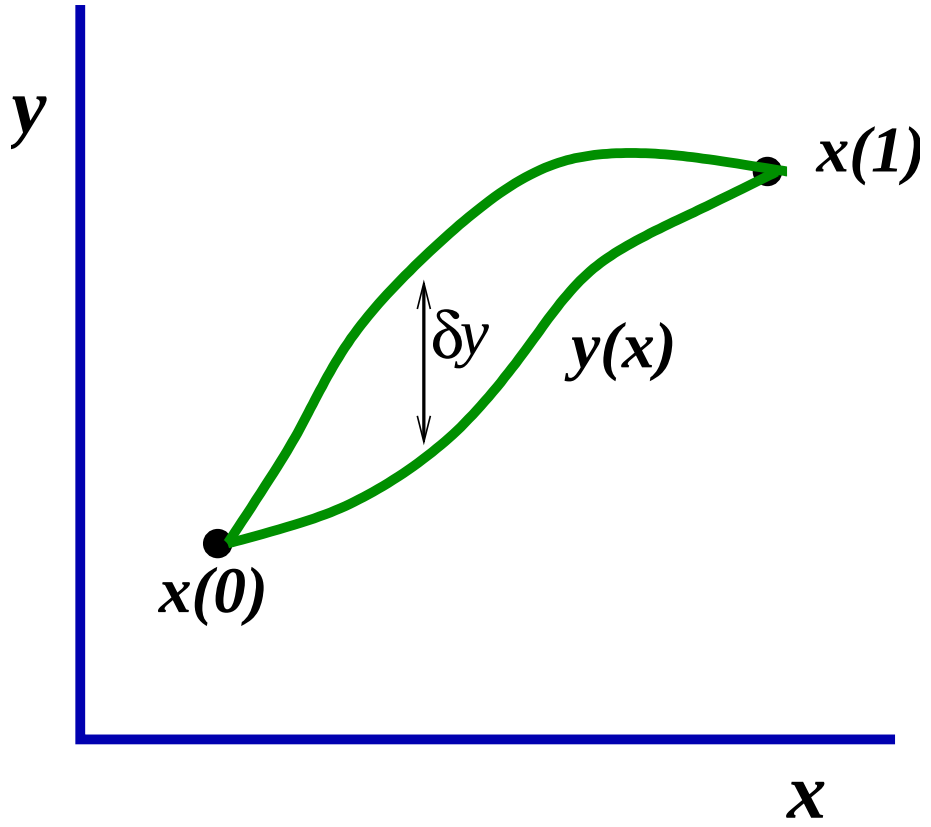
In order to derive these equations of motion, we first familiarize ourselves with the calculus of variations

Calculus of Variations I

We are interested in finding the stationary values of an integral of the form

$$I = \int_{x_0}^{x_1} f(y, \dot{y}) \mathrm{d}x$$

where $f(y, \dot{y})$ is a specified function of $y = y(x)$ and $\dot{y} = \mathrm{d}y/\mathrm{d}x$.



Consider a small variation $\delta y(x)$, which vanishes at the endpoints of the integration interval: $\delta y(x_0) = \delta y(x_1) = 0$

Calculus of Variations II

Using that

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial \dot{y}}\delta \dot{y}$$

with $\delta \dot{y} = \frac{\mathrm{d}}{\mathrm{d}x}\delta y(x)$**, the stationary values obey**

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial \dot{y}}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right] \mathrm{d}x = 0$$

Using integration by parts, and $\delta y(x_0) = \delta y(x_1) = 0$**, this reduces to**

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] \delta y \, \mathrm{d}x = 0$$

which yields the so-called Euler-Lagrange equations

$$\boxed{\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0}$$

These are second-order differential equations for $y(x)$**, whose solutions contain two arbitrary constants that may be determined form the known values of** y **at** x_0 **and** x_1**.**

The Lagrangian Formulation I

Application of the Euler-Lagrange equations to the Lagrangian $\mathcal{L}(q_i, \dot{q}_i)$ yields

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0}$$

which are the Lagrange equations (one for each degree of freedom), which represent the equations of motion according to Hamilton's principle. Note that they apply to any set of generalized coordinates

In addition to the generalized coordinates we also define the generalized momenta p_i (also called conjugate momenta) and the generalized forces F_i:

$$\boxed{p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad\qquad F_i \equiv \frac{\partial \mathcal{L}}{\partial q_i}}$$

With these definitions the Lagrange equations reduce to

$$\boxed{\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} = F_i}$$

NOTE: in general p_i and F_i are not components of the momentum vector $\vec{p}$ or the force vector $\vec{F}$!!! Whenever q_i is an angle, the conjugate momentum p_i is an angular momentum.

The Lagrangian Formulation II

As an example, let's consider once again motion in a central force field. Our generalized coordinates are the polar coordinates (r, θ), and the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}r^2\dot{\theta}^2 - \Phi(r)$$

The Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = 0 \quad \Rightarrow \quad r\dot{\theta}^2 - \frac{\partial \Phi}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}) = 0 \quad \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = -\frac{\partial \Phi}{\partial r}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \quad \Rightarrow \quad -\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0 \quad \Rightarrow \quad r^2\dot{\theta} = L = \mathrm{cst}$$

Note that the Lagrangian formulation allows you to write down the equations of motion much faster than using Newton's second law!

The Hamiltonian Formulation I

The Hamiltonian $\mathcal{H}(q_i, p_i)$ is related to the Lagrangian $\mathcal{L}(q_i, \dot{q}_i)$ via a Legendre Transformation

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In general, a Legendre Transformation is a transformation of a function $f(x, y)$ to $g(u, y)$, where $u = \frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial u} = x$

$$\boxed{g(u, y) = f - u\,x}$$

NOTE: You might be familiar with Legendre Transformations from Thermodynamics where they are used to compute different thermodynamic potentials from the internal energy $U = U(S, V)$, such as

enthalpy: $H = H(S, p) = U + p\,V$

Helmholtz free energy: $F = F(T, V) = U - T\,S$

Using a similar Legendre transformation we write the Hamiltonian as

$$\boxed{\mathcal{H}(\vec{q}, \vec{p}, t) = \sum_{i=1}^{n} p_i\, \dot{q}_i(\vec{q}, \vec{p}) - \mathcal{L}(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}), t)}$$

To compute $\mathcal{H}(\vec{q}, \vec{p}, t)$, first compute $\mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$, next compute the conjugate momenta $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, compute $\mathcal{H} = \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$ and finally express the $\dot{q}_i$ in terms of $\vec{p}$ and $\vec{q}$

The Hamiltonian Formulation II

Differentiating $\mathcal{H}$ with respect to the conjugate momenta yields

$$\frac{\partial \mathcal{H}}{\partial p_j} = \dot{q}_j + \sum_{i=1}^{n} p_i \frac{\partial \dot{q}_i}{\partial p_j} - \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial p_j}$$

The second and third terms vanish since $p_i = \partial \mathcal{L} / \partial \dot{q}_i$, so that we obtain that $\partial \mathcal{H} / \partial p_j = \dot{q}_j$. Similarly we obtain that

$$\frac{\partial \mathcal{H}}{\partial q_j} = \sum_{i=1}^{n} p_i \frac{\partial \dot{q}_i}{\partial q_j} - \frac{\partial \mathcal{L}}{\partial q_j} - \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial q_j}$$

Here the first and third terms cancel, and since the Lagrange equations tell us that $\partial \mathcal{L} / \partial q_j = \dot{p}_j$, we obtain that $\partial \mathcal{H} / \partial q_j = -\dot{p}_j$.

This yields the Hamiltonian equations of motion

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \qquad\qquad \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i$$

Note that whereas Lagrange's equations are a set of n second-order differential equations, Hamilton's equations are a set of $2n$ first-order differential equations. Although they are easier to solve, deriving the Hamiltonian itself is more involved.

The Hamiltonian Formulation III

The Hamiltonian description is especially useful for finding conserved quantities, which will play an important role in describing orbits.

If a generalized coordinate, say q_i, does not appear in the Hamiltonian, then the corresponding conjugate momentum p_i is a conserved quantity!!!

In the case of motion in a fixed potential, the Hamiltonian is equal to the total energy, i.e., $\mathcal{H} = E$

DEMONSTRATION: for a time-independent potential $\Phi = \Phi(\vec{x})$ the Lagrangian is equal to $\mathcal{L} = \frac{1}{2}\dot{\vec{x}}^2 - \Phi(\vec{x})$. Since $\vec{p} = \partial\mathcal{L}/\partial\dot{\vec{x}} = \dot{\vec{x}}$ we have that $\mathcal{H} = \dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{1}{2}\dot{\vec{x}}^2 + \Phi(\vec{x}) = \frac{1}{2}\dot{\vec{x}}^2 + \Phi(\vec{x}) = E$

The $2n$-dimensional phase-space of a dynamical system with n degrees of freedom can be described by the generalized coordinates and momenta $(\vec{q}, \vec{p})$. Since Hamilton's equations are first order differential equations, we can determine $\vec{q}(t)$ and $\vec{p}(t)$ at any time t once the initial conditions $(\vec{q}_0, \vec{p}_0)$ are given. Therefore, through each point in phase-space there passes a unique trajectory $\Gamma[\vec{q}(\vec{q}_0, \vec{p}_0, t), \vec{p}(\vec{q}_0, \vec{p}_0, t)]$. No two trajectories Γ_1 and Γ_2 can pass through the same $(\vec{q}_0, \vec{p}_0)$ unless $\Gamma_1 = \Gamma_2$.

The Hamiltonian Formulation IV

As an example, let's consider once more the motion in a central force field. Our generalized coordinates are the polar coordinates (r, θ), and, as we have seen before the Lagrangian is $\mathcal{L} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \Phi(r)$

The conjugate momenta are $p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r}$ and $p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2\dot{\theta}$

so that the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{p_\theta^2}{r^2} + \Phi(r)$$

Hamilton's equations now become

$$\frac{\partial \mathcal{H}}{\partial r} = -\frac{p_\theta^2}{r^3} + \frac{\partial \Phi}{\partial r} = -\dot{p}_r \qquad \frac{\partial \mathcal{H}}{\partial \theta} = 0 = -\dot{p}_\theta$$

$$\frac{\partial \mathcal{H}}{\partial p_r} = p_r = \dot{r} \qquad \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{r^2} = \dot{\theta}$$

which reduce to

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\partial \Phi}{\partial r} \qquad p_\theta = r^2\dot{\theta} = \text{cst}$$

Note that θ does not appear in the Hamiltonian: consequently p_θ is a conserved quantity

Noether's Theorem

In 1915 the German mathematician Emmy Noether proved an important theorem which plays a trully central role in theoretical physics.

Noether's Theorem: **If an ordinary Lagrangian posseses some continuous, smooth symmetry, then there will be a conservation law associated with that symmetry.**

- **Invariance of $\mathcal{L}$ under time translation $\longrightarrow$ energy conservation**
- **Invariance of $\mathcal{L}$ under spatial translation $\longrightarrow$ momentum conservation**
- **Invariance of $\mathcal{L}$ under rotational translation $\longrightarrow$ ang. mom. conservation**
- **Gauge Invariance of electric potential $\longrightarrow$ charge conservation**

Some of these symmetries are immediately evident from the Lagrangian:

- **If $\mathcal{L}$ does not explicitely depend on t then E is conserved**
- **If $\mathcal{L}$ does not explicitely depend on q_i then p_i is conserved**

Poisson Brackets I

DEFINITION: Let $A(\vec{q},\vec{p})$ and $B(\vec{q},\vec{p})$ be two functions of the generalized coordinates and their conjugate momenta, then the **Poisson bracket** of A and B is defined by

$$[A,B]=\sum_{i=1}^{n}\left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right]$$

Let $f=f(\vec{q},\vec{p},t)$ then

$$\mathrm{d}f=\frac{\partial f}{\partial q_i}\mathrm{d}q_i+\frac{\partial f}{\partial p_i}\mathrm{d}p_i+\frac{\partial f}{\partial t}\mathrm{d}t$$

where we have used the summation convention. This differential of f, combined with **Hamilton's equations**, allows us to write

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}$$

which reduces to

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+[f,\mathcal{H}]$$

This is often called **Poisson's equation of motion**. It shows that the time-evolution of any dynamical variable is governed by the Hamiltonian through the Poisson bracket of the variable with the Hamiltonian.

Poisson Brackets II

Using the Poisson brackets we can write

$$\boxed{\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} + [\mathcal{H}, \mathcal{H}] = \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{L}}{\partial t}}$$

where the latter equality follows from $\mathcal{H} = \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}$.

For an equilibrium system with a time-independent potential, $\partial \Phi / \partial t = 0$, we have that $\partial \mathcal{H} / \partial t = 0$ and thus also $\mathrm{d}\mathcal{H}/\mathrm{d}t = 0$. Since in this case the Hamiltonian is equal to the total energy, this simply reflects the energy conservation. Note that for any conservative system, $\mathcal{H}$ does not explicitely depend on time, and thus $\mathrm{d}\mathcal{H}/\mathrm{d}t = 0$

With the help of the Poisson brackets we can write Hamilton's equations in a more compact form

$$\boxed{\dot{q}_i = [q_i, \mathcal{H}] \qquad\qquad \dot{p}_i = [p_i, \mathcal{H}]}$$

Note that it is explicit that these equations of motion are valid in any system of generalized coordinated $(q_1, q_2, .., q_n)$ and their conjugate momenta $(p_1, p_2, .., p_n)$. As we will see next, in fact Hamilton's equations hold for any so-called canonical coordinate system.

Canonical Coordinate Systems

If we write $w_i = q_i$ and $w_{n+i} = p_i$ with $i = 1, .., n$ and we define the **symplectic matrix** c as

$$c_{\alpha\beta} \equiv [w_\alpha, w_\beta] = \begin{cases} \pm 1 & \text{if } \alpha = \beta \mp n \\ 0 & \text{otherwise} \end{cases}$$

with $\alpha, \beta \in [1, 2n]$, then

$$[A, B] = \sum_{\alpha,\beta=1}^{2n} c_{\alpha\beta} \frac{\partial A}{\partial w_\alpha} \frac{\partial B}{\partial w_\beta}$$

DEFINITION: Any set of $2n$ phase-space coordinates $\{w_\alpha, \alpha = 1, .., 2n\}$ is called **canonical** if $[w_\alpha, w_\beta] = c_{\alpha\beta}$.

Hamilton's equations can now be written in the extremely compact form:

$$\dot{w}_\alpha = [w_\alpha, \mathcal{H}]$$

which makes it explicit that they hold for any canonical coordinate system.

Note that the **generalized coordinates and momenta** $(\vec{q}, \vec{p})$ form a canonical coordinate system, since they obey the **canonical commutation relations**

$$[q_i, q_j] = [p_i, p_j] = 0 \qquad [p_i, q_j] = \delta_{ij}$$

Canonical Transformations I

Canonical Transformation: a transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ between two canonical coordinate systems that leaves the equations of motion invariant.

In order to reveal the form of these transformations, we first demonstrate the non-uniqueness of the Lagrangian.

Consider a transformation $\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}F}{\mathrm{d}t}$ where $F = F(\vec{q}, t)$

Under this transformation the action integral becomes

$$I' = \int_{t_0}^{t_1} \mathcal{L}' \mathrm{d}t = \int_{t_0}^{t_1} \mathcal{L} \mathrm{d}t + \int_{t_0}^{t_1} \frac{\mathrm{d}F}{\mathrm{d}t} \mathrm{d}t = I + F(t_1) - F(t_0)$$

Recall that the equations of motion correspond to $\delta I = 0$ (i.e., the action is stationary). Since the addition of $\mathrm{d}F/\mathrm{d}t$ only adds a constant, namely $F(t_1) - F(t_0)$ to the action, it leaves the equations of motion invariant.

Canonical Transformations II

Now consider our transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ with corresponding Lagrangians $\mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$ and $\mathcal{L}'(\vec{Q}, \dot{\vec{Q}}, t)$.

We start by writing the Lagrangians in terms of the corresponding Hamiltonians:

$$\boxed{\begin{aligned} \mathcal{L}(\vec{q}, \vec{p}, t) &= \vec{p} \cdot \dot{\vec{q}} - \mathcal{H}(\vec{q}, \vec{p}, t) \\ \mathcal{L}'(\vec{Q}, \vec{P}, t) &= \vec{P} \cdot \dot{\vec{Q}} - \mathcal{H}'(\vec{Q}, \vec{P}, t) \end{aligned}}$$

In order for the equations of motion to be invariant, we have the requirement that

$$\begin{aligned} & \mathcal{L}(\vec{q}, \vec{p}, t) = \mathcal{L}'(\vec{Q}, \vec{P}, t) + \frac{\mathrm{d}F}{\mathrm{d}t} \\ \Leftrightarrow \quad & \frac{\mathrm{d}F}{\mathrm{d}t} = \vec{p} \cdot \dot{\vec{q}} - \mathcal{H}(\vec{q}, \vec{p}, t) - \left[\vec{P} \cdot \dot{\vec{Q}} - \mathcal{H}'(\vec{Q}, \vec{P}, t)\right] \\ \Leftrightarrow \quad & \mathrm{d}F = p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t \end{aligned}$$

If we take $F = F(\vec{q}, \vec{Q}, t)$ then we also have that

$$\mathrm{d}F = \frac{\partial F}{\partial q_i}\mathrm{d}q_i + \frac{\partial F}{\partial Q_i}\mathrm{d}Q_i + \frac{\partial F}{\partial t}\mathrm{d}t$$

Canonical Transformations III

Equating the two expressions for the differential $\mathrm{d}F$ yields the transformation rules

$$p_i = \frac{\partial F}{\partial q_i} \qquad P_i = -\frac{\partial F}{\partial Q_i} \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial t}$$

The function $F(\vec{q}, \vec{Q}, t)$ is called the generating function of the canonical transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$

In order to transform $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ one proceeds as follows:

- **Find a function $F(\vec{q}, \vec{Q})$ so that $p_i = \partial F / \partial q_i$. This yields $Q_i(q_j, p_j)$**
- **Substitute $Q_i(q_j, p_j)$ in $P_i = \partial F / \partial Q_i$ to obtain $P_i(q_j, p_j)$**

As an example consider the generating function $F(\vec{q}, \vec{Q}) = q_i O_i$. According to the transformation rules we have that

$$p_i = \frac{\partial F}{\partial q_i} = Q_i \qquad P_i = -\frac{\partial F}{\partial Q_i} = -q_i$$

We thus have that $Q_i = p_i$ and $P_i = -q_i$: the canonical transformation has changed the roles of coordinates and momenta, eventhough the equations of motion have remained invariant! This shows that there is no special status to either generalized coordinates or their conjugate momenta

Canonical Transformations IV

For reasons that will become clear later, in practice it is more useful to consider a generating function of the form $S = S(\vec{q}, \vec{P}, t)$, i.e., one that depends on the old coordinates and the new momenta.

To derive the corresponding transformation rules, we start with the generating function $F = F(\vec{q}, \vec{Q}, t)$, and recall that

$$\mathrm{d}F = p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t$$

using that $P_i \mathrm{d}Q_i = \mathrm{d}(Q_i P_i) - Q_i \mathrm{d}P_i$, we obtain

$$\mathrm{d}(F + Q_i P_i) = p_i \mathrm{d}q_i + Q_i \mathrm{d}P_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t$$

Defining the new generator $S(\vec{q}, \vec{Q}, \vec{P}, t) \equiv F(\vec{q}, \vec{Q}, t) + \vec{Q} \cdot \vec{P}$, for which

$$\mathrm{d}S = \frac{\partial S}{\partial q_i}\mathrm{d}q_i + \frac{\partial S}{\partial Q_i}\mathrm{d}Q_i + \frac{\partial S}{\partial P_i}\mathrm{d}P_i + \frac{\partial S}{\partial t}\mathrm{d}t$$

Equating this to the above we find the transformation rules

$$p_i = \frac{\partial S}{\partial q_i} \qquad Q_i = \frac{\partial S}{\partial P_i} \qquad \frac{\partial S}{\partial Q_i} = 0 \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t}$$

Note that the third of these rules implies that $S = S(\vec{q}, \vec{P}, t)$ as intended.

Canonical Transformations V

The potential strength of canonical transformations becomes apparent from the following: Suppose one can find a canonical transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ such that $\mathcal{H}(\vec{q}, \vec{p}) \rightarrow \mathcal{H}'(\vec{P})$, i.e., such that the new Hamiltonian does not explicitely depend on the new coordinates Q_i.

Hamilton's equation of motion then become

$$\frac{\partial \mathcal{H}'}{\partial Q_i} = -\dot{P}_i = 0 \qquad \frac{\partial \mathcal{H}'}{\partial P_i} = -\dot{Q}_i$$

Thus, we have that all the conjugate momenta P_i are constant, and this in turn implies that none of $\dot{Q}_i$ can depend on time either. The equations of motion in our new, canonical coordinate system are therefore extremely simple:

$$Q_i(t) = \Omega_i t + k_i \qquad P_i = \text{constant}$$

Here $\Omega_i = \partial \mathcal{H}' / \partial P_i$ are constants and k_i are integration constants. Any generalized coordinate whose conjugate momentum is a conserved quantity, is called a cyclic variable. The question that remains now is how to find the generator $S(q, P, t)$ of the canonical transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ which leads to only cyclic variables Q_i.

The Hamilton-Jacobi Equation I

Recall the transformation rules for the generator $S(\vec{q}, \vec{P}, t)$:

$$p_i = \frac{\partial S}{\partial q_i} \qquad Q_i = \frac{\partial S}{\partial P_i} \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t}$$

If for simplicity we consider a generator that does not explicitely depend on time, i.e., $\partial S/\partial t = 0$ then we have that $\mathcal{H}(\vec{q}, \vec{p}) = \mathcal{H}'(\vec{P}) = E$. If we now substitute $\partial S/\partial q_i$ for p_i in the original Hamiltonian we obtain

$$\mathcal{H}\left(\frac{\partial S}{\partial q_i}, q_i\right) = E$$

This is the Hamilton-Jacobi equation, which is a partial differential equation.

If it can be solved for $S(\vec{q}, \vec{P})$ than, as we have seen above, basically the entire dynamics are solved.

Thus, for a dynamical system with n degrees of freedom, one can solve the dynamics in one of the three following ways:

- **Solve n second-order differential equations (Lagrangian formalism)**
- **Solve $2n$ first-order differential equations (Hamiltonian formalism)**
- **Solve a single partial differential equation (Hamilton-Jacobi equation)**

The Hamilton-Jacobi Equation II

Although it may seem an attractive option to try and solve the Hamilton-Jacobi equation, solving partial differential equations is in general much more difficult than solving ordinary differential equations, and the Hamilton-Jacobi equation is no exception.

However, in the specific case where the generator S is separable, i.e., if

$$S(\vec{q}, \vec{P}) = \sum_{i=1}^{n} f_i(q_i)$$

with f_i a set of n independent functions, then the Hamilton-Jacobi equation splits in a set of n ordinary differential equations which are easily solved by quadrature. The integration constants are related to the (constant) conjugate momenta P_i.

A Hamiltonian is called 'integrable' if the Hamilton-Jacobi equation is separable

Integrable Hamiltonians are extremely rare. Mathematically speaking they form a set of measure zero in the space of all Hamiltonians. In what follows, we establish the link between so-called isolating integrals of motion and whether or not a Hamiltonian is integrable.

Summary of the Above

- **Newton's second law:** $\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\vec{\nabla}\Phi(\vec{r})$
  - — **Complicated vector arithmetic & coordinate system dependence**
- **Lagrangian Formalism:** $\frac{\partial\mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) = 0$
  - — n **second-order differential equations**
- **Hamiltonian Formalism:** $\frac{\partial\mathcal{H}}{\partial p_i} = \dot{q}_i \qquad \frac{\partial\mathcal{H}}{\partial q_i} = -\dot{p}_i$
  - — $2n$ **first-order differential equations**
- **Hamilton-Jacobi equation:** $\mathcal{H}\left(\frac{\partial S}{\partial q_i}, q_i\right) = E$

$S(\vec{q},\vec{p})$ **is generator of canonical transformation** $(\vec{q},\vec{p}) \rightarrow (\vec{Q},\vec{P})$ **for which** $\mathcal{H}(\vec{q},\vec{p}) \rightarrow \mathcal{H}'(\vec{P})$**. If** $S(\vec{q},\vec{p})$ **is separable then the Hamilton-Jacobi equation breaks up in** n **ordinary differential equations which can be solved by simple quadrature. The resulting equations of motion are:**

$$P_i(t) = P_i(0) \qquad Q_i(t) = \left(\frac{\partial\mathcal{H}'}{\partial P_i}\right)t + k_i$$

Constants of Motion

Constants of Motion: any function $C(\vec{q},\vec{p},t)$ of the generalized coordinates, conjugate momenta and time that is constant along **every** orbit, i.e., if $\vec{q}(t)$ and $\vec{p}(t)$ are a solution to the equations of motion, then

$$C[\vec{q}(t_1),\vec{p}(t_1),t_1] = C[\vec{q}(t_2),\vec{p}(t_2),t_2]$$

for any t_1 and t_2. The value of the constant of motion depends on the orbit, but different orbits may have the same numerical value of C

A dynamical system with n **degrees of freedom** always has $2n$ independent **constants of motion**. Let $q_i = q_i[\vec{q}_0,\vec{p}_0,t]$ and $p_i = p_i[\vec{q}_0,\vec{p}_0,t]$ describe the solutions to the equations of motion. In principle, these can be inverted to $2n$ relations $q_{i,0} = q_{i,0}[\vec{q}(t),\vec{p}(t),t]$ and $p_{i,0} = p_{i,0}[\vec{q}(t),\vec{p}(t),t]$. By their very construction, these are $2n$ **constants of motion**.

If $\Phi(\vec{x},t) = \Phi(\vec{x})$, one of these $2n$ relations can be used to eliminate t. This leaves $2n-1$ non-trivial constants of motion, which restricts the system to a $2n-(2n-1)=1$-dimensional surface in **phase-space**, namely the **phase-space trajectory** $\Gamma(t)$

Note that the elimination of time reflects the fact that the physics are **invariant** to time translations $t \longrightarrow t+t_0$, i.e., the time at which we pick our initial conditions can not hold any information regarding our dynamical system.

Integrals of Motion I

Integrals of Motion: any function $I(\vec{x}, \vec{v})$ of the phase-space coordinates $(\vec{x}, \vec{v})$ alone that is constant along every orbit, i.e.

$$I[\vec{x}(t_1), \vec{v}(t_1)] = I[\vec{x}(t_2), \vec{v}(t_2)]$$

for any t_1 and t_2. The value of the integral of motion can be the same for different orbits. Note that an integral of motion can not depend on time. Thus, all integrals are constants, but not all constants are integrals.

Integrals of motion come in two kinds:

Isolating Integrals of Motion: these reduce the dimensionality of the trajectory $\Gamma(t)$ by one. Therefore, a trajectory in a dynamical system with n degrees of freedom and with i isolating integrals of motion is restricted to a $2n - i$ dimensional manifold in the $2n$-dimensional phase-space. Isolating integrals of motion are of great practical and theoretical importance.

Non-Isolating Integrals of Motion: these are integrals of motion that do not reduce the dimensionality of $\Gamma(t)$. They are of essentially no practical value for the dynamics of the system.

Integrals of Motion II

A stationary, Hamiltonian system (i.e., $\mathcal{H}(\vec{q},\vec{p},t) = \mathcal{H}(\vec{q},\vec{p})$) with n degrees of freedom always has $2n-1$ independent integrals of motion, which restrict the motion to the one-dimensional phase-space trajectory $\Gamma(t)$. The number of isolating integrals of motion can, depending on the Hamiltonian, vary between 1 and $2n-1$.

DEFINITION: Two functions I_1 and I_2 of the canonical phase-space coordinates $(\vec{q},\vec{p})$ are said to be in involution if their Poisson bracket vanishes, i.e., if

$$[I_1, I_2] = \frac{\partial I_1}{\partial q_i}\frac{\partial I_2}{\partial p_i} - \frac{\partial I_1}{\partial p_i}\frac{\partial I_2}{\partial q_i} = 0$$

A set of k integrals of motion that are in involution forms a set of k isolating integrals of motion.

Liouville's Theorem for Integrable Hamiltonians

A Hamiltonian system with n degrees of freedom which posseses n integrals of motion in involution, (and thus n isolating integrals of motion) is integrable by quadrature.

Integrable Hamiltonians I

LEMMA: If a system with n degrees of freedom has n constants of motion $P_i(\vec{q},\vec{p},t)$ [or integrals of motion $P_i(\vec{q},\vec{p})$] that are in involution, then there will also be a set of n functions $Q_i(\vec{q},\vec{p},t)$ [or $Q_i(\vec{q},\vec{p})$] which together with the P_i constitute a set of canonical variables.

Thus, given n isolating integrals of motion $I_i(\vec{q},\vec{p})$ we can make a canonical transformation $(\vec{q},\vec{p}) \rightarrow (\vec{Q},\vec{P})$ with $P_i = I_i(\vec{q},\vec{p}) =$constant and with $Q_i(t) = \Omega_i t + k_i$

An integrable, Hamiltonian system with n degrees of freedom always has a set of n isolating integrals of motion in involution. Consequently, the trajectory $\Gamma(t)$ is confined to a $2n - n = n$-dimensional manifold phase-space.

The surfaces specified by $(I_1, I_2, .., I_n) =$constant are topologically equivalent to n-dimensional tori. These are called invariant tori, because any orbit originating on one of them remains there indefinitely.

In an integrable, Hamiltonian system phase-space is completely filled (one says 'foliated') with invariant tori.

Integrable Hamiltonians II

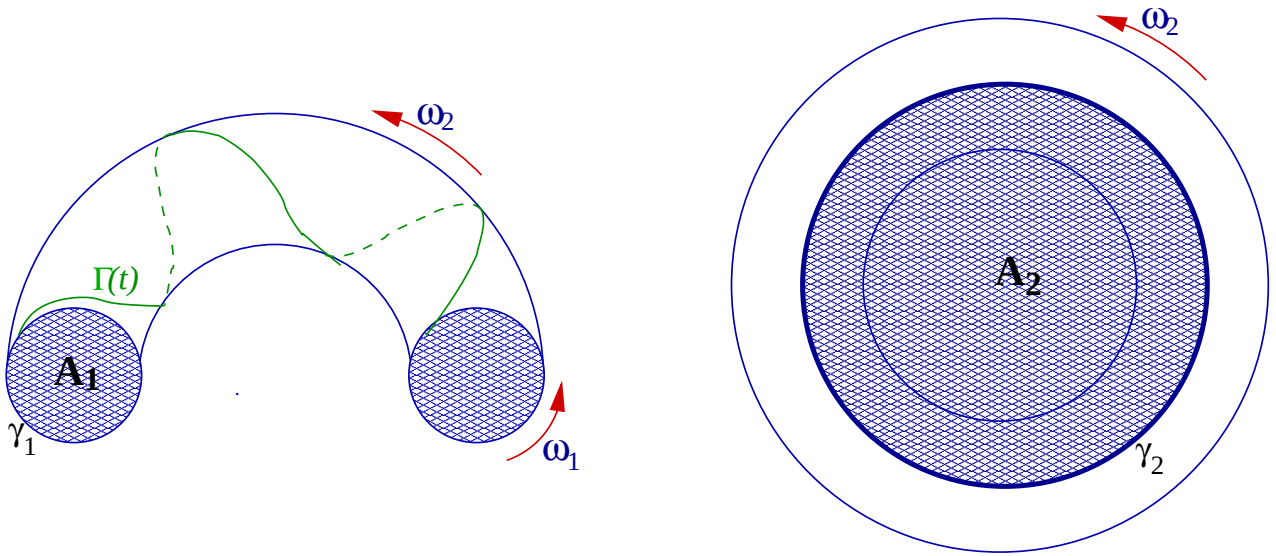
To summarize: if, for a system with n degrees of freedom, the Hamilton-Jacobi equation is separable, the Hamiltonian is integrable and there exist n isolating integrals of motion I_i in involution. In this case there exist canonical transformations $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ such that equations of motion reduce to:

$$\boxed{\begin{aligned} P_i(t) &= P_i(0) \\ Q_i(t) &= \left(\frac{\partial \mathcal{H}'}{\partial P_i}\right) t + k_i \end{aligned}}$$

One might think at this point, that one has to take $P_i = I_i$. However, this choice is not unique. Consider an integrable Hamiltonian with $n = 2$ degrees of freedom and let I_1 and I_2 be two isolating integrals of motion in involution. Now define $I_a = \frac{1}{2}(I_1 + I_2)$ and $I_b = \frac{1}{2}(I_1 - I_2)$, then it is straightforward to proof that $[I_a, I_b] = 0$, and thus that (I_a, I_b) is also a set of isolating integrals of motion in involution. In fact, one can construct infinitelly many sets of isolating integral of motion in involution. Which one should we choose, and in particular, which one yields the most meaningful description of the invariant tori?

Answer: The Action-Angle variables

Action-Angle Variables I



Let's be guided by the idea of our invariant tori. The figure illustrates a 2D-torus (in 4D-phase space), with a trajectory $\Gamma(t)$ on its surface. One can specify a location on this torus by the two position angles ω_1 and ω_2. The torus itself is characterized by the areas of the two (hatched) cross sections labelled A_1 and A_2. The action variables J_1 and J_2 are intimitally related to A_1 and A_2, which clearly are two integrals of motion.

The action variables are defined by:

$$J_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{p} \cdot \mathrm{d}\vec{q}$$

with γ_i the closed loop that bounds cross section A_i.

Action-Angle Variables II

The angle-variables ω_i follow from the canonical transformation rule $\omega_i = \frac{\partial S}{\partial J_i}$ with $S = S(\vec{q}, \vec{J})$ the generator of the canonical transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{\omega}, \vec{J})$. Since the actions J_i are isolating integrals of motion we have that the corresponding conjugate angle coordinates w_i obey

$$\omega_i(t) = \left(\frac{\partial \mathcal{H}'}{\partial J_i}\right) t + \omega_0$$

with $\mathcal{H}' = \mathcal{H}'(\vec{J})$ the Hamiltonian in action-angle variables $(\vec{\omega}, \vec{J})$.

We now give a detailed description of motion on invariant tori:

Orbits in integrable, Hamiltonian systems with n degrees of freedom are characterized by n constant frequencies

$$\boxed{\Omega_i \equiv \frac{\partial \mathcal{H}'}{\partial J_i}}$$

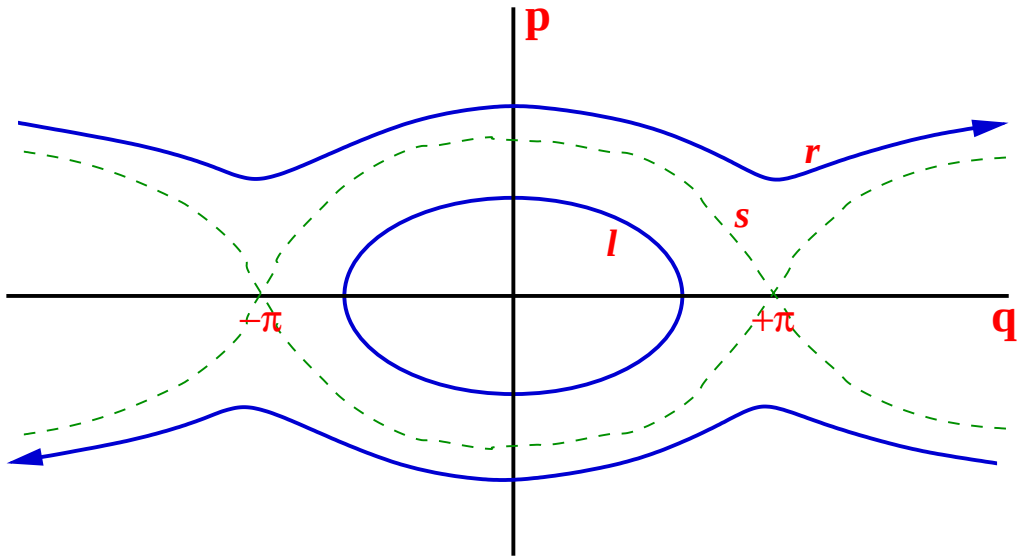
This implies that the motion along each of the n degrees of freedom, q_i, is periodic in time, and this can occur in two ways:

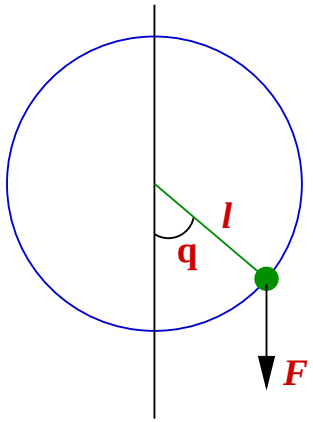
- **Libration: motion between two states of vanishing kinetic energy**
- **Rotation: motion for which the kinetic energy never vanishes**

The Pendulum

To get insight into libration and rotation consider a pendulum, which is a integrable Hamiltonian system with one degree of freedom, the angle q..

The figures below shows the corresponding phase-diagram.





l=libration
r=rotation
s=separatrix

- **Libration:** $q(\omega + 2\pi) = q(\omega)$.
- **Rotation:** $q(\omega + 2\pi) = q(\omega) + 2\pi$

To go from libration to rotation, one needs to cross the separatrix

Action-Angle Variables III

Why are action-angle variables the ideal set of isolating integrals of motion to use?

- **They are are the only conjugate momenta that enjoy the property of adiabatic invariance (to be discussed later)**
- **The angle-variables are the natural coordinates to label points on invariant tori.**
- **They are ideally suited for perturbation analysis, which is used to investigate near-integrable systems (see below)**
- **They are ideally suited to study the (in)-stability of a Hamiltonian system**

Example: Central Force Field

As an example, to get familiar with action-angle variables, let's consider once again motion in a central force field.

As we have seen before, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{p_\theta^2}{r^2} + \Phi(r)$$

where $p_r = \dot{r}$ and $p_\theta = r^2\dot{\theta} = L$.

In our planar description, we have two integrals of motion, namely energy $I_1 = E = \mathcal{H}$ and angular momentum $I_2 = L = p_\theta$.

These are classical integrals of motion, as they are associated with symmetries. Consequently, they are also isolating.

Let's start by checking whether they are in involution

$$[I_1, I_2] = \left[\frac{\partial I_1}{\partial r}\frac{\partial I_2}{\partial p_r} - \frac{\partial I_1}{\partial p_r}\frac{\partial I_2}{\partial r}\right] + \left[\frac{\partial I_1}{\partial \theta}\frac{\partial I_2}{\partial p_\theta} - \frac{\partial I_1}{\partial p_\theta}\frac{\partial I_2}{\partial \theta}\right]$$

Since $\frac{\partial I_2}{\partial p_r} = \frac{\partial I_2}{\partial r} = \frac{\partial I_1}{\partial \theta} = \frac{\partial I_2}{\partial \theta} = 0$ one indeed finds that the two integrals of motion are in involution.

Example: Central Force Field

The **actions** are defined by

$$J_r = \frac{1}{2\pi}\oint_{\gamma_r} p_r \mathrm{d}r \qquad\qquad J_\theta = \frac{1}{2\pi}\oint_{\gamma_\theta} p_\theta \mathrm{d}\theta$$

In the case of J_θ the θ-motion is one of **rotation**. Therefore the closed-line-integral is over an **angular** interval $[0, 2\pi]$.

$$J_\theta = \frac{1}{2\pi}\int_0^{2\pi} I_2 \mathrm{d}\theta = I_2$$

In the case of J_r, we need to realize that the r-motion is a **libration** between **apocenter** r_+ and **pericenter** r_-. Using that $I_1 = E = \mathcal{H}$ we can write

$$p_r = \sqrt{2[I_1 - \Phi(r)] - I_2^2/r^2}$$

The radial action then becomes

$$J_r = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2[I_1 - \Phi(r)] - I_2^2/r^2}\mathrm{d}r$$

Once we make a choise for the potential $\Phi(r)$ then J_r can be solved as function of I_1 and J_θ. Since $I_1 = \mathcal{H}$, this in turn allows us to write the Hamiltonian as function of the actions: $\mathcal{H}(J_r, J_\theta)$.

Example: Central Force Field

As an example, let's consider a potential of the form

$$\Phi(r) = -\frac{\alpha}{r} - \frac{\beta}{r^2}$$

with α and β two constants. Substituting this in the above, one finds:

$$J_r = \alpha \left(\frac{1}{2|I_1|}\right)^{1/2} - \sqrt{J_\theta^2 - 2\beta}$$

Inverting this for $I_1 = \mathcal{H}$ yields

$$\boxed{\mathcal{H}(J_r, J_\theta) = \frac{\alpha^2}{2}\left(J_r + \sqrt{J_\theta^2 - 2\beta}\right)^{-2}}$$

Since the actions are isolating integrals of motion, and we have an expression for the Hamiltonian in terms of these actions, the generalized coordinates that correspond to these actions (the angles w_r and w_θ) evolve as $w_i(t) = \Omega_i\, t + w_{i,0}$

The radial and angular frequencies are

$$\Omega_r = \frac{\partial \mathcal{H}}{\partial J_r} = -\alpha^2 \left(J_r + \sqrt{J_\theta^2 - 2\beta}\right)^{-3}$$

$$\Omega_\theta = \frac{\partial \mathcal{H}}{\partial J_\theta} = -\alpha^2 \left(J_r + \sqrt{J_\theta^2 - 2\beta}\right)^{-3} \frac{J_\theta}{\sqrt{J_\theta^2 - 2\beta}}$$

Example: Central Force Field

The ratio of these frequencies is

$$\frac{\Omega_r}{\Omega_\theta} = \left(1 - \frac{2\beta}{J_\theta^2}\right)^{1/2}$$

Note that for $\beta = 0$, for which $\Phi(r) = -\frac{\alpha}{r}$, and thus the potential is of the Kepler form, we have that $\Omega_r = \Omega_\theta$ independent of the actions (i.e., for each individual orbit).

In this case the orbit is closed, and there is an additional isolating integral of motion (in addition to E and L). We may write this 'third' integral as

$$I_3 = w_r - w_\theta = \Omega_r t + w_{r,0} - \Omega_\theta t - w_{\theta,0} = w_{r,0} - w_{\theta,0}$$

Without loosing generality, we can pick the zero-point of time, such that $w_{r,0} = 0$. This shows that we can think of the third integral in a Kepler potential as the angular phase of the line connecting apo- and peri-center.

Quasi-Periodic Motion

In general, in an integrable Hamiltonian system with the canonical transformation $(\vec{q},\vec{p}) \rightarrow (\vec{Q},\vec{P})$ one has that $q_k = q_k(\omega_1,..,\omega_n)$ with $k=(1,..,n)$. If one changes ω_i by 2π, while keeping the other $\omega_j (j \neq i)$ fixed, then q_i then performs a complete libration or rotation.

The Cartesian phase-space coordinates $(\vec{x},\vec{v})$ must be periodic functions of the angle variables ω_i with period 2π. Any such function can be expressed as a Fourier series

$$\vec{x}(\vec{\omega},\vec{J}) = \sum_{l,m,n=-\infty}^{\infty} X_{lmn}(\vec{J}) \exp\left[i(l\omega_1 + m\omega_2 + n\omega_3)\right]$$

Using that $\omega_i(t) = \Omega_i t + k_i$ we thus obtain that

$$\boxed{\vec{x}(t) = \sum_{l,m,n=-\infty}^{\infty} \tilde{X}_{lmn} \exp\left[i(l\Omega_1 + m\Omega_2 + n\Omega_3)t\right]}$$

with $\tilde{X}_{lmn} = X_{lmn} \exp\left[i(lk_1 + mk_2 + nk_3)\right]$

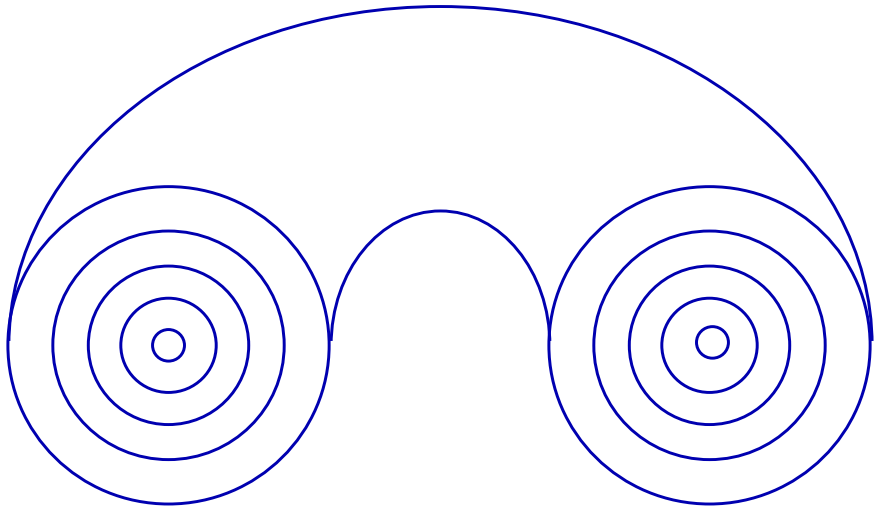
Functions of the form of $\vec{x}(t)$ are said to be quasi-periodic functions of time. Hence, in an integrable systems, all orbits are quasi-periodic, and confined to an invariant torus.

Integrable Hamiltonians III

When one integrates a trajectory $\Gamma(t)$ in an integrable system for sufficiently long, it will come infinitessimally close to any point $\vec{\omega}$ on the surface of its torus. In other words, the trajectory densely fills the entire torus.

Since no two trajectories $\Gamma_1(t)$ and $\Gamma_2(t)$ can intersect the same point in phase-space, we thus immediately infer that two tori are not allowed to intersect.

In an integrable, Hamiltonian system phase-space is completely foliated with non-intersecting, invariant tori



Integrable Hamiltonians IV

In an integrable, Hamiltonian system with n degrees of freedom, all orbits are confined to, and densely fill the surface of n-dimensional invariant tori.

These orbits, which have (at least) as many isolating integrals as spatial dimensions are called regular

Regular orbits have n frequencies Ω_i which are functions of the corresponding actions J_i. This means that one can always find suitable values for J_i such that two of the n frequencies Ω_i are commensurable, i.e. for which

$$\boxed{l\ \Omega_i = m\ \Omega_j}$$

with $i \neq j$ and l, m both integers.

A regular orbit with commensurable frequencies is called a resonant orbit (also called closed or periodic orbit), and has a dimensionality that is one lower than that of the non-resonant, regular orbits. This implies that there is an extra isolating integral of motion, namely

$$I_{n+1} = l\omega_i - m\omega_j$$

Note: since $\omega_i(t) = \Omega_i t + k_i$, one can obtains that $I_{n+1} = lk_i - mk_j$, and thus is constant along the orbit.

Near-Integrable Systems I

Thus far we have focussed our attention on integrable, Hamiltonian systems.

Given a Hamiltonian $\mathcal{H}(\vec{q},\vec{p})$, how can one determine whether the system is integrable, or whether the Hamilton-Jacobi equation is separable?

Unfortunately, there is no real answer to this question: In particular, there is no systematic method for determining if a Hamiltonian is integrable or not!!!

However, if you can show that a system with n degrees of freedom has n independent integrals of motion in involution then the system is integrable.

Unfortunately, the explicit expression of the integrals of motion in terms of the phase-space coordinates is only possible in a very so called classical integrals of motion, those associated with a symmetry of the potential and/or with an invariance of the coordinate system.

In what follows, we only consider the case of orbits in 'external' potentials for which $n = 3$. In addition, we only consider stationary potentials $\Phi(\vec{x})$, so that the Hamiltonian does not explicitely on time and $\mathcal{H}(\vec{q},\vec{p}) = E =$constant. Therefore

Energy is always an isolating integral of motion.

Note: this integral is related to the invariance of the Lagrangian $\mathcal{L}$ under time translation, i.e., to the homogeneity of time.

Near-Integrable Systems II

Integrable Hamiltonians are extremely rare. As a consequence, it is extremely unlikely that the Hamiltonian associated with a typical galaxy potential is integrable.

One can prove that even a slight perturbation away from an integrable potential will almost always destroy any integral of motion other than E.

So why have we spent so must time discussing integrable Hamiltonians?
▷ Because most galaxy-like potentials turn out to be near-integrable.

Definition: A Hamiltonian system is near-integrable if a large fraction of phase-space is still occupied by regular orbits (i.e., by orbits on invariant tori).

The dynamics of near-integrable Hamiltonians is the subject of the Kolmogorov-Arnold-Moser (KAM) Theorem which states:

If $\mathcal{H}_0$ is an integrable Hamiltonian whose phase-space is completely foliated with regular orbits on invariant tori, then in a perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \varepsilon\mathcal{H}_1$ most orbits will still lie on such tori for sufficiently small ε. The fraction of phase-space covered by these tori $\longrightarrow 1$ for $\varepsilon \longrightarrow 0$ and the perturbed tori are deformed versions of the unperturbed ones.

Near-Integrable Systems III

The stability of the original tori to a perturbation can be proven everywhere except in small regions around the resonant tori of $\mathcal{H}_0$. The width of these regions depends on ε and on the order of the resonance.

According to the Poincaré-Birkhoff Theorem the tori around unstable resonant tori break up and the corresponding regular orbits become irregular and stochastic.

Definition: **A resonant orbit is stable if an orbit starting close to it remains close to it. They parent orbit families (see below)**

Definition: **An irregular orbit is an orbit that is not confined to a n-dimensional torus. In general it can wander through the entire phase-space permitted by conservation of energy.**

Consequently, an irregular orbit is restricted to a higher-dimensional manifold than a regular orbit. Irregular orbits are stochastic in that they are extremely sensitive to initial conditions: two stochastic trajectories $\Gamma_1(t)$ and $\Gamma_2(t)$ which at $t = t_0$ are infinitesimally close together will diverge with time.

Increasing ε, increases the widths of the stochastic zones, which may eventually 'eat up' a large fraction of phase-space.

Near-Integrable Systems IV

Note that unperturbed resonant tori form a dense set in phase-space, just like the rational numbers are dense on the real axis.

Just like you can always find a rational number in between two real numbers, in a near-integrable system there will always be a resonant orbit in between any two tori. Since many of these will be unstable, they create many stochastic regions

As long as the resonance is of higher order (i.e, $16:23$ rather than $1:2$) the corresponding chaotic regions are very small, and tightly bound by their surrounding tori.

Since two trajectories can not cross, an irregular orbit is bounded by its neighbouring regular orbits. The irregular orbit is therefore still (almost) confined to a n-dimensional manifold, and it behaves as if it has n isolating integrals of motion.

▷ It may be very difficult to tell whether an orbit is regular or irregular.

However, iff $n > 2$ an irregular orbit may slip through a 'crack' between two confining tori, a process know as Arnold diffusion.

Because of Arnold diffusion the stochasticity will be larger than 'expected'. However, the time-scale for Arnold diffusion to occus is long, and it is unclear how important it is for Galactic Dynamics.

Near-Integrable Systems V

A few words on nomenclature:

Recall that an (isolating) integral of motion is defined as a function of phase-space coordinates that is constant along every orbit.

A near-integrable systems in principle has only one isolating integral of motion, namely energy E.

Nevertheless, according to the KAM Theorem, many orbits in a near-integrable system are confined to invariant tori.

Although in conflict with the definition, astronomers often say that the regular orbits in near-integrable systems admit n isolating integrals of motion.

Astronomers also often use KAM Theorem to the extreme, by assuming that they can ignore the irregular orbits, and that the Hamiltonians that correspond to their 'galaxy-like' potentials are integrable. Clearly the validity of this approximation depends on the fraction of phase-space that admits three isolating integrals of motion. For most potentials used in Galactic Dynamics, it is still unclear how large this fraction really is, and thus, how reliable the assumption of integrability is.

Surfaces of Section I

Consider a system with $n = 2$ degrees of freedom (e.g., planar motion), and with a Hamiltonian

$$\mathcal{H}(\vec{x}, \vec{p}) = \tfrac{1}{2}(p_x^2 + p_y^2) + \Phi(x, y)$$

Conservation of energy, $E = \mathcal{H}$, restricts the motion to a three-dimensional hyper-surface $\mathcal{M}_3$ in four-dimensional phase-space.

To investigate whether the orbits admit any additional (hidden) isolating integrals of motion, Poincaré introduced the surface-of-section (SOS)

Consider the intersection of $\mathcal{M}_3$ with the surface $y = 0$. Integrate the orbit, and everytime it crosses the surface $y = 0$ with $\dot{y} > 0$, record the position in the (x, p_x)-plane. After many orbital periods. the accumulated points begin to show some topology that allows one to discriminate between regular, irregular and resonance orbits.

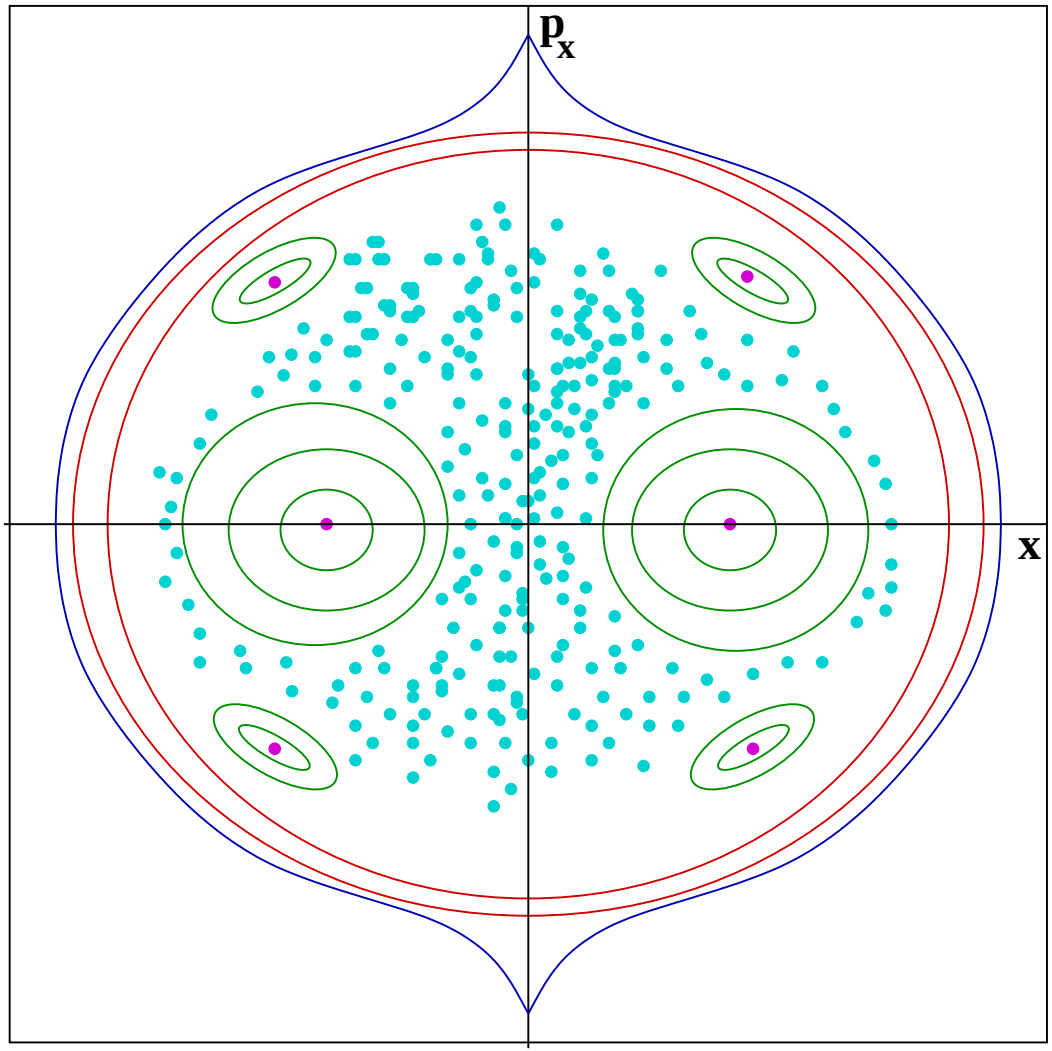
Given (x, p_x) and the condition $y = 0$, we can determine p_y from

$$p_y = +\sqrt{2[E - \Phi(x, 0)] - p_x^2}$$

where the $+$-sign is chosen because $\dot{y} > 0$.

To get insight, and relate orbits to their SOSs, see JAVA-Applet at:
http://burro.astr.cwru.edu/JavaLab/SOSweb/backgrnd.html

Surfaces of Section II



= energy surface

= regular box orbit

= regular loop orbit

= irregular (stochastic) orbit

= periodic (resonance) orbit

NOTE: Each resonance orbit creates a family of regular orbits.

Loop orbit: has fixed sense of rotation about the center; never has x^-

Box orbit: no fixed sense of rotation about the center. Orbit comes arbitrarily close to center.

This figure is only an illustration of the topology of various orbits in a SOS. It does not correspond to an existing Hamiltonian.

Orbits in Spherical Potentials I

A spherical potential has four classical, isolating integrals of motion: Energy E, associated with time-invariance of the Lagrangian, and the three components of the angular momentum vector, L_x, L_y, and L_z, associated with rotational invariance of the potential.

NOTE: Since

$$[L_x, L_y] = L_z, \quad [L_y, L_z] = L_x, \quad [L_z, L_x] = L_y$$

the set (L_x, L_y, L_z) is not in involution. However, the absolute value of the angular momentum

$$|L| = \sqrt{L_x^2 + L_y^2 + L_z^2}$$

is in involution with any of its components. We can thus define a set of three isolating integrals of motion in involution, e.g., $(E, |L|, L_z)$. The values for these three integrals uniquely determine the motion, and specify a unique invariant torus.

Orbits in Spherical Potentials II

We have seen before that for motion in a central force field:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{2[E-\Phi(r)]-\frac{L^2}{r^2}} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{L}{r^2}$$

From this we immediately infer the nature of the motion:

- θ**-motion is rotation (**$\mathrm{d}\theta/\mathrm{d}t$ **is never zero)**
- r**-motion is libration (**$\mathrm{d}r/\mathrm{d}t = 0$ **at apo- and pericenter)**

The radial period is

$$T_r = 2\int_{r_-}^{r_+}\frac{\mathrm{d}r}{\mathrm{d}r/\mathrm{d}t} = 2\int_{r_-}^{r_+}\frac{\mathrm{d}r}{\sqrt{2[E-\Phi(r)]-L^2/r^2}}$$

In the same period the polar angle θ increases by an amount

$$\Delta\theta = 2\int_{r_-}^{r_+}\frac{\mathrm{d}\theta}{\mathrm{d}r}\mathrm{d}r = 2\int_{r_-}^{r_+}\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r = 2\int_{r_-}^{r_+}\frac{L\mathrm{d}r}{r^2\sqrt{2[E-\Phi(r)]-L^2/r^2}}$$

The azimuthal period can thus be written as $T_\theta = \frac{2\pi}{\Delta\theta}T_r$. From this we see that the orbit will be closed (resonant) if

$$\frac{T_\theta}{T_r} = \frac{2\pi}{\Delta\theta} = \frac{n}{m} \qquad \text{with } n \text{ and } m \text{ integers}$$

Orbits in Spherical Potentials III

In general $\Delta\theta/2\pi$ will not be a rational number. ▷ orbit not closed.

Instead, a typical orbit resembles a rosette and eventually passes through every point in between the annuli bounded by the apo- and pericenter.

However, there are two special potentials for which all orbits are closed:

- **Spherical Harmonic Oscillator Potential: $\Phi(r) = \frac{1}{2}\Omega^2 r^2$**
  - **—Orbits are ellipses centered on center of attraction**
  - **— $T_\theta : T_r = 2 : 1$**
- **Kepler Potential: $\Phi(r) = -\frac{G\,M}{r}$**
  - **—Orbits are ellipses with attracting center at one focal point**
  - **— $T_\theta : T_r = 1 : 1$**

Since galaxies are less centrally concentrated than point masses and more centrally concentrated than homogeneous spheres, a typical star in a spherical galaxy changes its angular coordinate by $\Delta\theta$ during a radial libration, where $\pi < \Delta\theta < 2\pi$

QUESTION: What is the resonance that corresponds to a circular orbit?

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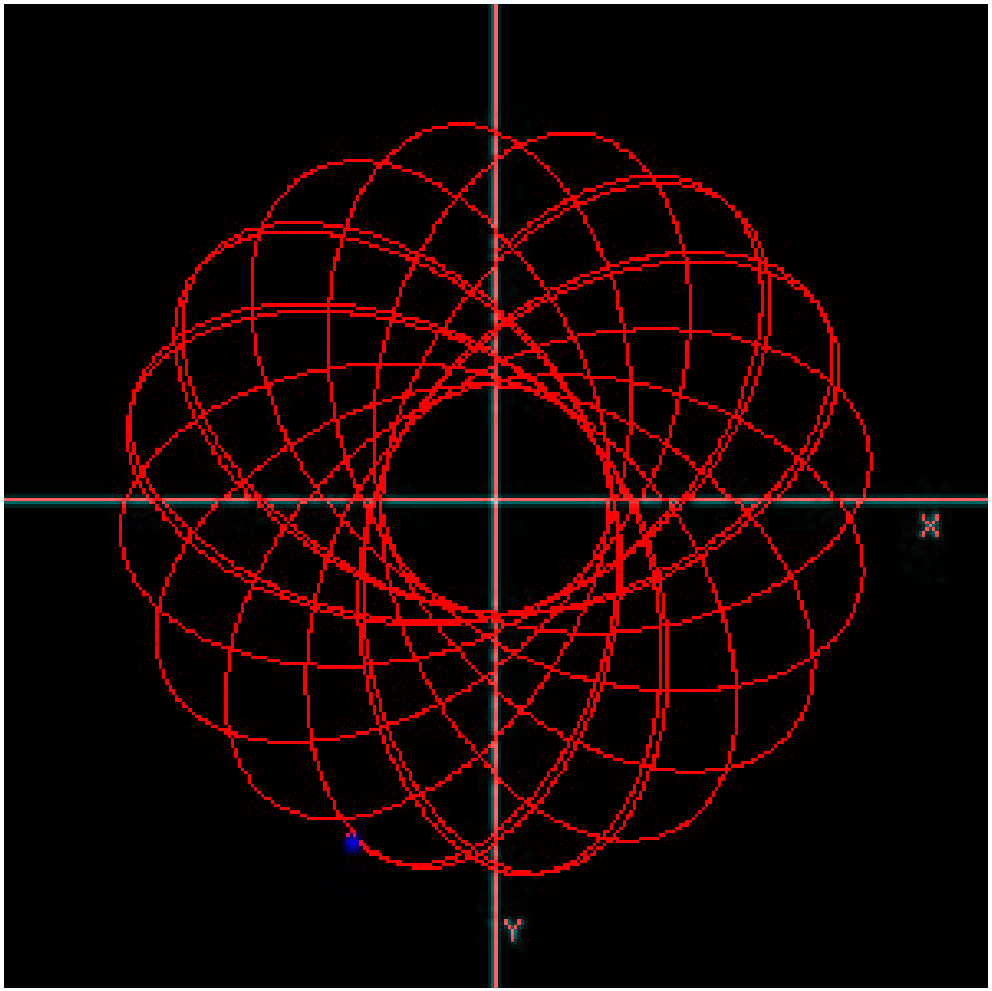
QUESTION: What is the resonance that corresponds to a circular orbit?
ANSWER: Although closed, a circular orbit is NOT a resonance orbit

Orbits in Spherical Potentials IV



An example of a rosette orbit with non-commensurable frequencies. Virtually all orbits in spherical potentials are of this form. The more general name for this type of orbits is loop orbits. They have a net sense of rotation around the center.

Orbits in Planar Potentials I

Before we discuss orbits in less symmetric, three-dimensional potentials, we first focus our attention on Planar Potentials $\Phi(x, y)$. This is useful for the following reasons:

- **There are cases in which the symmetry of the potential allows a reduction of the number of degrees of freedom by means of the effective potential Φ_{eff}. Examples are axisymmetric potentials where Φ_{eff} allows a study of motion in the so-called meridional plane.**
- **Motion confined to the symmetry-planes of ellipsoidal, spheroidal and spherical potentials is planar.**
- **There are mass distributions of astronomical interest with potentials that are reasonably well approximated by planar potentials (disks).**
- **To get insight into the various orbit families.**

Orbits in Planar Potentials II

As an example, we consider motion in the planar, logarithmic potential

$$\Phi_L(x,y) = \tfrac{1}{2}\, v_0^2 \ln\left(R_c^2 + x^2 + \frac{y^2}{q^2}\right) \qquad (q \leq 1)$$

This potential has the following properties:

(i) Equipotentials have constant axial ratio q so that influence of non-axisymmetry is similar at all radii

(ii) For $R = \sqrt{x^2+y^2} \ll R_c$ a power-series expansion gives

$$\Phi_L \simeq \frac{v_0^2}{2R_c^2}\left(x^2 + \frac{y^2}{q^2}\right)$$

which is similar to that of a two-dimensional harmonic oscillator, which corresponds to a homogeneous density distribution.

(iii) For $R \gg R_c$ and $q = 1$ we have that $\Phi_L = \frac{1}{2}v_0^2 \ln R$. One can easily verify that this corresponds to a circular velocity curve $v_{\rm circ}(R) = v_0$; i.e., at large radii Φ_L yields a flat rotation curve, similar to that of disk galaxies.

Orbits in Planar Potentials III

We start our investigation of orbits in $\Phi_L(x,y)$ with those that are confined to $R \ll R_c$, i.e., those confined to the constant density core.

Using series expansion, we can approximate the potential by

$$\Phi_L \simeq \frac{v_0^2}{2R_c^2}\left(x^2 + \frac{y^2}{q^2}\right) = \Phi_1(x) + \Phi_2(y)$$

Note that we can separate the potential. This allows us to immediately identify two isolating integrals of motion in involution from the Hamiltonian:

$$I_1 = p_x^2 + 2\Phi_1(x) \qquad I_2 = p_y^2 + 2\Phi_2(y)$$

The motion of the system is given by the superposition of the librations along the two axes, which are the solutions of the decoupled system of equations

$$\ddot{x} = -\omega_x^2 x \qquad \ddot{y} = -\omega_y^2 y$$

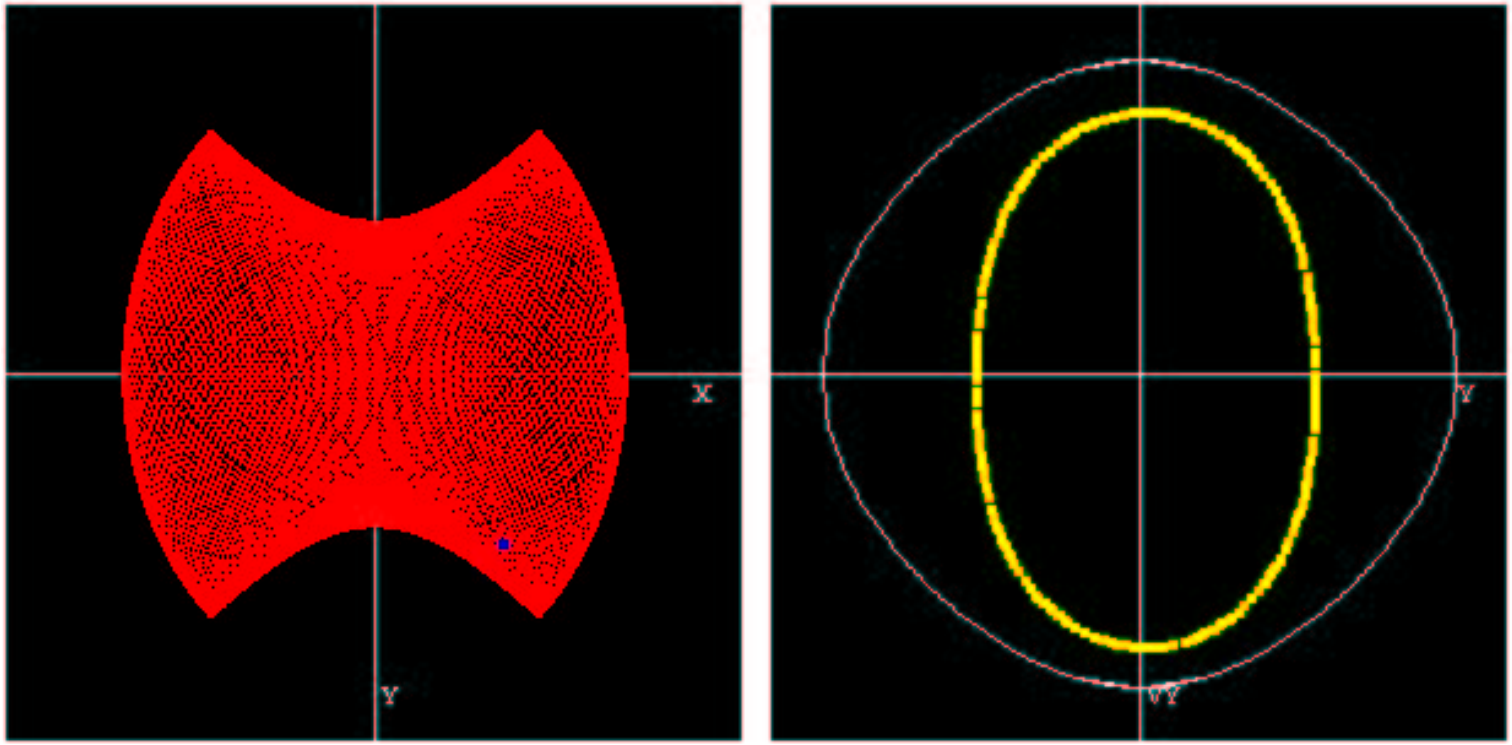
which corresponds to a two-dimensional harmonic oscillator with frequencies $\omega_x = v_0/R_c$ and $\omega_y = v_0/qR_c$. Unless these are incommensurable (i.e., unless $\omega_x/\omega_y = n/m$ for some integers n and m) the star passes close to every point inside a rectangular box.

These orbits are therefore known as box orbits. Such orbits have no net sense of circulation about the center.

Orbits in Planar Potentials IV

For orbits at larger radii $R \gtrsim R_c$ one has to resort to numerical integration.

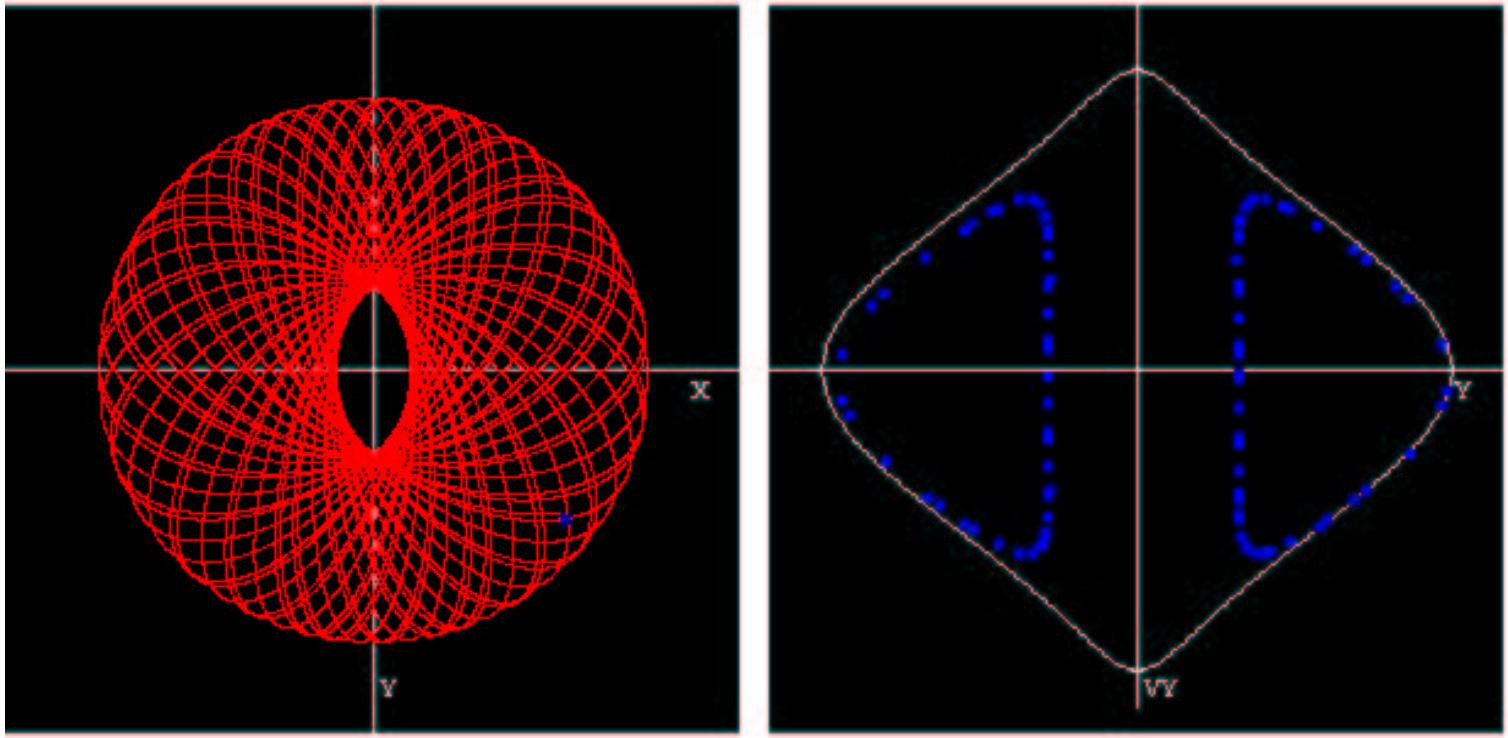
This reveals two major orbit families: The first is the family of box orbits, which have no net sense of cirulation about the center, and which, in the course of time, will pass arbitrarily close to the center of the potential



Note that the orbit completes a filled curve in the SOS, indicating that it admits a second isolating integral of motion, I_2. This is not a classical integral, as it is not associated with a symmetry of the system. We can, in general, not express I_2 in the phase-space coordinates.

Orbits in Planar Potentials V

The second main family is that of loop orbits. These do have a net sense of circulation, and always maintain a minimum distance from the center of the potential. Any star launched from $R \gg R_c$ in the tangential direction with a speed of the order of v_0 will follow such a loop orbit.



Once again, the fact that the orbit completes a filled curve in the SOS, indicates that it admits a second, (non-classical) isolating integral of motion. Since we don't know what this integral is (in terms of the phase-space coordinates) it is simply called I_2.

Orbits in Planar Potentials VI

In $\Phi_L(x,y)$ there are two main orbit families: loop orbits and box orbits

Each family of orbits is closely associated with a corresponding closed orbit. This closed orbit is called the parent of the orbit family. All closed orbits that are parents to families are said to be stable, in that members of their family that are initially close to them remain close to them at all times.
Unstable, closed orbits also exist, but they don't parent an orbit family.

Modulo the irregular orbits, one can obtain a good consensus of the orbits in a system by finding the stable periodic orbits at each energy.

For our planar, logarithmic potential, the parent of the loop orbits is the closed loop orbit (which intersects SOS at a single point on $\dot{x}=0$ axis).

The parent of the box orbits is the closed long-axis orbit. This is the orbit that is confined to the x-axis with $y=\dot{y}=0$. In the SOS (x vs. p_x) this is the orbit associated with the boundary curve which corresponds to

$$\tfrac{1}{2}\dot{x}^2+\Phi_L(x,0)=E \qquad \text{i.e. } y=\dot{y}=0$$

Finally, there is the closed short-axis orbit. This is the orbit that is confined to the y-axis $x=\dot{x}=0$. In the SOS this is the orbit associated with a single dot exactly at the center of the SOS. Clearly, this orbit is unstable: rather than parenting an orbit family, it marks the transition between loop and box orbits.

Orbits in Planar Potentials VII

If we set the core radius $R_c = 0$ in our planar, logarithmic potential, we remove the homogeneous core and introduce a singular R^{-2} cusp.

This singular logarithmic potential admits a number of new orbit families, which are associated with resonant parents, which we ID by the frequency ratio $\omega_x : \omega_y$.

The family associated with the $2:1$ resonance are called the banana orbits

The family associated with the $3:2$ resonance are called the fish orbits

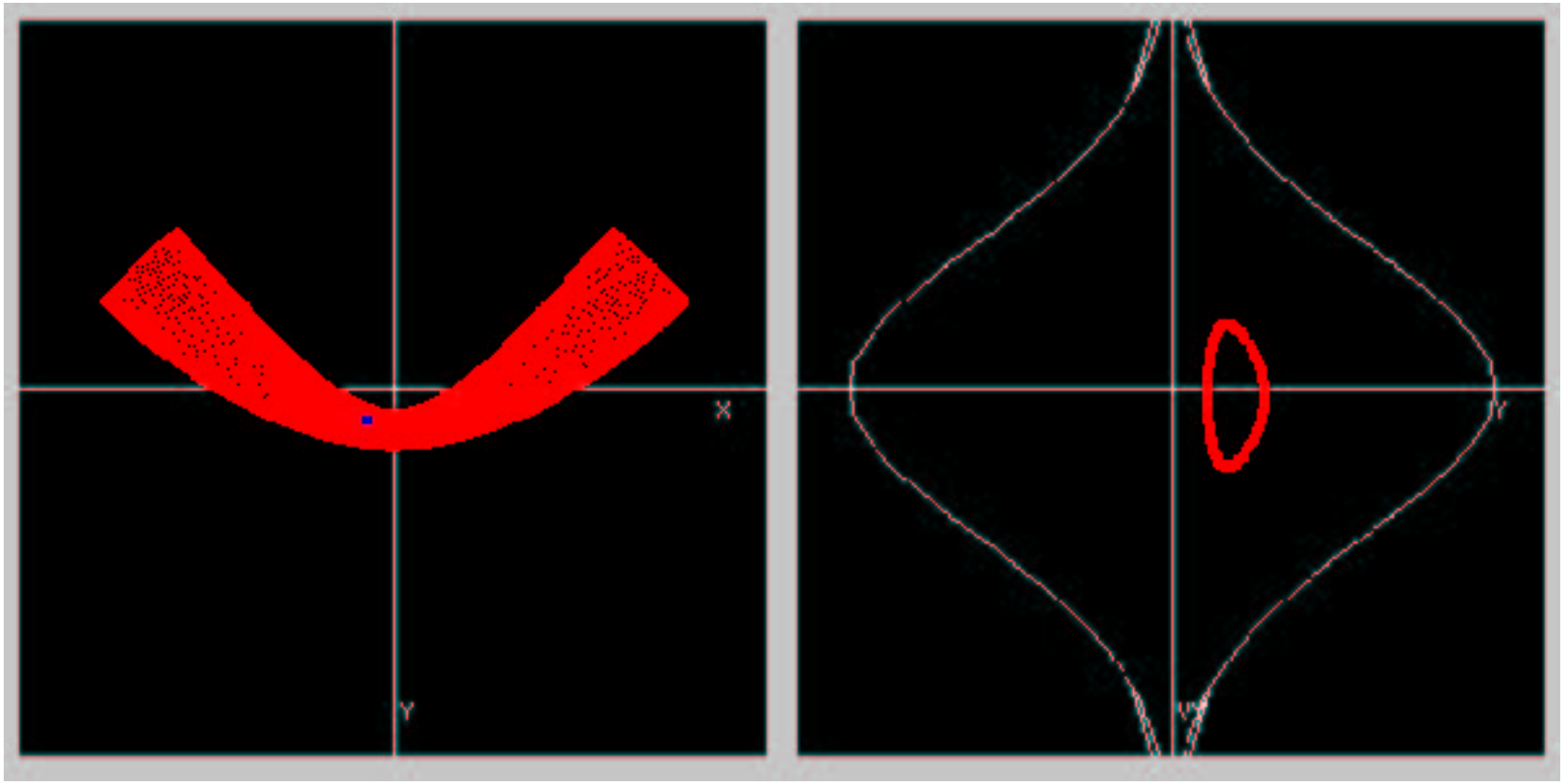
The family associated with the $4:3$ resonance are called the pretzel orbits

All these families together are called boxlet orbits.

The (singular) logarithmic potential is somewhat special in that it shows a surprisingly regular orbit structure. Virtually the entire phase-space admits two isolating integrals of motion in involution (E and I_2). Clearly, the logarithmic potential must be very near-integrable.

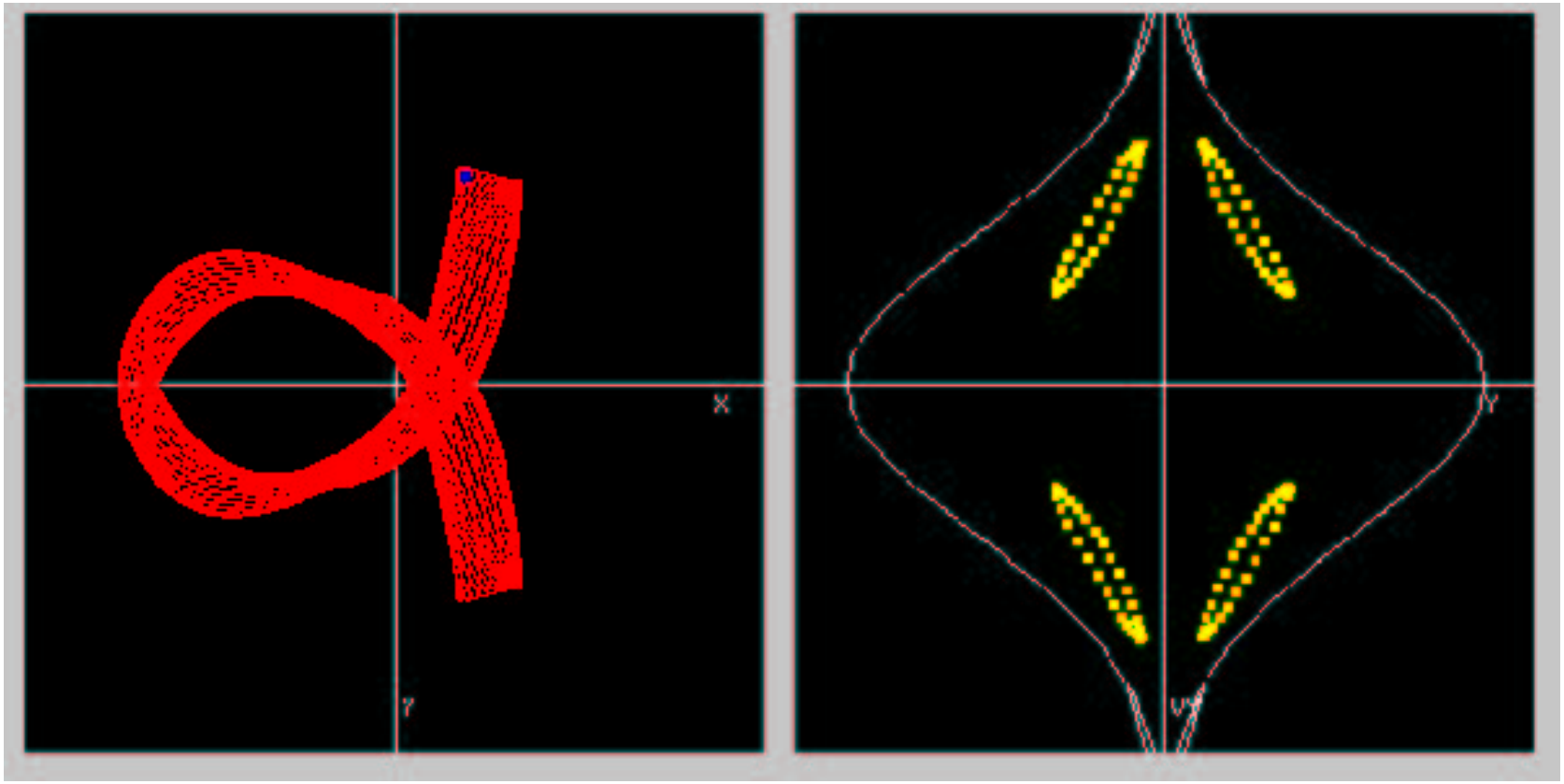
However, upon introducing a massive black hole in the center of the logarithmic potential, many of the box-orbits become stochastic. One says that the BH destroys the box-orbits. Recall that each box orbit comes arbitrarily close to the BH.

Orbits in Singular Logarithmic Potentials



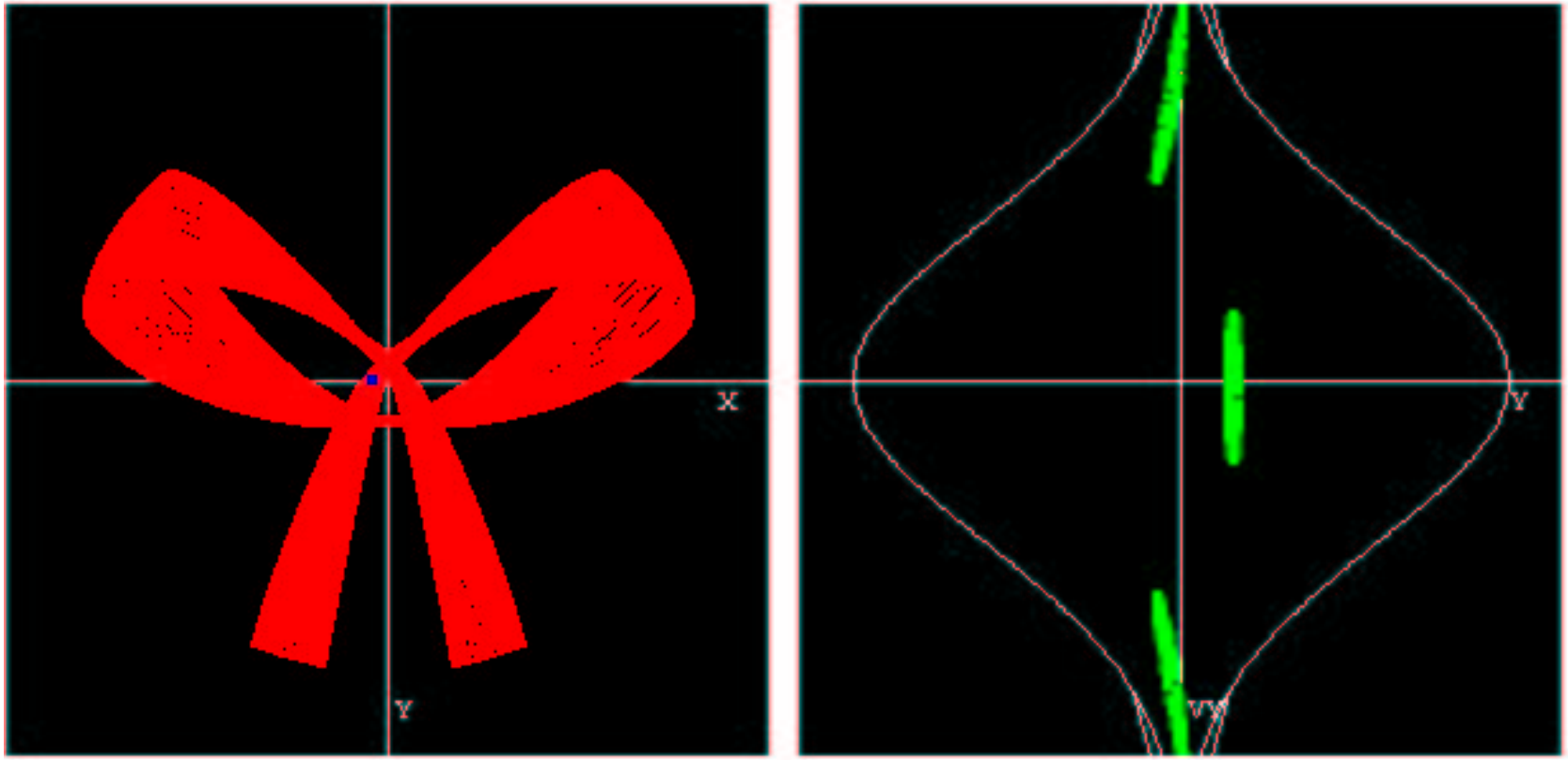
Here is an example of a banana-orbit (member of the $2:1$ resonance family).

Orbits in Singular Logarithmic Potentials



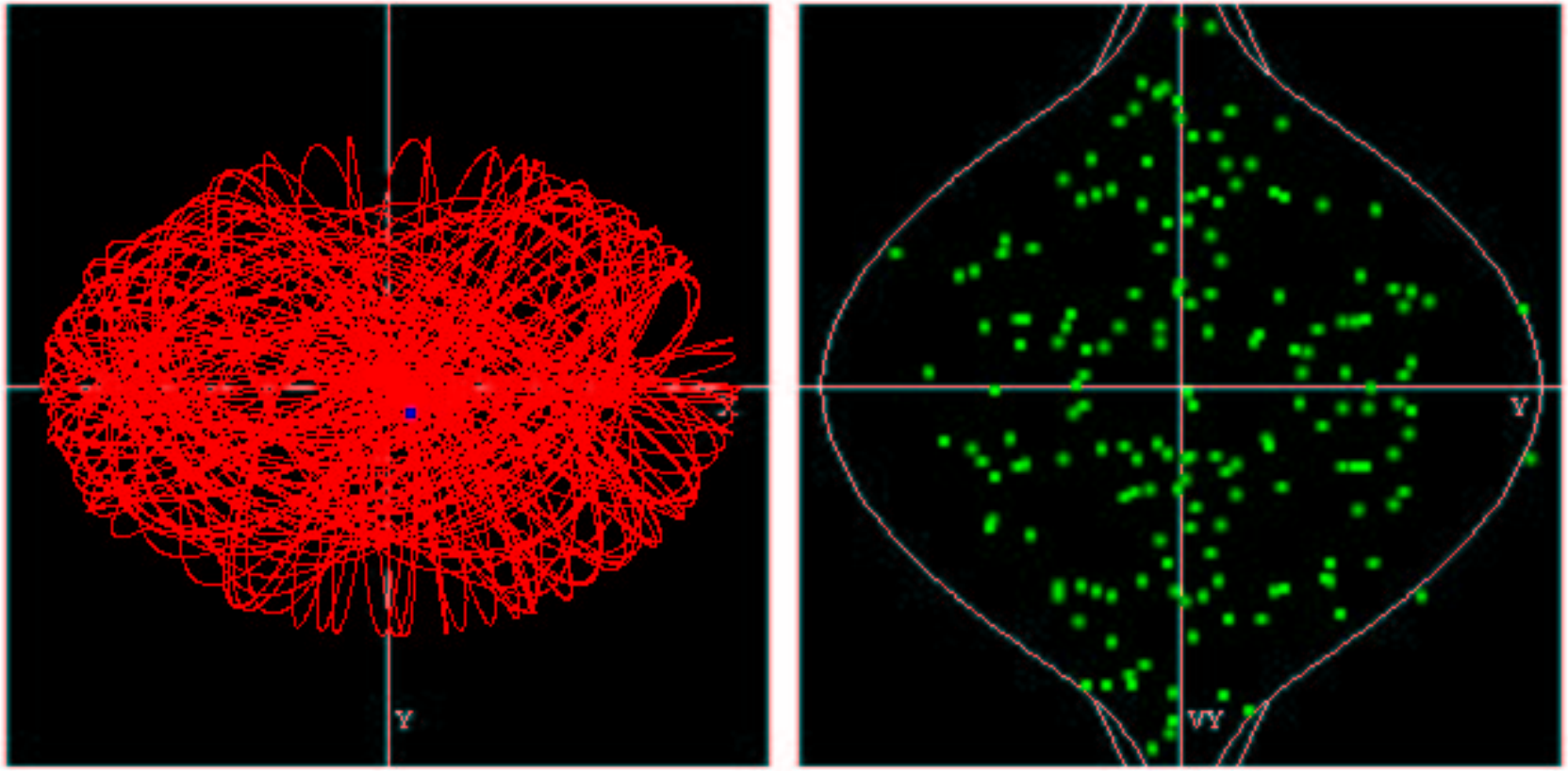
Here is an example of a fish-orbit (member of the $3:2$ resonance family).

Orbits in Singular Logarithmic Potentials



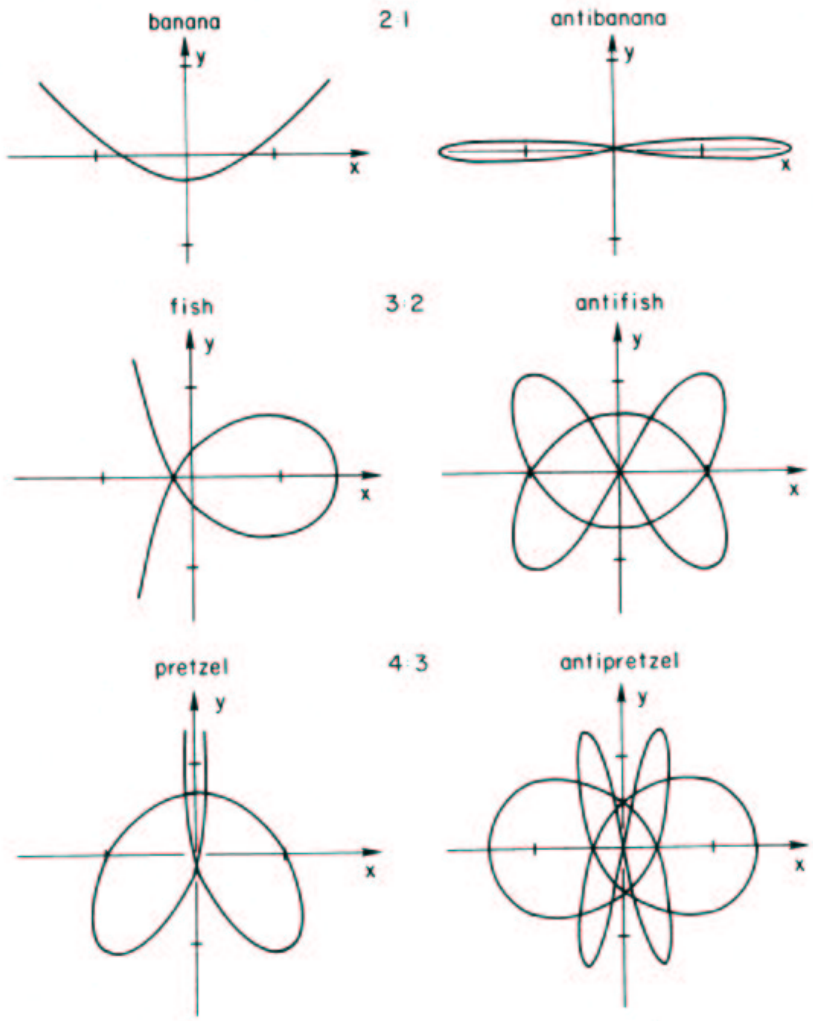
Here is an example of a pretzel-orbit (member of the $4:3$ resonance family).

Orbits in Logarithmic Potentials with BH



Here is an example of a stochastic orbit in a (cored) logarithmic potential with a central black hole.

Centrophobic versus Centrophilic



The closed boxlets comes in two kinds: centrophobic, which avoid the center and are stable, and centrophilic, which go through the center and are unstable. The centrophilic versions of the banana and fish orbits are called the antibanana and antifish orbits, etc. Because they are unstable, they don't parent any families.

Orbits in Planar Potentials: Summary

If $\Phi(x, y)$ has a homogeneous core, at sufficiently small E potential is indistinguishable from that of 2D harmonic oscillator

- **x-axial and y-axial closed orbits are stable**
- **they parent a family of box orbits (filling a rectangular box)**

At larger radii, for orbits with larger E:

- **y-axial orbit becomes unstable and bifurcates into two families of loop orbits (with opposite sense of rotation)**
- **close to this unstable, closed y-axial orbit a small layer of stochastic orbits is present**
- **x-axial orbit still stable, and parents family of box orbits**

If $\Phi(x, y)$ is scale-free and/or has central cusp that is sufficiently steep

- **x-axial orbit may become unstable. Its family of box orbits then becomes family of boxlets associated with (higher-order) resonances.**

If central BH is present, discrete scattering events can turn box orbits into stochastic orbits

Orbits in Axisymmetric Potentials I

Axisymmetric potentials (oblate or prolate) are far more realistic examples to consider in astronomy. Elliptical galaxies might well be spheroidal (but could also be ellipsoidal), while disk galaxies almost certainly are axisymmetric (though highly flattened).

For axisymmetric systems the coordinate system of choise are the cylindrical coordinates (R, ϕ, z), and $\Phi = \Phi(R, z)$.

Solving Newton's equation of motion in cylindrical coordinates yields:

$$\begin{aligned} \ddot{R} - R\dot{\phi}^2 &= -\frac{\partial \Phi}{\partial R} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(R^2\dot{\phi}\right) &= 0 \\ \ddot{z} &= -\frac{\partial \Phi}{\partial z} \end{aligned}$$

The second of these expresses conservation of the component of angular momentum about the z-axis; $L_z = R^2\dot{\phi}$, while the other two equations describe the coupled oscillations in the R and z-directions.

NOTE: For stars confined to equatorial plane $z = 0$, equations of motion are identical to that of motion in spherical density distribution (not surprising, since in this case the motion is once again central). Therefore, orbits confined to equatorial plane are rosette orbits.

Orbits in Axisymmetric Potentials II

As for the spherical case, we can reduce the equations of motion to

$$\ddot{R} = -\frac{\partial \Phi_{\text{eff}}}{\partial R} \qquad \ddot{z} = -\frac{\partial \Phi_{\text{eff}}}{\partial z}$$

with $\Phi_{\text{eff}}(R,z) = \Phi(R,z) + \frac{L_z^2}{2R^2}$ the **effective potential**. The L_z^2/R^2-term serves as a **centrifugal barrier**, only allowing orbits with $L_z = 0$ near the symmetry-axis.

This allows us to reduce the $3D$ motion to $2D$ motion in **Meridional Plane** (R,z), which rotates non-uniformly around the symmetry axis according to $\dot{\phi} = L_z/R^2$.

In addition to simplifying the problem, it also allows the use of **surfaces-of-section** to investigate the orbital properties.

For the energy we can write

$$E = \frac{1}{2}\left[\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2\right] + \Phi = \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \Phi_{\text{eff}}$$

so that the orbit is restricted to the area in the **meridional plane** satisfying $E \geq \Phi_{\text{eff}}$. The curve bounding this area is called the **zero-velocity curve (ZVC)** (since for a point on it $\vec{v} = 0$).

Epicycle Approximation I

We have defined the effective potential $\Phi_{\rm eff} = \Phi + \frac{L_z^2}{2R^2}$. This has a minimum at $(R, z) = (R_g, 0)$, where

$$\frac{\partial \Phi_{\rm eff}}{\partial R} = \frac{\partial \Phi}{\partial R} - \frac{L_z^2}{R^3} = 0$$

The radius $R = R_g$ corresponds to the radius of a circular orbit with energy

$E = \Phi(R_g, 0) + \frac{1}{2}v_c^2 = \Phi(R_g, 0) + \frac{L_z^2}{2R_g^2} = \Phi_{\rm eff}$.

If we define $x = R - R_g$ and expand $\Phi_{\rm eff}$ around the point $(x, y) = (0, 0)$ in a Taylor series we obtain

$$\begin{aligned}\Phi_{\rm eff} \quad = \quad & \Phi_{\rm eff}(R_g, 0) + (\Phi_x)x + (\Phi_y)y + (\Phi_{xy})xy + \tfrac{1}{2}(\Phi_{xx})x^2 + \\ & \tfrac{1}{2}(\Phi_{yy})y^2 + \mathcal{O}(xz^2) + \mathcal{O}(x^2z) + \text{etc}\end{aligned}$$

where

$$\Phi_x = \left(\frac{\partial \Phi_{\rm eff}}{\partial x}\right)_{(R_g,0)} \qquad \Phi_{xx} = \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial x^2}\right)_{(R_g,0)} \qquad \Phi_{xy} = \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial x \partial y}\right)_{(R_g,0)}$$

By definition of R_g, and by symmetry considerations, we have that

$$\Phi_x = \Phi_y = \Phi_{xy} = 0$$

Epicycle Approximation II

In the epicycle approximation only terms up to second order are considered: all terms of order xz^2, x^2z or higher are considered negligble. Defining

$$\kappa^2 \equiv \Phi_{xx} \qquad \nu^2 \equiv \Phi_{yy}$$

we thus have that, in the epicycle approximation,

$$\Phi_{\rm eff} = \Phi_{\rm eff}(R_g, 0) + \tfrac{1}{2}\kappa^2 x^2 + \tfrac{1}{2}\nu^2 y^2$$

so that the equations of motion in the meridional plane become

$$\ddot{x} = -\kappa^2\, x \qquad \ddot{y} = -\nu^2\, y$$

Thus, the x- and y-motions are simple harmonic oscillations with the epicycle frequency κ and the vertical frequency ν.

In addition, we have the circular frequency

$$\Omega(R) = \frac{v_c(R)}{R} = \sqrt{\frac{1}{R}\left(\frac{\partial \Phi}{\partial R}\right)_{(R,0)}} = \frac{L_z}{R^2}$$

which allows us to write

$$\kappa^2 = \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right)_{(R_g,0)} = \left(R\frac{{\rm d}\Omega^2}{{\rm d}R} + 4\Omega^2\right)_{R_g}$$

Epicycle Approximation III

As we have seen before, for a realistic galactic potential $\Omega < \kappa < 2\Omega$, where the limits correspond to the homogeneous mass distribution ($\kappa = 2\Omega$) and the Kepler potential ($\kappa = \Omega$)

In the epicycle approximation the motion is very simple:

$$\begin{aligned} R(t) &= A\cos(\kappa t + a) + R_g \\ z(t) &= B\cos(\nu t + b) \\ \phi(t) &= \Omega_g t + \phi_0 - \frac{2\Omega_g A}{\kappa R_g}\sin(\kappa t + a) \end{aligned}$$

with A, B, a , b, and ϕ_0 all constants. The ϕ-motion follows from

$$\dot{\phi} = \frac{L_z}{R^2} = \frac{L_z}{R_g^2}\left(1 + \frac{x}{R_g}\right)^{-2} \simeq \Omega_g\left(1 - \frac{2x}{R_g}\right)$$

Note that there are three frequencies (Ω, κ, ν) and also three isolating integrals of motion in involution: (E_R, E_z, L_z) with $E_R = \frac{1}{2}(\dot{x}^2 + \kappa^2 x^2)$ and $E_z = \frac{1}{2}(\dot{z}^2 + \nu^2 z^2)$ ▷ all orbits are regular.

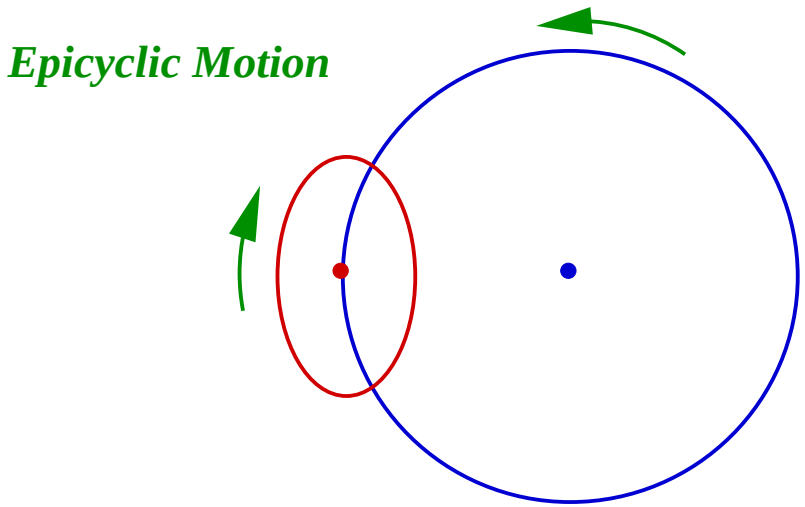
The motion in (R, ϕ) can be described as retrogate motion on an ellipse (the epicycle), whose guiding center (or epicenter) is in prograde motion around the center of the system.

Epicycle Approximation IV

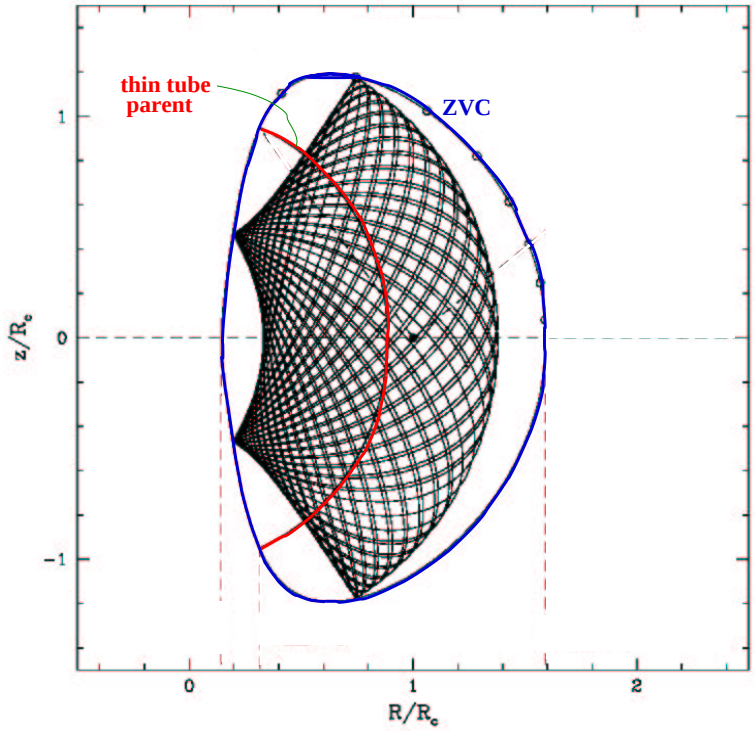
An important question is: "When is the epicycle approximation valid?"

First consider the z-motion: The equation of motion, $\ddot{z} = -\nu^2 z$ implies a constant density in the z-direction. Hence, the epicycle approximation is valid as long as $\rho(z)$ is roughly constant. This is only approximately true very close to equatorial plane. In general, however, epicycle approx. is poor for motion in z-direction.

In the radial direction, we have to realize that the Taylor expansion is only accurate sufficiently close to $R = R_g$. Hence, the epicycle approximation is only valid for small librations around the guiding center; i.e., for orbits with an angular momentum that is close to that of the corresponding circular orbit.



Orbits in Axisymmetric Potentials III

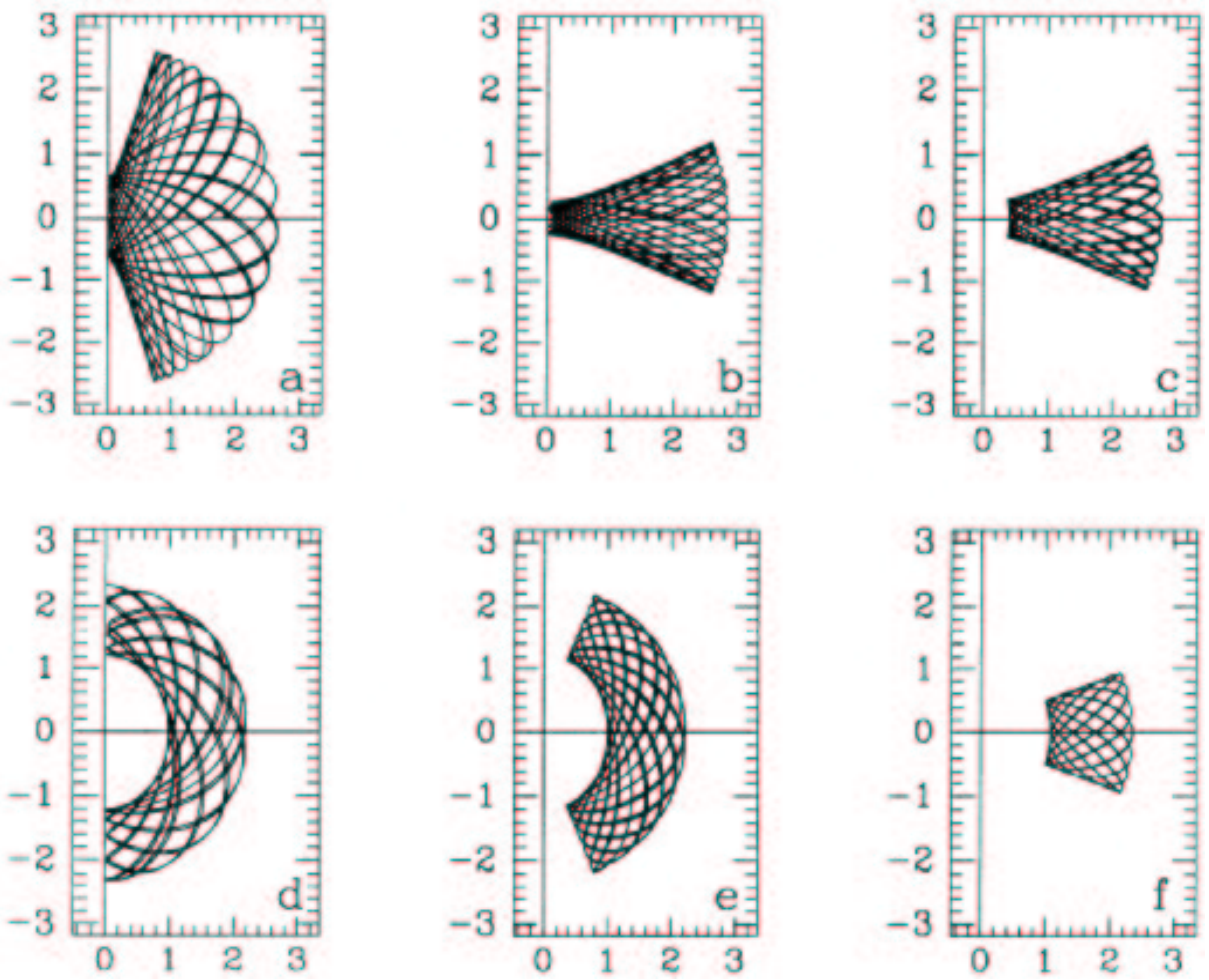


Typical orbit in axisymmetric potential. If orbit admits two isolating integrals of motion, it would (ultimately) fill entire area within ZVC. Rather, orbit is restricted to sub-area within ZVC, indicating that orbit admits a third isolating integral of motion.

Since this is not a classical integral of motion, and we don't know how to express it in terms of the phase-space coordinates, it is simply called I_3.

Note that the point where orbit touches ZVC can be used to 'label' I_3: The set (E, L_z, I_3) uniquely defines an orbit.

Orbits in Axisymmetric Potentials IV



The orbit shown on the previous page is a so-called short-axis tube orbit. This is the main orbit family in oblate potentials, and is associated with (parented by) the circular orbits in equatorial plane.

Orbits (c), (e) and (f) above are from the same orbit family. Orbits (a), (b) and (d) are special in that $L_z = 0$.

Orbits in Axisymmetric Potentials V

Because of the centrifugal barrier only orbits with $L_z = 0$ will be able to come arbitrarily close to the center.

However, not all orbits with $L_z = 0$ are box orbits. There is another family of zero-angular momentum orbits, namely the two-dimensional loop orbits (e.g., orbit (d) on previous page). Their meriodional plane is stationary (i.e., $\dot{\phi} = 0$) and their angular momentum vector is perpendicular to the z-axis. Hence, $I_3 = L$; note that $[L, L_z] = 0$.

Numerous authors have investigated orbits in axisymmetric potentials using numerical techniques. The main conclusions are:

- **Most orbits in axisymmetric potentials designed to model elliptical galaxies are regular and appear to respect an effective third integral I_3.**
- **The principal orbit family in oblate potentials is the short-axis tube family, while two families of inner and outer long-axis tube orbits dominate in prolate potentials.**
- **In scale-free or cusped potentials several minor orbits families become important. These are the (boxlets) associated with resonant parents.**
- **The fraction of phase-space occupied by stochastic, irregular orbits is generally (surprisingly) small.**

Orbits in Triaxial Potentials I

Consider a triaxial density distribution with the major, intermediate, and minor axes aligned with the x, y, and z axes, respectively.

In general, triaxial galaxies have four main orbit families: box orbits, and three tube orbits: short axis tubes, inner long-axis tubes, and outer long-axis tubes.

Orbit structure different in cusp, core, main body, and outer part (halo).

In central core, potential is harmonic, and motion is that of a 3D harmonic oscillator. ▷ all orbits are box orbits, parented by stable long-axial orbit

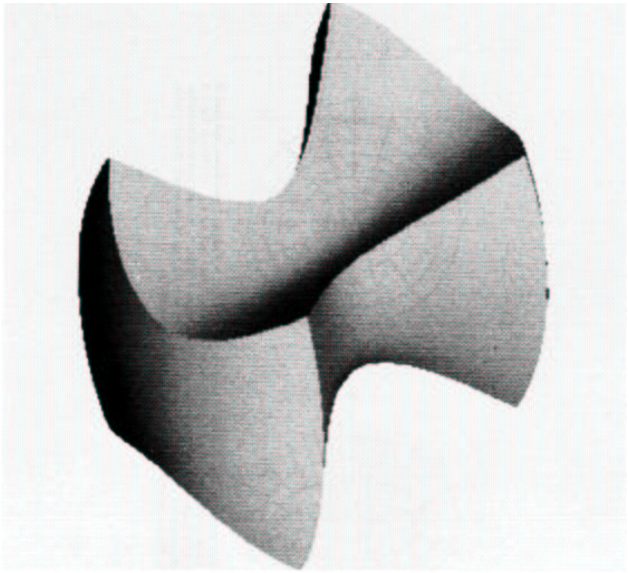
Outside of core region, frequencies become strongly radius (energy) dependent. There comes an energy where $\omega_x = \omega_y$. At this $1:1$-resonance the y-axial orbit becomes unstable and bifurcates into short-axis tube family (two subfamilies with opposite sense of rotation).

At even higher E the $\omega_y : \omega_z = 1:1$ resonance makes z-axial orbit unstable $\longrightarrow$ inner and outer long-axis tube families (each with two subfamilies with opposite sense of rotation).

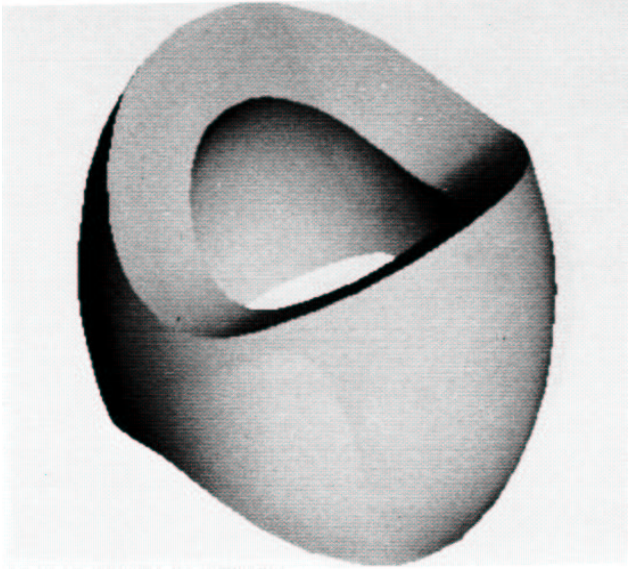
At even larger radii (in 'halo' of triaxial system) the x-axial orbit becomes unstable ▷ box orbits are replaced by boxlets and stochastic orbits. The three families of tube orbits are also present

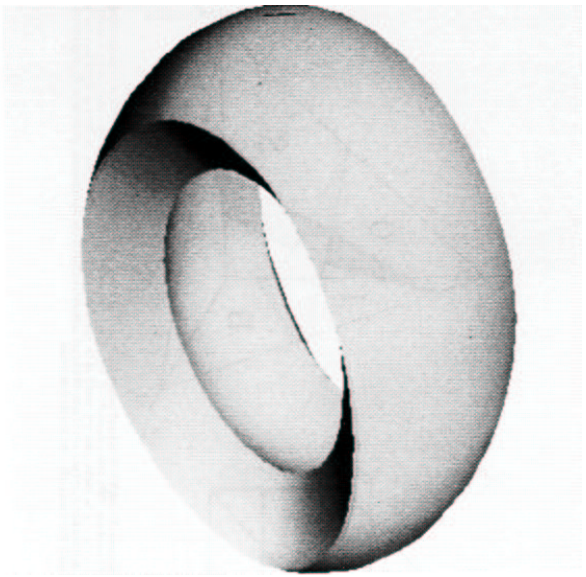
Orbits in Triaxial Potentials II

box orbit

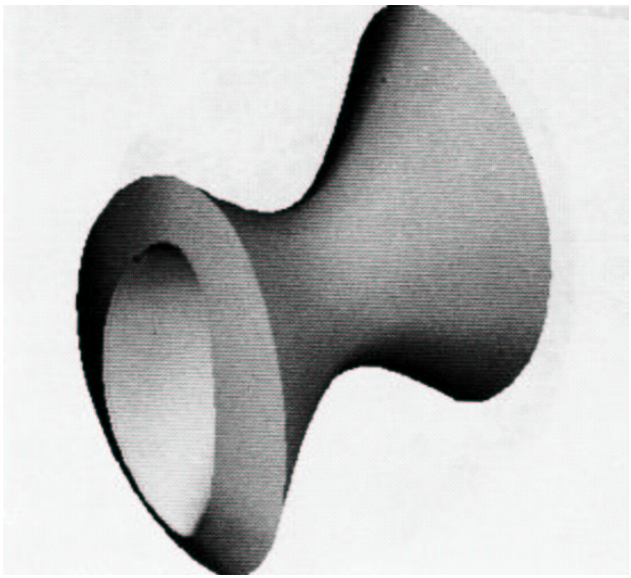


short−axis tube orbit





outer long−axis tube orbit



inner long−axis tube orbit

Orbits in Triaxial Potentials III

If center is cusped rather than cored, resonant orbits families (boxlets) and stochastic orbits take over part of phase-space formerly held by box orbits. The extent to which this happens depends on cusp slope.

Short-axis tubes contribute angular momentum in z-direction; Long-axis tubes contribute angular momentum in x-direction ▷ total angular momentum vector may point anywhere in plane containing long and short axes. NOTE: this can serve as kinematic signature of triaxiality.

The closed loop orbit around the intermediate y-axis is unstable ▷ no family of intermediate-axis tubes.

Gas moves on closed, non-intersecting orbits. The only orbits with these properties are the stable loop orbits around x- and z-axes. Consequently, gas and/or dust disks in triaxial galaxies can exist in xy-plane and yz-plane, but not in xz-plane. NOTE: these disks must be ellipsoidal rather than circular, and the velocity varies along ellipsoids.

Stäckel Potentials I

Useful insight may be obtained from separable Stäckel models. These are the only known triaxial potentials that are completely integrable.

In Stäckel potentials all orbits are regular and part of one of the four main families.

Stäckel potentials are separable in ellipsoidal coordinates (λ, μ, ν)

(λ, μ, ν) are the roots for τ of

$$\frac{x^2}{\tau+\alpha} + \frac{y^2}{\tau+\beta} + \frac{z^2}{\tau+\gamma} = 1$$

Here $\alpha < \beta < \gamma$ are constants and $-\gamma \leq \nu \leq -\beta \leq \mu \leq -\alpha \leq \lambda$

Surfaces of constant λ are ellipsoids

Surfaces of constant μ are hyperboloids of one sheet

Surfaces of constant ν are hyperboloids of two sheets

Stäckel potentials are of the form:

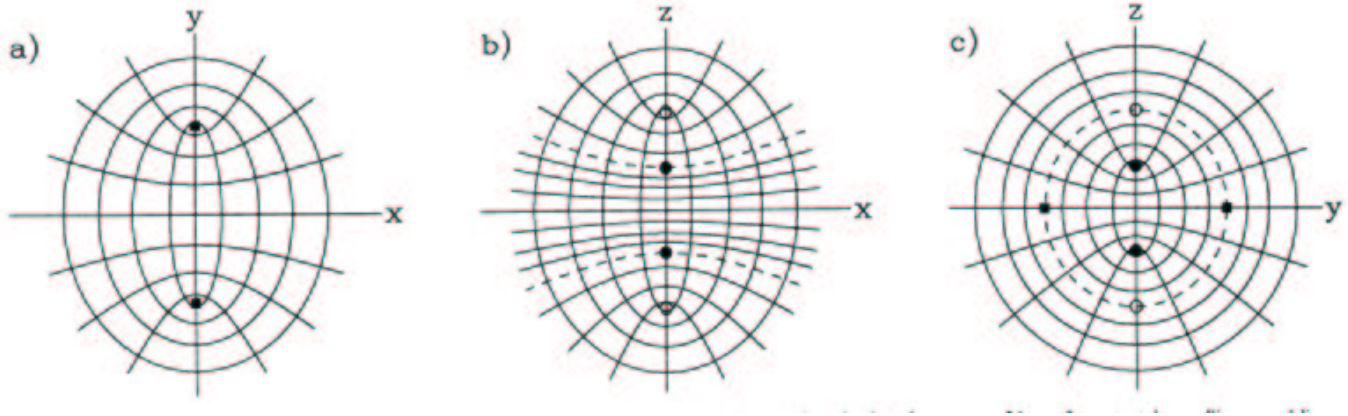
$$\Phi(\vec{r}) = \Phi(\lambda, \mu, \nu) = -\frac{F_1(\lambda)}{(\lambda-\mu)(\lambda-\nu)} - \frac{F_2(\mu)}{(\mu-\nu)(\mu-\lambda)} - \frac{F_3(\nu)}{(\nu-\lambda)(\nu-\mu)}$$

with F_1, F_2 and F_3 arbitrary functions

Stäckel Potentials II

The figure below shows contours of constant (λ, μ, ν) plotted in the three planes (from left to right) xy, xz and yz

At large distances, the ellipsoidal coordinates become close to spherical. Near the origin they are close to cartesian. For more details, see de Zeeuw, 1985, MNRAS, 216, 273

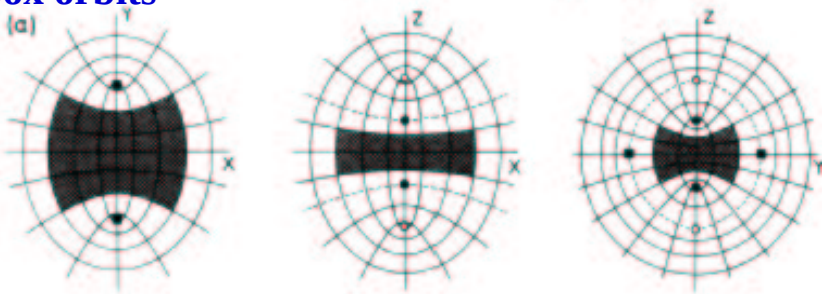


In triaxial Stäckel potentials all three integrals (E, I_2, I_3) are analytical, and the orbits are confined by contours of constant ellipsoidal coordinates (see next page).

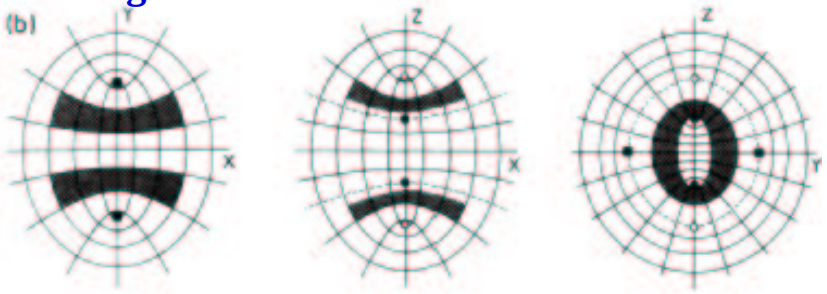
Although Stäckel are a very special class, the fact that they are separable makes them ideally suited to get insight. Most triaxial potentials that do not have a Stäckel form have orbital structures that are similar to that of Stäckel potentials.

Stäckel Potentials III

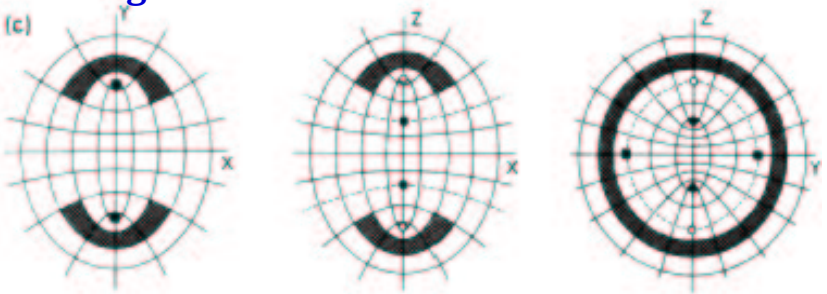
box orbits



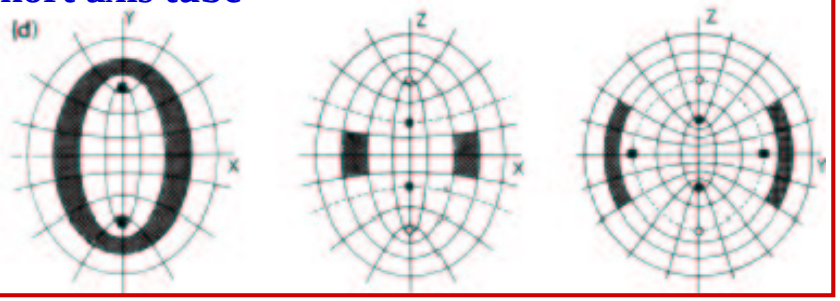
inner long axis tube



outer long axis tube



short axis tube



Orbits in Ellipsoid Land; Summary

System	Dim	Orbit Families
Oblate	$3D$	S
Prolate	$3D$	$I + O$
Triaxial	$3D$	$S + I + O + B$
Elliptic Disk	$2D$	$S + B$

$B =$ **box orbits**

$S =$ **short-axis tubes**

$I =$ **inner long-axis tubes**

$O =$ **outer long-axis tubes**

Libration versus Rotation

Three-dimensional orbits

All tube orbits are build up from 2 librations and 1 rotation.

All box orbits are build up from 3 librations.

All boxlets are build up from 2 librations and 1 rotation.

Two-dimensional orbits

All loop orbits are build up from 1 libration and 1 rotation.

All box orbits are build up from 2 librations.

All boxlets are build up from 2 librations.

Rotating Potentials I

The figures of non-axisymmetric potentials may rotate with respect to inertial space.

The example of interest for astronomy are barred potentials, which are rotating with a certain pattern speed.

We express the pattern speed in angular velocity $\vec{\Omega}_p = \Omega_p \vec{e}_z$

In what follows we denote by $\mathrm{d}\vec{a}/\mathrm{d}t$ **the rate of change of a vector** $\vec{a}$ **as measured by an inertial observer, and by** $\dot{\vec{a}}$ **the rate of change as measured by an observer corotating with the figure.**

It is straightforward to show that

$$\boxed{\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \dot{\vec{a}} + \vec{\Omega}_p \times \vec{a}}$$

Applying this twice to the position vector $\vec{r}$**, we obtain**

$$\begin{aligned}
\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}} + \vec{\Omega}_p \times \vec{r}\right) \\
&= \ddot{\vec{r}} + \vec{\Omega}_p \times \dot{\vec{r}} + \vec{\Omega}_p \times \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \\
&= \ddot{\vec{r}} + \vec{\Omega}_p \times \dot{\vec{r}} + \vec{\Omega}_p \times \left(\dot{\vec{r}} + \vec{\Omega}_p \times \vec{r}\right) \\
&= \ddot{\vec{r}} + 2\left(\vec{\Omega}_p \times \dot{\vec{r}}\right) + \vec{\Omega}_p \times \left(\vec{\Omega}_p \times \vec{r}\right)
\end{aligned}$$

Rotating Potentials II

Since Newton's laws apply to inertial frames we have that

$$\ddot{\vec{r}} = -\vec{\nabla}\Phi - 2\left(\vec{\Omega}_p \times \dot{\vec{r}}\right) - \vec{\Omega}_p \times \left(\vec{\Omega}_p \times \vec{r}\right)$$

Note the two extra terms: $-2\left(\vec{\Omega}_p \times \dot{\vec{r}}\right)$ **represents the Coriolis force and** $-\vec{\Omega}_p \times \left(\vec{\Omega}_p \times \vec{r}\right)$ **the centrifugal force.**

The energy is given by

$$\begin{aligned} E &= \tfrac{1}{2}\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)^2 + \Phi(\vec{r}) \\ &= \tfrac{1}{2}\left(\dot{\vec{r}} + \vec{\Omega}_p \times \vec{r}\right)^2 + \Phi(\vec{r}) \\ &= \tfrac{1}{2}\dot{\vec{r}}^2 + \dot{\vec{r}} \cdot \left(\vec{\Omega}_p \times \vec{r}\right) + \tfrac{1}{2}\left(\vec{\Omega}_p \times \vec{r}\right)^2 + \Phi(\vec{r}) \\ &= \tfrac{1}{2}\dot{\vec{r}}^2 + \dot{\vec{r}} \cdot \left(\vec{\Omega}_p \times \vec{r}\right) + \tfrac{1}{2}|\vec{\Omega}_p \times \vec{r}|^2 + \Phi(\vec{r}) \\ &= E_J + \dot{\vec{r}} \cdot \left(\vec{\Omega}_p \times \vec{r}\right) + |\vec{\Omega}_p \times \vec{r}|^2 \end{aligned}$$

Where we have defined Jacobi's Integral

$$E_J \equiv \tfrac{1}{2}\dot{\vec{r}}^2 + \Phi(\vec{r}) - \tfrac{1}{2}|\vec{\Omega}_p \times \vec{r}|^2$$

Rotating Potentials III

The importance of E_J becomes apparent from the following:

$$\begin{aligned}\frac{\mathrm{d}E_J}{\mathrm{d}t} &= \dot{\vec{r}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}}\right) + \frac{\mathrm{d}\Phi}{\mathrm{d}t} - (\vec{\Omega}_p \times \vec{r}) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\Omega}_p \times \vec{r}) \\ &= \dot{\vec{r}}\left[\ddot{\vec{r}} + (\vec{\Omega}_p \times \dot{\vec{r}})\right] + \vec{\nabla}\Phi \cdot \dot{\vec{r}} - (\vec{\Omega}_p \times \vec{r}) \cdot (\vec{\Omega}_p \times \dot{\vec{r}}) \\ &= \dot{\vec{r}} \cdot \ddot{\vec{r}} + \dot{\vec{r}} \cdot \vec{\nabla}\Phi - (\vec{\Omega}_p \times \vec{r}) \cdot (\vec{\Omega}_p \times \dot{\vec{r}})\end{aligned}$$

Here we have used that $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$ and that $\mathrm{d}\vec{\Omega}_p/\mathrm{d}t = 0$. If we multiply the equation of motion with $\dot{\vec{r}}$ we obtain that

$$\begin{aligned}&\dot{\vec{r}} \cdot \ddot{\vec{r}} + \dot{\vec{r}} \cdot \vec{\nabla}\Phi + 2\dot{\vec{r}} \cdot (\vec{\Omega}_p \times \dot{\vec{r}}) + \dot{\vec{r}} \cdot \left[\vec{\Omega}_p \times (\vec{\Omega}_p \times \vec{r})\right] = 0 \\ \Leftrightarrow\quad &\dot{\vec{r}} \cdot \ddot{\vec{r}} + \dot{\vec{r}} \cdot \vec{\nabla}\Phi + (\vec{\Omega}_p \times \vec{r}) \cdot (\dot{\vec{r}} \times \vec{\Omega}_p) = 0\end{aligned}$$

Where we have used that $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$. Since $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ we have that

$$\boxed{\frac{\mathrm{d}E_J}{\mathrm{d}t} = 0}$$

The Jacobi Integral is a conserved quantity, i.e. an integral of motion.

Rotating Potentials IV

For comparison, since $\Phi = \Phi(t)$ the Hamiltonian is explictely time-dependent; consequently, the total energy E is not a conserved quantity (i.e., is not an integral of motion).

The angular momentum is given by

$$\vec{L} = \vec{r} \times \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{r} \times \dot{\vec{r}} + \vec{r} \times (\vec{\Omega}_p \times \vec{r})$$

This allows us to write

$$\begin{aligned} \vec{\Omega}_p \cdot \vec{L} &= \vec{\Omega}_p \cdot (\vec{r} \times \dot{\vec{r}}) + \vec{\Omega}_p \cdot \left[\vec{r} \times (\vec{\Omega}_p \times \vec{r})\right] \\ &= \dot{\vec{r}} \cdot (\vec{\Omega}_p \times \vec{r}) + |\vec{\Omega}_p \times \vec{r}|^2 \end{aligned}$$

from which we obtain that

$$\boxed{E_J = E - \vec{\Omega}_p \cdot \vec{L}}$$

Thus, in a rotating, non-axisymmetric potential neither energy E not angular momentum L are conserved, but the Jacobi integral $E_J = E - \vec{\Omega}_p \cdot \vec{L}$ is.

Note that E_J is the sum of $\frac{1}{2}\dot{\vec{r}}^2 + \Phi$, which would be the energy if the frame were not rotating, and the quantity $-\frac{1}{2}|\vec{\Omega}_p \times \vec{r}|^2 = -\frac{1}{2}\Omega_p^2 R^2$, which can be thought of as the potential energy corresponding to the centrifugal force.

Rotating Potentials V

If we now define the effective potential

$$\Phi_{\rm eff} = \Phi - \tfrac{1}{2}\Omega_p^2 R^2$$

the equation of motion becomes

$$\ddot{\vec{r}} = -\vec{\nabla}\Phi_{\rm eff} - 2(\vec{\Omega}_b \times \dot{\vec{r}})$$

and the Jacobi integral is $E_J = \frac{1}{2}|\dot{\vec{r}}|^2 + \Phi_{\rm eff}$

An orbit with a given value for it's Jacobi Integral is restricted in its motion to regions in which $E_J \leq \Phi_{\rm eff}$. The surface $\Phi_{\rm eff} = E_J$ is therefore often called the zero-velocity surface.

The effective potential has five points at which both $\partial\Phi_{\rm eff}/\partial x$ and $\partial\Phi_{\rm eff}/\partial y$ vanish. These points, L_1 to L_5, are called the Lagrange Points (cf. restricted three-body problem).

- Motion around L_3 (minimum of $\Phi_{\rm eff}$) always stable.
- Motion around L_1 and L_2 (saddle points of $\Phi_{\rm eff}$) always unstable.
- Motion around L_4 and L_5 (maxima of $\Phi_{\rm eff}$) can be stable or unstable depending on potential.

NOTE: stable/unstable refers to whether orbits remain close to Lagrange points or not.

Lagrange Points

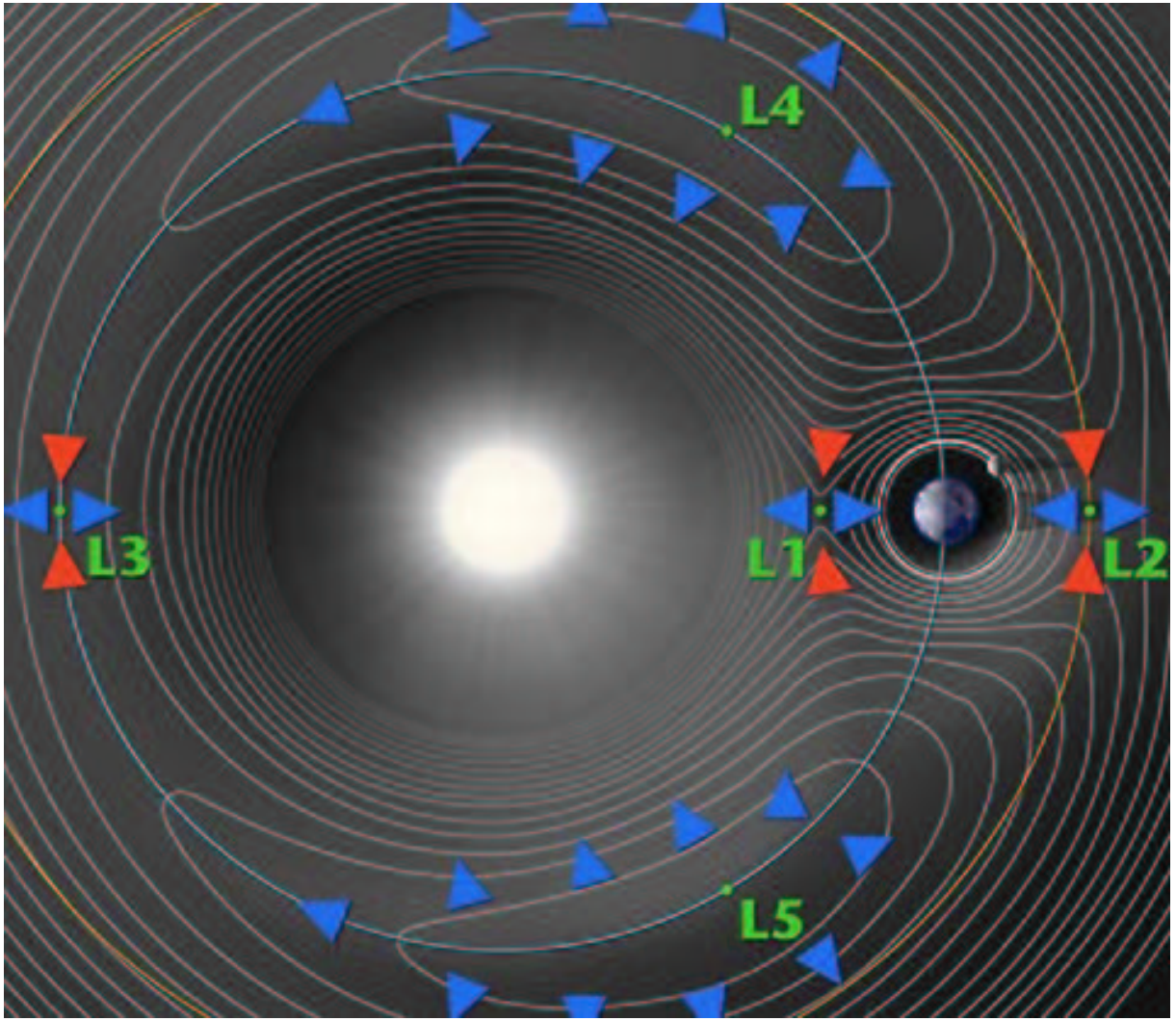


Illustration of Lagrange points (L_1 to L_5) in Sun-Earth-Moon system.

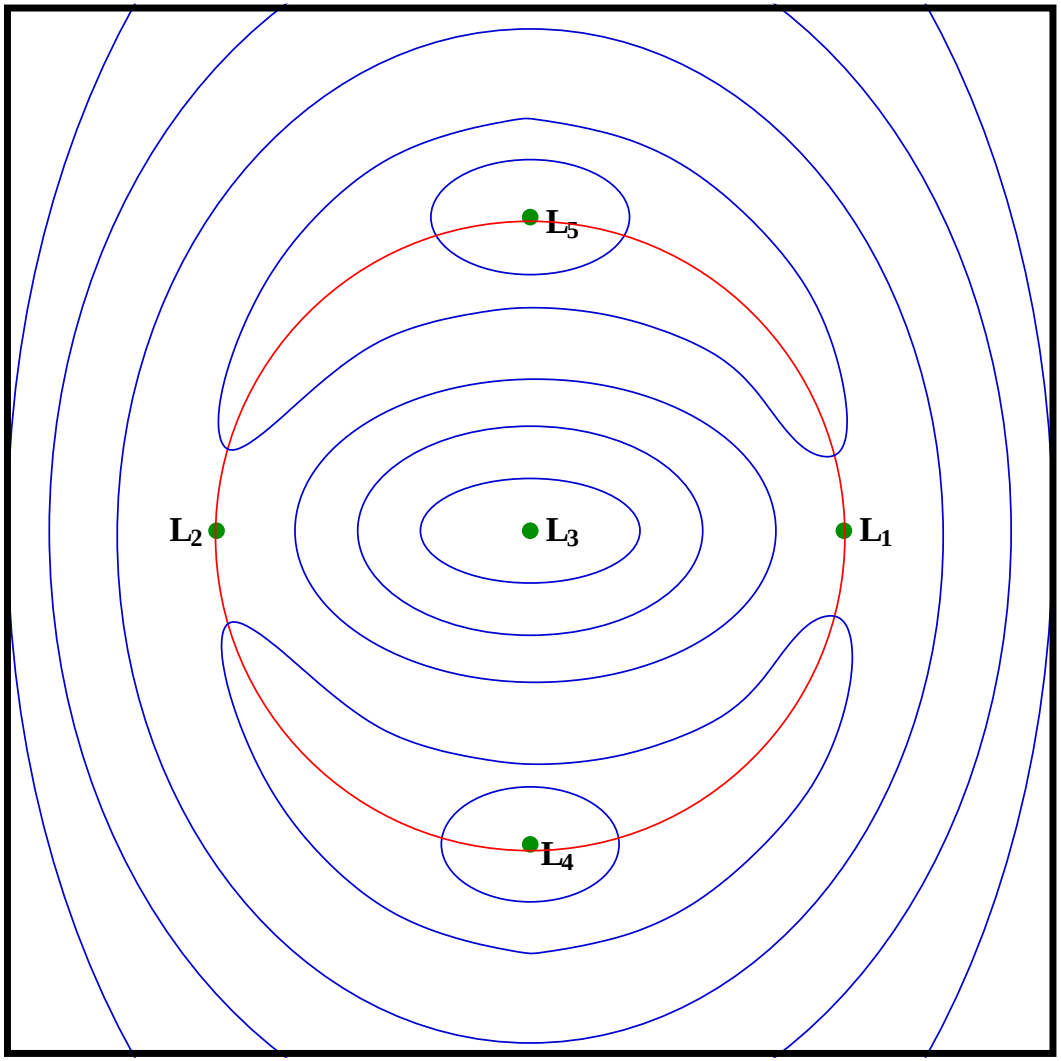


Illustration of Lagrange points (L_1 to L_5) in logarithmic potential. The annulus bounded by circles through L_1, L_2 and L_3, L_4 (depicted as red circle) is called the region of corotation.

Lindblad Resonances I

Let (R, θ) be the polar coordinates that are corotating with the planar potential $\Phi(R, \theta)$. If the non-axisymmetric distortions of the potential, which has a pattern speed Ω_p, is sufficiently small then we may write

$$\Phi(R, \theta) = \Phi_0(R) + \Phi_1(R, \theta) \qquad |\Phi_1/\Phi_0| \ll 1$$

It is useful to consider the following form for Φ_1

$$\Phi_1(R, \theta) = \Phi_p(R) \cos(m\theta)$$

where $m = 2$ corresponds to a (weak) bar.

In the epicycle approximation the motion in $\Phi_0(R)$ is that of an epicycle, with frequency $\kappa(R)$, around a guiding center which rotates with frequency $\Omega(R) = \sqrt{\frac{1}{R}\frac{\mathrm{d}\Phi_0}{\mathrm{d}R}}$.

In presence of $\Phi_1(R, \theta)$, movement of guiding center is $\theta_0(t) = [\Omega(R) - \Omega_p]\, t$. In addition to natural frequencies $\Omega(R)$ and $\kappa(R)$ there is new frequency Ω_p. Because $\Phi_1(R, \theta)$ has m-fold symmetry, guiding center at R finds itself at effectively same location in (R, θ)-plane with frequency $m\,[\Omega(R) - \Omega_p]$.

Lindblad Resonances II

Motion in R-direction becomes that of harmonic oscillator of natural frequency $\kappa(R)$ that is driven by frequency $m\,[\Omega(R) - \Omega_p]$.

At several R the natural and driving frequencies are in resonance.

(1) Corotation: $\Omega(R) = \Omega_p$

(Guiding center corotates with potential).

(2) Lindblad Resonances: $m\,[\Omega(R) - \Omega_p] = \pm\kappa(R)$

Most important of these are:

$\Omega(R) - \frac{\kappa}{2} = \Omega_p$: **Inner Lindblad Resonance**

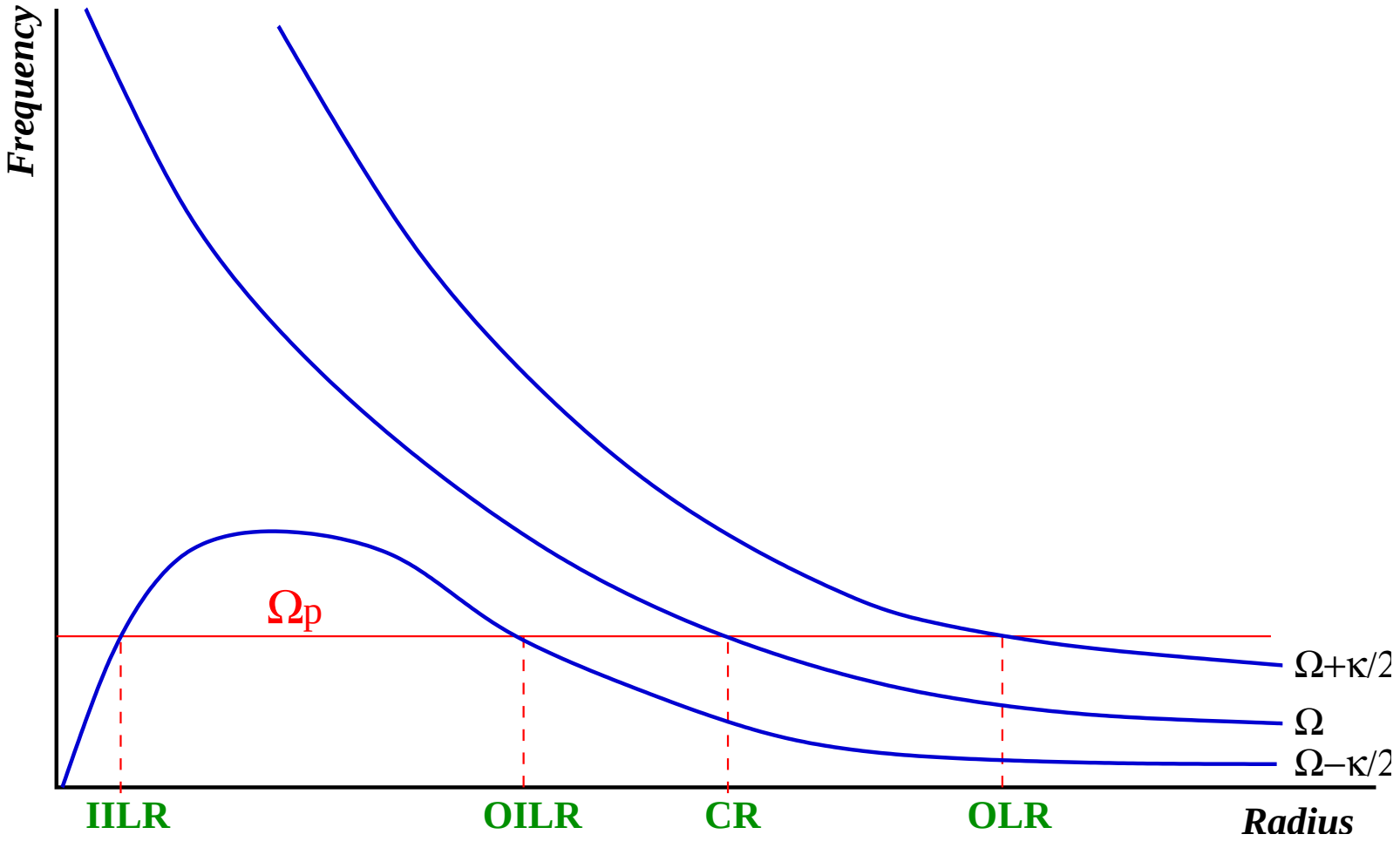
$\Omega(R) + \frac{\kappa}{2} = \Omega_p$: **Outer Lindblad Resonance**

$\Omega(R) - \frac{\kappa}{4} = \Omega_p$: **Ultra Harmonic Resonance**

Depending on $\Phi(R, \theta)$ and Ω_p one can have 0, 1, or 2 ILRs. If there are two, we distinguish between Inner Inner Lindblad Resonance (IILR) and Outer Inner Lindblad Resonance (OILR).

If cusp (or BH) is present there is always 1 ILR, because $\Omega(R) - \kappa(R)/2$ increases monotically with decreasing R.

Lindblad Resonances III



Lindblad Resonances play important role for orbits in barred potentials.

Lindblad Resonances IV

As an example, we discus the orbital families in a planar, rotating, logarithmic potential

(a) Long-axial orbit $\longrightarrow$ stable, oval, prograde, and oriented $\|$ to $\Phi_{\rm eff}$. (x_1-family).

(b) Short-axial orbit $\longrightarrow$ stable, oval, retrograde, and oriented $\perp$ to $\Phi_{\rm eff}$.

At $E > E_1$ (at IILR), family (b) becomes unstable and bifurcates into two prograde loop families that are oriented perpendicular to $\Phi_{\rm eff}$. The stable (unstable) family is called the x_2 (x_3) family. At the same energy the x_1-orbits develop self-intersecting loops.

At $E > E_2$ (at OILR) the x_2 and x_3 families dissapear. The x_1 family looses its self-intersecting loops.

In vicinity of corotation annulus there are families of orbits around L_4 and L_5 (if these are stable).

At large radii beyond CR $\Omega_p \gg \Omega(R)$. Consequently, the orbits effectively see a circular potential and the orbits become close to circular rosettes.

The Distribution Function

As we have seen before the distribution function (or phase-space density) $f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{v}$ gives a full description of the state of any collisionless system.

Here $f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{v}$ specifies the number of stars having positions in the small volume $\mathrm{d}^3\vec{x}$ centered on $\vec{x}$ and velocities in the small range $\mathrm{d}^3\vec{v}$ centered on $\vec{v}$, or, when properly normalized, expresses the probability that a star is located in $\mathrm{d}^3\vec{x}\mathrm{d}^3\vec{v}$.

Define the 6-dimensional phase-space vector

$$\vec{w} = (\vec{x},\vec{v}) = (w_1, w_2, ..., w_6)$$

The velocity of the flow in phase-space is then

$$\dot{\vec{w}} = (\dot{\vec{x}},\dot{\vec{v}}) = (\vec{v}, -\vec{\nabla}\Phi)$$

Note that $\dot{\vec{w}}$ has the same relationship to $\vec{w}$ as the 3D fluid flow velocity $\vec{u} = \dot{\vec{x}}$ has to $\vec{x}$ in an ordinary fluid.

In the absence of collisions (long-range, short-range, or direct) and under the assumption that stars are neither created nor destroyed, the flow in phase-space must conserve mass.

The Continuity Equation

Consider an ordinary fluid in an arbitrary closed volume V bounded by a surface S.

The mass of fluid within V is $M(t) = \int_V \rho(\vec{x}, t) \mathrm{d}^3\vec{x}$ and

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_V \left(\frac{\partial \rho}{\partial t}\right) \mathrm{d}^3\vec{x}$$

The mass flowing out of V through an area element $\mathrm{d}^2 S$ per unit time is given by $\rho\vec{v} \cdot \mathrm{d}^2\vec{S}$ with $\mathrm{d}^2\vec{S}$ an outward pointing vector normal to the surface S. Thus

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\int_S \rho\, \vec{v} \cdot \mathrm{d}^2\vec{S}$$

so that we obtain

$$\int_V \frac{\partial \rho}{\partial t}\, \mathrm{d}^3\vec{x} + \int_S \rho\, \vec{v} \cdot \mathrm{d}^2\vec{S} = 0$$

Using the divergence theorem $\int_V \vec{\nabla} \cdot \vec{F} \mathrm{d}^3\vec{x} = \int_S \vec{F} \cdot \mathrm{d}^2\vec{S}$ we obtain

$$\int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v})\right] \mathrm{d}^3\vec{x} = 0$$

Since this must hold for any volume V we obtain the continuity equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v}) = \frac{\partial \rho}{\partial t} + \rho\, \vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{\nabla}\rho = 0}$$

The Collisionless Boltzmann Equation I

Similarly, for our 6D flow in phase-space the continuity equation is given by

$$\boxed{\frac{\partial f}{\partial t} + \vec{\nabla} \cdot (f \dot{\vec{w}}) = 0}$$

and is called the Collisionless Boltzmann Equation (hereafter CBE).

To simplify this equation we first write out the second term:

$$\vec{\nabla} \cdot (f \dot{\vec{w}}) = \sum_{i=1}^{6} \frac{\partial (f \dot{w}_i)}{\partial w_i} = f \sum_{i=1}^{3} \left[\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right] + \sum_{i=1}^{3} \left[v_i \frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i} \right]$$

Since $\partial v_i / \partial x_i = 0$ (x_i and v_i are independent phase-space coordinates) and $\partial \dot{v}_i / \partial v_i = \frac{\partial}{\partial v_i} \left(-\frac{\partial \Phi}{\partial x_i} \right) = 0$ (gradient of potential does not depend on velocities), we may (using summation convention rewrite the CBE as

$$\boxed{\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0}$$

or in vector notation

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f - \vec{\nabla} \Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0}$$

The Collisionless Boltzmann Equation II

Note: Since $f = f(\vec{x}, \vec{v}, t)$ we have that

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x_i}\mathrm{d}x_i + \frac{\partial f}{\partial v_i}\mathrm{d}v_i$$

and thus

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}v_i - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}$$

Using this we can write the CBE in its compact form:

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = 0}$$

$\mathrm{d}f/\mathrm{d}t$ expresses the Lagrangian derivative along trajectories through phase-space, and the CBE expresses that this flow is incompressible. In other words, the phase-space density f around the phase-point of a give star always remains the same.

In the presence of collisions it is no longer true that $\dot{\vec{v}} = -\vec{\nabla}\Phi$, and the CBE no longer holds. Rather, collisions result in an additional collision term:

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \Gamma(t)}$$

This equation is called the Master Equation. If $\Gamma(t)$ describes long-range collisions only, then we call it the Fokker-Planck Equation.

Coarse-Grained Distribution Function

We defined the DF as the phase-space density of stars in a volume $\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{v}$. However, in our assumption of a smooth $\rho(\vec{r})$ and $\Phi(\vec{r})$, the only meaningful interpretation of the DF is that of a probability density.

Note that this probability density is also well defined in the discrete case, even though it may vary rapidly. Since it has an infinitely high resolution, it is often called the fine-grained DF.

Just as the wave-functions in quantum mechanics, the fine-grained DF is not measurable. However, we can use it to compute the expectation value of any phase-space function $Q(\vec{x},\vec{v})$.

A measurable DF, one that is actually related to counting objects in a given phase-space volume, is the so-called coarse-grained DF, $\bar{f}$, defined as the average value of the fine-grained DF, f, in some specified small volume:

$$\bar{f}(\vec{x}_0,\vec{v}_0) = \int\int w(\vec{x}-\vec{x}_0,\vec{v}-\vec{v}_0)\, f(\vec{x},\vec{v})\,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{v}$$

with $w(\vec{x},\vec{v})$ some (properly normalized) kernel which rapidly falls to zero for $|\vec{x}| > \epsilon_x$ and $|\vec{v}| > \epsilon_v$.

NOTE: the fine-grained DF does satisfy the CBE.
the coarse-grained DF does not satisfy the CBE.

Moment Equations I

Although the CBE looks very simple ($\mathrm{d}f/\mathrm{d}t = 0$), solving it for the DF is virtually impossible. It is more practical to consider moment equations.

The resulting Stellar-Hydrodynamics Equations are obtained by multiplying the CBE by powers of velocity and then integrating over all of velocity space.

Consider moment equations related to $v_i^l v_j^m v_k^n$ where the indices (i, j, k) refer to one of the three generalized coordinates, and (l, m, n) are integers.

Recall that

$$\rho = \int f \, \mathrm{d}^3\vec{v} \qquad\qquad \rho\langle v_i^l v_j^m v_k^n \rangle = \int v_i^l v_j^m v_k^n \, f \, \mathrm{d}^3\vec{v}$$

The $(l+m+n)^{\mathrm{th}}$ moment equation of the CBE is

$$\int v_i^l v_j^m v_k^n \frac{\partial f}{\partial t} \mathrm{d}^3\vec{v} + \int v_i^l v_j^m v_k^n \, v_a \frac{\partial f}{\partial x_a} \mathrm{d}^3\vec{v} - \int v_i^l v_j^m v_k^n \frac{\partial \Phi}{\partial x_a} \frac{\partial f}{\partial v_a} \mathrm{d}^3\vec{v} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t} \int v_i^l v_j^m v_k^n \, f \, \mathrm{d}^3\vec{v} + \frac{\partial}{\partial x_i} \int v_i^l v_j^m v_k^n \, v_a f \mathrm{d}^3\vec{v} - \frac{\partial \Phi}{\partial x_a} \int v_i^l v_j^m v_k^n \frac{\partial f}{\partial v_a} \mathrm{d}^3\vec{v} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t} \left[\rho \langle v_i^l v_j^m v_k^n \rangle \right] + \frac{\partial}{\partial x_i} \left[\rho \langle v_i^l v_j^m v_k^n v_a \rangle \right] - \frac{\partial \Phi}{\partial x_a} \int v_i^l v_j^m v_k^n \frac{\partial f}{\partial v_a} \mathrm{d}^3\vec{v} = 0$$

1^{st} term: integration range doesn't depend on t so $\frac{\partial}{\partial t}$ may be taken outside

2^{nd} term: $\frac{\partial}{\partial x_i}$ doesn't depend on v_i, so derivative may be taken outside.

3^{rd} term: $\frac{\partial \Phi}{\partial x_i}$ doesn't depend on v_i, so may be taken outside.

Moment Equations II

Let's consider the zeroth moment: $l = m = n = 0$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\langle v_i\rangle)}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i}\mathrm{d}^3\vec{v} = 0$$

Using the divergence theorem we can write

$$\int \frac{\partial f}{\partial v_i}\mathrm{d}^3\vec{v} = \int f\mathrm{d}^2\vec{S} = 0$$

where the last equality follows from the fact that $f \to 0$ **if** $|v| \to \infty$**. The zeroth moment of the CBE therefore reduces to**

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\langle v_i\rangle)}{\partial x_i} = 0}$$

Note that this is the continuity equation, identical to that of fluid dynamics.

Just a short remark regarding notation:

$$\langle v_i\rangle = \int v_i f\mathrm{d}^3\vec{v}$$

is used as short-hand for

$$\langle v_i(\vec{x})\rangle = \int v_i(\vec{x}) f(\vec{x},\vec{v})\mathrm{d}^3\vec{v}$$

Thus, $\langle v_i\rangle$ **is a local expectation value. For brevity we do not explicitely write the** $\vec{x}$**-dependence.**

Moment Equations III

Next we consider the first-order moment equations $(l,m,n)=(1,0,0)$ or $(0,1,0)$ or$(0,0,1)$

$$\int v_j \frac{\partial f}{\partial t}\mathrm{d}^3\vec{v} + \int v_j v_i \frac{\partial f}{\partial x_i}\mathrm{d}^3\vec{v} - \int v_j \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}\mathrm{d}^3\vec{v} = 0$$

$$\Leftrightarrow \frac{\partial(\rho\langle v_j\rangle)}{\partial t} + \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i}\mathrm{d}^3\vec{v} = 0$$

Using integration by parts we write

$$\begin{aligned}\int v_j \frac{\partial f}{\partial v_i}\mathrm{d}^3\vec{v} &= \int \frac{\partial(v_j f)}{\partial v_i}\mathrm{d}^3\vec{v} - \int \frac{\partial v_j}{\partial v_i} f \mathrm{d}^3\vec{v} \\ &= \int v_j f \mathrm{d}^2 S - \int \delta_{ij} f \mathrm{d}^3\vec{v} \\ &= -\delta_{ij}\rho\end{aligned}$$

so that we obtain

$$\boxed{\frac{\partial(\rho\langle v_j\rangle)}{\partial t} + \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i} + \rho\frac{\partial \Phi}{\partial x_j} = 0}$$

These are called the momentum equations. Note that this represents a set of three equations (for $j=1,2,3$), and that a summation over i is implied.

The Jeans Equations I

We can obtain the so-called **Jeans Equations** by subtracting $\langle v_j \rangle$ times the **continuity equation** from the **momentum equations**:

First we write $\langle v_j \rangle$ times the **continuity equation**:

$$\langle v_j \rangle \frac{\partial \rho}{\partial t} + \langle v_j \rangle \frac{\partial (\rho \langle v_i \rangle)}{\partial x_i} = 0$$

$$\Leftrightarrow \frac{\partial (\rho \langle v_j \rangle)}{\partial t} - \rho \frac{\partial \langle v_j \rangle}{\partial t} + \frac{\partial (\rho \langle v_i \rangle \langle v_j \rangle)}{\partial x_i} - \rho \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = 0$$

Subtracting this from the **momentum equations** yields

$$\frac{\partial (\rho \langle v_i v_j \rangle)}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} + \rho \frac{\partial \langle v_j \rangle}{\partial t} - \frac{\partial (\rho \langle v_i \rangle \langle v_j \rangle)}{\partial x_i} + \rho \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = 0$$

If we define $\boxed{\sigma_{ij}^2 = \langle v_i v_j \rangle - \langle v_i \rangle \cdot \langle v_j \rangle}$ then we obtain

$$\boxed{\rho \frac{\partial \langle v_j \rangle}{\partial t} + \rho \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\rho \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\rho \sigma_{ij}^2)}{\partial x_i}}$$

These are called the **Jeans Equations**. Once again, this represents a set of three equations (for $j = 1, 2, 3$), and a summation over i is implied.

The Jeans Equations II

We can derive a very similar equation for fluid dynamics. The equation of motion of a fluid element in the fluid is

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla} P - \rho \vec{\nabla} \Phi_{\mathrm{ext}}$$

with P the pressure and Φ_{ext} some external potential.

Since $\vec{v} = \vec{v}(\vec{x}, t)$ we have that $\mathrm{d}\vec{v} = \frac{\partial \vec{v}}{\partial t}\mathrm{d}t + \frac{\partial \vec{v}}{\partial x_i}\mathrm{d}x_i$, and thus

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\partial v}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}$$

which allows us to write the equations of motion as

$$\boxed{\rho \frac{\partial v}{\partial t} + \rho \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla} P - \rho \vec{\nabla} \Phi_{\mathrm{ext}}}$$

These are the so-called Euler Equations. A comparison with the Jeans Equations shows that $\rho\sigma_{ij}^2$ has a similar effect as the pressure. However, now it is not a scalar but a tensor.

$\rho\sigma_{ij}^2$ is called the stress-tensor

The stress-tensor is manifest symmetric ($\sigma_{ij} = \sigma_{ji}$) and there are thus 6 independent terms.

The Jeans Equations III

The stress tensor σ_{ij}^2 measures the random motions of the stars around the streaming part $\langle v_i \rangle \langle v_j \rangle$.

Note that the stress-tensor is a local quantity $\sigma_{ij}^2 = \sigma_{ij}^2(\vec{x})$. At each point $\vec{x}$ it defines the velocity ellipsoid; an ellipsoid whose principal axes are defined by the orthogonal eigenvectors of σ_{ij}^2 with lengths that are proportional to the square roots of the respective eigenvalues.

The incompressible stellar fluid experiences anisotropic pressure-like forces.

Note that the Jeans Equations have 9 unknowns (3 streaming motions $\langle v_i \rangle$ and 6 terms of the stress-tensor). With only three equations, this is not a closed set.

For comparison, in fluid dynamics there are only 4 unknowns (3 streaming motions and the pressure). The 3 Euler Equations combined with the Equation of State forms a closed set.

The Jeans Equations IV

One might think that adding higher-order moment equations of the CBE will allow to obtain a closed set of equations. However, adding more equations also adds more unknowns such as $\langle v_i v_j v_k \rangle$, etc. The set of CBE moment equations never closes!

In practice one therefore makes some assumptions, such as assumptions regarding the form of the stress-tensor, in order to be able to solve the Jeans Eequations.

If, with this approach, a solution is found, the solution may not correspond to a physical (i.e., everywhere positive) DF. Thus, although any real DF obeys the Jeans equations, not every solution to the Jeans equations cprresponds to a physical DF!!!

Cylindrically Symmetric Jeans Equations

As a worked out example we derive the Jeans equations under cylindrical symmetry. We therefore write the Jeans equations in the cylindrical coordinate system (R, ϕ, z).

The first step is to write the CBE in cylindrical coordinates

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z}$$

First we recall from vector calculus that

$$\vec{v} = \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z = v_R\vec{e}_R + v_\phi\vec{e}_\phi + v_z\vec{e}_z$$

from which we obtain that

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \ddot{R}\vec{e}_R + \dot{R}\dot{\vec{e}}_R + \dot{R}\dot{\phi}\vec{e}_\phi + R\ddot{\phi}\vec{e}_\phi + R\dot{\phi}\dot{\vec{e}}_\phi + \ddot{z}\vec{e}_z + \dot{z}\dot{\vec{e}}_z$$

Using that $\dot{\vec{e}}_R = \phi\vec{e}_\phi$, $\dot{\vec{e}}_\phi = -\dot{\phi}\vec{e}_R$, and $\dot{\vec{e}}_z = 0$ we have that

$$\boxed{\vec{a} = \left[\ddot{R} - R\dot{\phi}^2\right]\vec{e}_R + \left[2\dot{R}\dot{\phi} + R\ddot{\phi}\right]\vec{e}_\phi + \ddot{z}\vec{e}_z}$$

$$\begin{aligned} v_R = \dot{R} \quad &\Rightarrow \quad \dot{v}_R = \ddot{R} \\ v_\phi = R\dot{\phi} \quad &\Rightarrow \quad \dot{v}_\phi = \dot{R}\dot{\phi} + R\ddot{\phi} \\ v_z = \dot{z} \quad &\Rightarrow \quad \dot{v}_z = \ddot{z} \end{aligned}$$

Cylindrically Symmetric Jeans Equations

This allows us to write

$$\vec{a} = \left[\dot{v}_R - \frac{v_\phi^2}{R}\right]\vec{e}_R + \left[\frac{v_R v_\phi}{R} + \dot{v}_\phi\right]\vec{e}_\phi + \dot{v}_z\vec{e}_z$$

Newton's equation of motion in vector form reads

$$\vec{a} = -\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial R}\vec{e}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\phi}\vec{e}_\phi + \frac{\partial\Phi}{\partial z}\vec{e}_z$$

Combining the above we obtain

$$\begin{aligned}
\dot{v}_R &= -\frac{\partial\Phi}{\partial R} + \frac{v_\phi^2}{R} \\
\dot{v}_\phi &= -\frac{1}{R}\frac{\partial\Phi}{\partial R} + \frac{v_R v_\phi}{R} \\
\dot{v}_z &= -\frac{\partial\Phi}{\partial z}
\end{aligned}$$

Which allows us to the write the CBE in cylindrical coordinates as

$$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial\phi} + v_z\frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial\Phi}{\partial R}\right]\frac{\partial f}{\partial v_R} - \frac{1}{R}\left[v_R v_\phi + \frac{\partial\Phi}{\partial\phi}\right]\frac{\partial f}{\partial v_\phi} - \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0$$

Cylindrically Symmetric Jeans Equations

The Jeans equations follow from multiplication with v_R, v_ϕ, and v_z and integration over velocity space. Note that the symmetry requires that all derivatives with respect to ϕ must vanish.

The remaining terms are:

$$\int v_R \frac{\partial f}{\partial t} \mathrm{d}^3\vec{v} = \frac{\partial}{\partial t} \int v_R f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R\rangle)}{\partial t}$$

$$\int v_R^2 \frac{\partial f}{\partial R} \mathrm{d}^3\vec{v} = \frac{\partial}{\partial R} \int v_R^2 f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R^2\rangle)}{\partial R}$$

$$\int v_R v_z \frac{\partial f}{\partial z} \mathrm{d}^3\vec{v} = \frac{\partial}{\partial z} \int v_R v_z f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z}$$

$$\int \frac{v_R v_\phi^2}{R} \frac{\partial f}{\partial v_R} \mathrm{d}^3\vec{v} = \frac{1}{R}\left[\int \frac{\partial(v_R v_\phi^2 f)}{\partial v_R} \mathrm{d}^3\vec{v} - \int \frac{\partial(v_R v_\phi^2)}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\langle v_\phi^2\rangle}{R}$$

$$\int v_R \frac{\partial \Phi}{\partial R} \frac{\partial f}{\partial v_R} \mathrm{d}^3\vec{v} = \frac{\partial \Phi}{\partial R}\left[\int \frac{\partial(v_R f)}{\partial v_R} \mathrm{d}^3\vec{v} - \int \frac{\partial v_R}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\partial \Phi}{\partial R}$$

$$\int \frac{v_R^2 v_\phi}{R} \frac{\partial f}{\partial v_\phi} \mathrm{d}^3\vec{v} = \frac{1}{R}\left[\int \frac{\partial(v_R^2 v_\phi f)}{\partial v_\phi} \mathrm{d}^3\vec{v} - \int \frac{\partial(v_R^2 v_\phi)}{\partial v_\phi} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\langle v_R^2\rangle}{R}$$

$$\int v_R \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} \mathrm{d}^3\vec{v} = \frac{\partial \Phi}{\partial z}\left[\int \frac{\partial(v_R f)}{\partial v_z} \mathrm{d}^3\vec{v} - \int \frac{\partial v_z}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = 0$$

Cylindrically Symmetric Jeans Equations

Working out the similar terms for the other Jeans equations we obtain the

Jeans Equations in Cylindrical Coordinates

$$\frac{\partial(\rho\langle v_R\rangle)}{\partial t}+\frac{\partial(\rho\langle v_R^2\rangle)}{\partial R}+\frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z}+\rho\left[\frac{\langle v_R^2\rangle-\langle v_\phi^2\rangle}{R}+\frac{\partial\Phi}{\partial R}\right]=0$$
$$\frac{\partial(\rho\langle v_\phi\rangle)}{\partial t}+\frac{\partial(\rho\langle v_R v_\phi\rangle)}{\partial R}+\frac{\partial(\rho\langle v_\phi v_z\rangle)}{\partial z}+2\rho\frac{\langle v_R v_\phi\rangle}{R}=0$$
$$\frac{\partial(\rho\langle v_z\rangle)}{\partial t}+\frac{\partial(\rho\langle v_R v_z\rangle)}{\partial R}+\frac{\partial(\rho\langle v_z^2\rangle)}{\partial z}+\rho\left[\frac{\langle v_R v_z\rangle}{R}+\frac{\partial\Phi}{\partial z}\right]=0$$

Note that there are indeed 9 unknowns in these 3 equations. Only if we make additional assumptions can we solve these equations. In particular, one often makes the following assumptions:

(1) **System is static** $\Rightarrow \frac{\partial}{\partial t}$**-terms are zero and** $\langle v_R\rangle=\langle v_z\rangle=0$

(2) **Stress Tensor is diagonal** $\Rightarrow \langle v_i v_j\rangle=0$ **if** $i\neq j$

(3) **Meridional Isotropy** $\Rightarrow \langle v_R^2\rangle=\langle v_z^2\rangle=\sigma_R^2=\sigma_z^2\equiv\sigma^2$

Under these assumptions we have 3 unknowns left: $\langle v_\phi\rangle$, $\langle v_\phi^2\rangle$, **and** σ^2.

Cylindrically Symmetric Jeans Equations

Under the assumptions listed on the previous page the Jeans equation reduce to

$$\frac{\partial(\rho\sigma^2)}{\partial R} + \rho\left[\frac{\sigma^2 - \langle v_\phi^2\rangle}{R} + \frac{\partial\Phi}{\partial R}\right] = 0$$
$$\frac{\partial(\rho\sigma^2)}{\partial z} + \rho\frac{\partial\Phi}{\partial z} = 0$$

Note that we have only 2 equations left: the system is still not closed.

If from the surface brightness we can estimate the mass density $\rho(R,z)$ and hence the potential $\Phi(R,z)$, we can solve the second of these Jeans equations for the meridional velocity dispersion

$$\sigma^2(R,z) = \frac{1}{\rho}\int_z^\infty \rho\frac{\partial\Phi}{\partial z}\mathrm{d}z$$

and the first Jeans equation then gives the mean square azimuthal velocity

$$\langle v_\phi^2\rangle = \langle v_\phi\rangle^2 + \sigma_\phi^2$$

$$\langle v_\phi^2\rangle(R,z) = \sigma^2(R,z) + R\frac{\partial\Phi}{\partial R} + \frac{R}{\rho}\frac{\partial(\rho\sigma^2)}{\partial R}$$

Thus, although $\langle v_\phi^2\rangle$ is uniquely specified by the Jeans equations, we don't know how it splits in the actual azimuthal streaming $\langle v_\phi\rangle$ and the azimuthal dispersion σ_ϕ^2. Additional assumptions are needed for this.

Spherically Symmetric Jeans Equations

A similar analysis but for a spherically symmetric system, using the spherical coordinate system (r, θ, ϕ), gives the following set of Jeans equations

$$\frac{\partial(\rho\langle v_r\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + \frac{\rho}{r}\left[2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right] + \rho\frac{\partial\Phi}{\partial r} = 0$$
$$\frac{\partial(\rho\langle v_\theta\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\theta\rangle)}{\partial r} + \frac{\rho}{r}\left[3\langle v_r v_\theta\rangle + \left(\langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right)\cot\theta\right] = 0$$
$$\frac{\partial(\rho\langle v_\phi\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\phi\rangle)}{\partial r} + \frac{\rho}{r}\left[3\langle v_r v_\phi\rangle + 2\langle v_\theta v_\phi\rangle\cot\theta\right] = 0$$

If we now make the additional assumptions that the system is static and that also the kinematic properties of the system are spherical symmetric then there can be no streaming motions and all mixed second-order moments vanish. Consequently, the stress tensor is diagonal with $\sigma_\theta^2 = \sigma_\phi^2$. Under these assumptions only one of the three Jeans equations remains:

$$\frac{\partial(\rho\sigma_r^2)}{\partial r} + \frac{2\rho}{r}\left[\sigma_r^2 - \sigma_\theta^2\right] + \rho\frac{\partial\Phi}{\partial r} = 0$$

Notice once again how the spherical Jeans equation is not sufficient to determine the dynamics: if the density and potential are presumed known, it contains two unknown functions $\sigma_r^2(r)$ and $\sigma_\theta^2(r)$ which can therefore not be determined both.

Spherically Symmetric Jeans Equations

It is useful to define the anisotropy parameter

$$\beta(r) \equiv 1 - \frac{\sigma_\theta^2(r)}{\sigma_r^2(r)}$$

With β thus defined the Jeans equation can be written as

$$\frac{1}{\rho}\frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + 2\frac{\beta\langle v_r^2\rangle}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

If we now use that $\mathrm{d}\Phi/\mathrm{d}r = GM(r)/r$ then we obtain

$$M(r) = -\frac{r\langle v_r^2\rangle}{G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\langle v_r^2\rangle}{\mathrm{d}\ln r} + 2\beta\right]$$

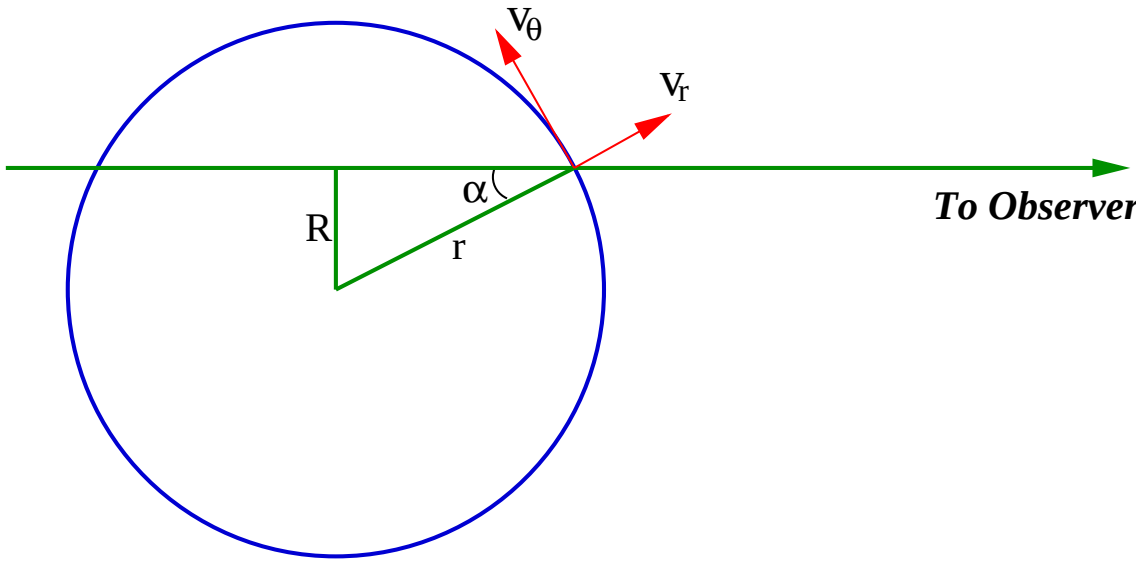
Thus, if we can measure $\rho(r)$, $\langle v_r^2\rangle(r)$, and $\beta(r)$, we can use the Jeans equations to infer the mass profile $M(r)$.

Consider an external, spherical galaxy. Observationally, we can measure the projected surface brightness profile, $\Sigma(R)$, which is related to the luminosity density $\nu(r) = \rho(r)/\Upsilon(r)$ as

$$\Sigma(R) = 2\int_R^\infty \frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2 - R^2}}$$

with $\Upsilon(r)$ the mass-to-light ratio.

Spherically Symmetric Jeans Equations



Similarly, the line-of-sight velocity dispersion is an observationally accessible quantity. As the figure illustrates, it is related to both $\langle v_r^2 \rangle (r)$ and $\beta(r)$ according to

$$\begin{aligned}
\Sigma(R)\sigma_p^2(R) &= 2\int_R^\infty \langle (v_r \cos\alpha - v_\theta \sin\alpha)^2 \rangle \frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2-R^2}} \\
&= 2\int_R^\infty \left(\langle v_r^2 \rangle \cos^2\alpha + \langle v_\theta^2 \rangle \sin^2\alpha \right) \frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2-R^2}} \\
&= 2\int_R^\infty \left(1 - \beta \frac{R^2}{r^2} \right) \frac{\nu \langle v_r^2 \rangle\, r\, \mathrm{d}r}{\sqrt{r^2-R^2}}
\end{aligned}$$

Spherically Symmetric Jeans Equations

The 3D luminosity density is trivially obtained from the observed $\Sigma(R)$:

$$\nu(r) = -\frac{1}{\pi} \int_r^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R} \frac{\mathrm{d}R}{\sqrt{R^2 - r^2}}$$

In general, we have three unknowns: $M(r)$ (or equivalently $\rho(r)$ or $\Upsilon(r)$), $\langle v_r^2 \rangle(r)$ and $\beta(r)$.

With our two observables $\Sigma(R)$ and $\sigma_p^2(R)$ these can only be determined if we make additional assumptions.

EXAMPLE 1: Assume isotropy ($\beta(r) = 0$). In this case we can use the Abel inversion technique to obtain

$$\nu(r)\langle v_r^2 \rangle(r) = -\frac{1}{\pi} \int_r^\infty \frac{\mathrm{d}(\Sigma\sigma_p^2)}{\mathrm{d}R} \frac{\mathrm{d}R}{\sqrt{R^2 - r^2}}$$

and the enclosed mass follows from the Jeans equation

$$M(r) = -\frac{r\langle v_r^2 \rangle}{G} \left[\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\langle v_r^2 \rangle}{\mathrm{d}\ln r} \right]$$

Note that the first term uses the luminosity density $\nu(r)$ rather than the mass density $\rho(r)$, because σ_p^2 is weighted by light rather than mass.

Spherically Symmetric Jeans Equations

The **mass-to-light ratio** now follows from

$$\Upsilon(r) = \frac{M(r)}{4\pi \int_0^r \nu(r)\, r^2\, \mathrm{d}r}$$

which can be used to investigate whether system contains **dark matter halo** or central **black hole**, but always under assumption that system is **isotropic**.

EXAMPLE 2: Assume a constant mass-to-light ratio: $\Upsilon(r) = \Upsilon_0$. In this case the luminosity density $\nu(r)$ immediately yields the enclosed mass:

$$M(r) = 4\pi\Upsilon_0 \int_0^r \nu(r)\, r^2\, \mathrm{d}r$$

We can now use the **Jeans Equation** to write $\beta(r)$ in terms of $M(r)$, $\nu(r)$ and $\langle v_r^2 \rangle(r)$. Substituting this in the equation for $\Sigma(R)\sigma_p^2(R)$ allows a solution for $\langle v_r^2 \rangle(r)$, and thus for $\beta(r)$. As long as $0 \leq \beta(r) \leq 1$ the model is said to be **self-consistent** witin the context of the Jeans equations.

Almost always, radically different models (based on radically different assumptions) can be constructed, that are all consistent with the data and the Jeans equations. This is often referred to as the **mass-anisotropy degeneracy**. Note, however, that none of these models need to be physical: they can still have $f < 0$.

The Virial Equations I

We can obtain an important tensor equation relating global properties of the system, by multiplying the CBE by both v_j and x_k and then integrating over the entire phase-space.

The first step of this has already been performed in our derivation of the Jeans equations, and yielded the momentum equations

$$\boxed{\frac{\partial(\rho\langle v_j\rangle)}{\partial t} + \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i} + \rho\frac{\partial\Phi}{\partial x_j} = 0}$$

Multiplying all terms with x_k and integrating over real space yields

$$\frac{\partial}{\partial t}\int \rho x_k \langle v_j\rangle \mathrm{d}^3x = -\int x_k \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} - \int \rho\, x_k\, \frac{\partial\Phi}{\partial x_j}\mathrm{d}^3\vec{x}$$

Using integration by parts the first term on the r.h.s. becomes

$$\begin{aligned}\int x_k \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} &= \int \frac{\partial(\rho x_k\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} - \int \rho\langle v_i v_j\rangle \frac{\partial x_k}{\partial x_i}\mathrm{d}^3\vec{x} \\ &= -\int \delta_{ki}\rho\langle v_i v_j\rangle \mathrm{d}^3\vec{x} \\ &= -\int \rho\langle v_k v_j\rangle \mathrm{d}^3\vec{x} \\ &= -2\mathcal{K}_{kj}\end{aligned}$$

where we have defined the kinetic energy tensor

$$\boxed{\mathcal{K}_{ij} = \tfrac{1}{2}\int \rho\langle v_i v_j\rangle \mathrm{d}^3\vec{x}}$$

The Virial Equations II

It is customary to split the kinetic energy tensor into contributions from ordered and random motions:

$$\mathcal{K}_{ij} \equiv \mathcal{T}_{ij} + \tfrac{1}{2}\Pi_{ij}$$

where

$$\mathcal{T}_{ij} \equiv \tfrac{1}{2}\int \rho \langle v_i \rangle \langle v_j \rangle \, \mathrm{d}^3\vec{x} \qquad\qquad \Pi_{ij} \equiv \int \rho \, \sigma_{ij}^2 \, \mathrm{d}^3\vec{x}$$

In addition to the $\mathcal{K}$ we also define the potential energy tensor

$$\mathcal{W}_{ij} \equiv -\int \rho x_i \frac{\partial \Phi}{\partial x_j} \mathrm{d}^3\vec{x}$$

Combining the above we obtain

$$\frac{\partial}{\partial t}\int \rho \, x_k \langle v_j \rangle \mathrm{d}^3 x = 2\mathcal{K}_{kj} + \mathcal{W}_{kj}$$

which allows us to write

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho \left[x_k \langle v_j \rangle + x_j \langle v_k \rangle \right] = 2\mathcal{K}_{jk} + \mathcal{W}_{jk}$$

where we have used that $\mathcal{K}$ and $\mathcal{W}$ are symmetric.

The Virial Equations III

Finally we also define the moment of inertia tensor

$$\mathcal{I}_{ij} \equiv \int \rho \, x_i \, x_j \, \mathrm{d}^3\vec{x}$$

Differentiating with respect to time, and using the continuity equation (i.e., the zeroth moment equation of the CBE) yields

$$\begin{aligned}
\frac{\mathrm{d}I_{jk}}{\mathrm{d}t} &= \int \frac{\partial \rho}{\partial t} x_j \, x_k \, \mathrm{d}^3\vec{x} \\
&= -\int \frac{\partial \rho \langle v_i \rangle}{\partial x_i} x_j \, x_k \, \mathrm{d}^3\vec{x} \\
&= -\int \frac{\partial (\rho \langle v_i \rangle x_j x_k)}{\partial x_i} \mathrm{d}^3\vec{x} + \int \rho \langle v_i \rangle \frac{\partial (x_j x_k)}{\partial x_i} \mathrm{d}^3\vec{x} \\
&= \int \rho \langle v_i \rangle \left[x_j \delta_{ik} + x_k \delta_{ij} \right] \mathrm{d}^3\vec{x} \\
&= \int \rho \left[x_j \langle v_k \rangle + x_k \langle v_j \rangle \right] \mathrm{d}^3\vec{x}
\end{aligned}$$

so that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho \left[x_k \langle v_j \rangle + x_j \langle v_k \rangle \right] = \frac{1}{2} \frac{\mathrm{d}^2 \mathcal{I}_{jk}}{\mathrm{d}t^2}$$

which allows us to write the Tensor Virial Theorem as

$$\frac{1}{2} \frac{\mathrm{d}^2 \mathcal{I}_{jk}}{\mathrm{d}t^2} = 2\mathcal{T}_{jk} + \Pi_{jk} + \mathcal{W}_{jk}$$

which relates the gross kinematic and structural properties of gravitational systems.

The Virial Equations IV

If the system is in a steady-state the moment of inertia tensor is stationary, and the Tensor Virial Theorem reduces to $2\mathcal{K}_{ij} + \mathcal{W}_{ij} = 0$.

Of particular interest is the trace of the Tensor Virial Theorem, which relates the total kinetic energy $K = \frac{1}{2} M \langle v^2 \rangle$ to the total potential energy $W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, \mathrm{d}^3\vec{x}$.

$$\begin{aligned} \mathrm{tr}(\mathcal{K}) \equiv \sum_{i=1}^{3} \mathcal{K}_{ii} &= \frac{1}{2} \int \rho(\vec{x}) \left[\langle v_1^2 \rangle(\vec{x}) + \langle v_2^2 \rangle(\vec{x}) + \langle v_3^2 \rangle(\vec{x}) \right] \mathrm{d}^3\vec{x} \\ &= \frac{1}{2} \int \rho(\vec{x})\, \langle v^2 \rangle(\vec{x}) \mathrm{d}^3\vec{x} \\ &= \frac{1}{2} M \langle v^2 \rangle = K \end{aligned}$$

where we have used that

$$\langle v^2 \rangle = \frac{1}{M} \int \rho(\vec{x}) \langle v^2 \rangle(\vec{x}) \mathrm{d}^3\vec{x}$$

Similarly, the trace of the potential energy tensor is equal to the total potential energy (see next page for derivation):

$$\mathrm{tr}(\mathcal{W}) = W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, \mathrm{d}^3\vec{x}$$

We thus obtain the scalar virial theorem

$$\boxed{2K + W = 0}$$

The Potential Energy Tensor I

We have defined the potential energy tensor as

$$\mathcal{W}_{ij} \equiv -\int \rho x_i \frac{\partial \Phi}{\partial x_j} \mathrm{d}^3\vec{x}$$

Using that $\Phi(\vec{x}) = -G\int \frac{\rho(\vec{x})}{|\vec{x}'-\vec{x}|}\mathrm{d}^3\vec{x}$ **we obtain**

$$\mathcal{W}_{ij} = G\int\int \rho(\vec{x})\,\rho(\vec{x}')\frac{x_i(x'_j-x_j)}{|\vec{x}'-\vec{x}|^3}\mathrm{d}^3\vec{x}'\mathrm{d}^3\vec{x}$$

Using that $\vec{x}$ **and** $\vec{x}'$ **are dummy variables, we may relabel them, and write**

$$\mathcal{W}_{ij} = G\int\int \rho(\vec{x}')\,\rho(\vec{x})\frac{x'_j(x_k-x'_k)}{|\vec{x}-\vec{x}'|^3}\mathrm{d}^3\vec{x}\mathrm{d}^3\vec{x}'$$

Interchanging the order of integration and summing the above two equations yields the manifestly symmetric expression

$$\mathcal{W}_{ij} = -\frac{G}{2}\int\int \rho(\vec{x})\,\rho(\vec{x}')\frac{(x'_j-x_j)(x'_k-x_k)}{|\vec{x}'-\vec{x}|^3}\mathrm{d}^3\vec{x}'\mathrm{d}^3\vec{x}$$

This expression allows us to write

$$\begin{aligned}\mathrm{tr}(\mathcal{W}) &\equiv \sum_{i=1}^{3}\mathcal{W}_{ii} = -\frac{G}{2}\int\int \rho(\vec{x})\rho(\vec{x}')\frac{|\vec{x}'-\vec{x}|^2}{|\vec{x}'-\vec{x}|^3}\mathrm{d}^3\vec{x}'\mathrm{d}^3\vec{x}\\ &= -\frac{G}{2}\int\rho(\vec{x})\int\frac{\rho(\vec{x}')}{|\vec{x}'-\vec{x}|}\mathrm{d}^3\vec{x}'\mathrm{d}^3\vec{x} = \frac{1}{2}\int\rho(\vec{x})\Phi(\vec{x})\mathrm{d}^3\vec{x} = W\end{aligned}$$

The Surface Pressure Term

In our derivation on the previous pages we obtained

$$\begin{aligned}\int x_k \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} &= \int \frac{\partial(\rho x_k\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} - \int \rho\langle v_i v_j\rangle \frac{\partial x_k}{\partial x_i}\mathrm{d}^3\vec{x} \\ &= -\int \rho\langle v_k v_j\rangle \mathrm{d}^3\vec{x} = -2\mathcal{K}_{kj}\end{aligned}$$

where we have used that

$$\int \frac{\partial(\rho x_k\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} = \int \rho x_k\langle v_k v_j\rangle \mathrm{d}^2 S = 0$$

based on the assumption that $\rho(r) = 0$ when $r \longrightarrow \infty$. However, this is only true for an isolated system with 'vacuum' boundary conditions.

In reality, a halo or galaxy is embedded in a cosmological density field, often with ongoing infall. This yields a non-zero surface pressure. In its most general form the scalar virial theorem therefore reads

$$\boxed{2K + W + S_p = 0}$$

with the surface pressure term

$$S_p = -\int \langle v^2\rangle \vec{r}\cdot\vec{n}\mathrm{d}^2\vec{S}$$

As long as $S_p \neq 0$ we thus expect that $2K/|W| \neq 1$.
See Shapiro et al. (astro-ph/0409173) for a detailed discussion.

The Virial Equations V

From a simple dimensional analysis one finds that $|W| \propto GM^2/R$ with M the system's mass and R a characteristic radius.

A useful characteristic radius is the so-called gravitational radius defined by

$$r_g \equiv \frac{GM^2}{|W|}$$

One can relate the gravitational radius to the half-mass radius r_h, defined as radius enclosing half the total mass. As shown by Spitzer (1969), typical stellar systems have $r_g \simeq 2.5 r_h$.

Combining this with the scalar virial theorem we can write that

$$M \simeq 2.5 \frac{r_h \langle v^2 \rangle}{G}$$

which is a useful equation to obtain a (rough) estimate of the virial mass from a measure of the half-mass radius and the rms motion

The Virial Equations VI

Using the scalar virial theorem we obtain

$$\boxed{E = K + W = -K = \tfrac{1}{2}W}$$

Consider the formation of a virialized object. If the system forms by collecting material from large radii, the initial conditions are well approximated by $K_{\rm init} = W_{\rm init} = E_{\rm init} = 0$.

Because of gravity the matter starts to collapse. Since $W = -GM^2/r_g$ this makes W more negative. At the same time K increases. Initially, during the early collapse, $E = T + W = 0$.

After the first shell crossing, the system starts to virialize. When virialization is complete, $2T + W = 0$ and $E = W/2$.

Therefore, half the gravitational energy released by collapse is invested in kinetic form. The system somehow disposes of the other half in order to achieve a binding energy $E_b = -E$.

QUESTION Where does the other half of the energy go?

Application: M/L of Spherical Systems

As an application of the Virial Theorem, consider spherical, non-rotating systems (spherical galaxies or globulars)

If the mass-to-light ratio Υ does not depend on radius then

$$K_{xx} = \int \tfrac{1}{2}\rho(\vec{x})\langle v_x^2\rangle \mathrm{d}^3\vec{x} = \tfrac{\Upsilon}{2}\int \nu(\vec{x})\langle v_x^2\rangle \mathrm{d}^3\vec{x}$$

where $\nu(\vec{x}) = \rho(\vec{x})/\Upsilon$ is the 3D luminosity distribution, and K_{xx} is the kinetic energy associated with motion in the x-direction

Since a spherical, non-rotating system is isotropic we have that

$$K = K_{xx} + K_{yy} + K_{zz} = 3K_{xx}$$

If one has observationally determined the surface brightness profile $\Sigma(R)$ and the line-of-sight velocity dispersion $\sigma_p^2(R)$ then it is easy to see that

$$K = 3\tfrac{\Upsilon}{2}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\mathrm{d}R R\Sigma(R)\sigma_p^2(R) = 3\pi\Upsilon\int_0^{\infty}\mathrm{d}R R\Sigma(R)\sigma_p^2(R) \equiv \Upsilon J$$

where we defined the observationally accessible $J = J(\Sigma,\sigma_p^2)$

Application: M/L of Spherical Systems

As seen in exersizes, for spherical system: $W = -\frac{G}{2}\int_0^\infty \frac{M^2(r)}{r^2}\mathrm{d}r$

Using that $M(r) = 4\pi\int_0^r \rho(r')r'^2\mathrm{d}r'$

where the density profile is related to $\Sigma(R)$ **according to**

$$\rho(r) = -\frac{\Upsilon}{\pi}\int_r^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2-r^2}}$$

we obtain that

$$W = -8\Upsilon^2\int_0^\infty \frac{\mathrm{d}r}{r^2}\left[\int_0^r \mathrm{d}r'\, r'^2\int_{r'}^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2-r^2}}\right]^2 \equiv \Upsilon^2\tilde{J}$$

where we have defined the observationally accessible integral $\tilde{J} = \tilde{J}(\Sigma)$

According to the virial theorem $2K + W = 0$**, and thus** $-2K/W = 1$**.**

Substituting $K = \Upsilon J$ **and** $W = \Upsilon^2\tilde{J}$ **we thus obtain that**

$$\boxed{\Upsilon = -\frac{2J}{\tilde{J}}}$$

Flattening of Oblate Spheroids I

As another application of the virial theorem we relate the flattening of an oblate spheroid to its kinematics.

Consider an oblate system with it's symmetry axis along the z-direction. Because of symmetry considerations we have that

$$\langle v_R \rangle = \langle v_z \rangle = 0 \qquad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0$$

If we write that

$$\langle v_x \rangle = \langle v_\phi \rangle \sin\phi \qquad \langle v_y \rangle = \langle v_\phi \rangle \cos\phi$$

we obtain

$$\begin{aligned} \mathcal{T}_{xy} &= \tfrac{1}{2} \int \rho \langle v_x \rangle \langle v_y \rangle \mathrm{d}^3\vec{x} \\ &= \tfrac{1}{2} \int_0^{2\pi} \mathrm{d}\phi \sin\phi \cos\phi \int_0^\infty \mathrm{d}R \int_{-\infty}^{\infty} \mathrm{d}z \rho(R,z) \langle v_\phi \rangle^2(R,z) \\ &= 0 \end{aligned}$$

A similar analysis shows that all other non-diagonal elements of $\mathcal{T}$, Π, and $\mathcal{W}$ have to be zero.

In addition, because of symmetry considerations we must have that $\mathcal{T}_{xx} = \mathcal{T}_{yy}$, $\Pi_{xx} = \Pi_{yy}$, and $\mathcal{W}_{xx} = \mathcal{W}_{yy}$.

Flattening of Oblate Spheroids II

Given these symmetries, the only independent, non-trivial virial equations are

$$2\mathcal{T}_{xx} + \Pi_{xx} + \mathcal{W}_{xx} = 0, \qquad 2\mathcal{T}_{zz} + \Pi_{zz} + \mathcal{W}_{zz} = 0$$

Taking the ratio we find that

$$\boxed{\frac{2\mathcal{T}_{xx}+\Pi_{xx}}{2\mathcal{T}_{zz}+\Pi_{zz}} = \frac{\mathcal{W}_{xx}}{\mathcal{W}_{zz}}}$$

The usefulness of this equation lies in the fact that, for density distributions that are constant on similar concentric spheroids, i.e., $\rho = \rho(m^2)$, the ratio $\mathcal{W}_{xx}/\mathcal{W}_{zz}$ depends only on the axis ratio c/a of the spheroids, and is independent of the density profile! For an oblate body, to good approximation

$$\boxed{\frac{\mathcal{W}_{xx}}{\mathcal{W}_{zz}} \simeq \left(\frac{c}{a}\right)^{-0.9}}$$

Let us start by considering isotropic, oblate rotators.

Then $\Pi_{xx} = \Pi_{zz} = M\tilde{\sigma}^2$, $\mathcal{T}_{zz} = 0$ and $\mathcal{T}_{xx} + \mathcal{T}_{yy} = 2\mathcal{T}_{xx} = \frac{1}{2}M\tilde{v}^2$.

Here M is the total mass, $\tilde{\sigma}^2$ is the mass-weighted rms-average of the intrinsic one-dimensional velocity dispersion, and $\tilde{v}^2$ is the mass-weighted rms rotation velocity.

Flattening of Oblate Spheroids III

Thus, for an isotropic, oblate rotators we have that

$$\frac{\frac{1}{2}M\tilde{v}^2+M\tilde{\sigma}^2}{M\tilde{\sigma}^2} \simeq \left(\frac{c}{a}\right)^{-0.9}$$

which reduces to

$$\boxed{\frac{\tilde{v}}{\tilde{\sigma}} \simeq \sqrt{2[(c/a)^{-0.9}-1]}}$$

This specifies the relation between the flattening of the spheroid and the ratio of streaming motion to random motion. Note that you need a rather large amount of rotation to achieve only modest flattening: $c/a = 0.7$ requires $\tilde{v} \sim 0.9\tilde{\sigma}$

Next consider a non-rotating, anisotropic, oblate system:

In this case $\Pi_{xx} = M\tilde{\sigma}_{xx}^2$ and $\Pi_{zz} = M\tilde{\sigma}_{zz}^2$, and the virial theorem gives that

$$\boxed{\frac{\tilde{\sigma}_{zz}}{\tilde{\sigma}_{xx}} \simeq \left(\frac{c}{a}\right)^{0.45}}$$

Now a flattening of $c/a = 0.7$ requires only a small anisotropy of $\tilde{\sigma}_{zz}/\tilde{\sigma}_{xx} \simeq 0.85$

Flattening of Oblate Spheroids IV

Finally, consider the general case of rotating, anisotropic, oblate systems

Now we have $\Pi_{zz} = (1-\delta)\Pi_{xx} = (1-\delta)M\tilde{\sigma}^2$, $\mathcal{T}_{zz} = 0$ and $2\mathcal{T}_{xx} = \frac{1}{2}M\tilde{v}^2$, where we have introduced the anistropy parameter $\delta < 1$.

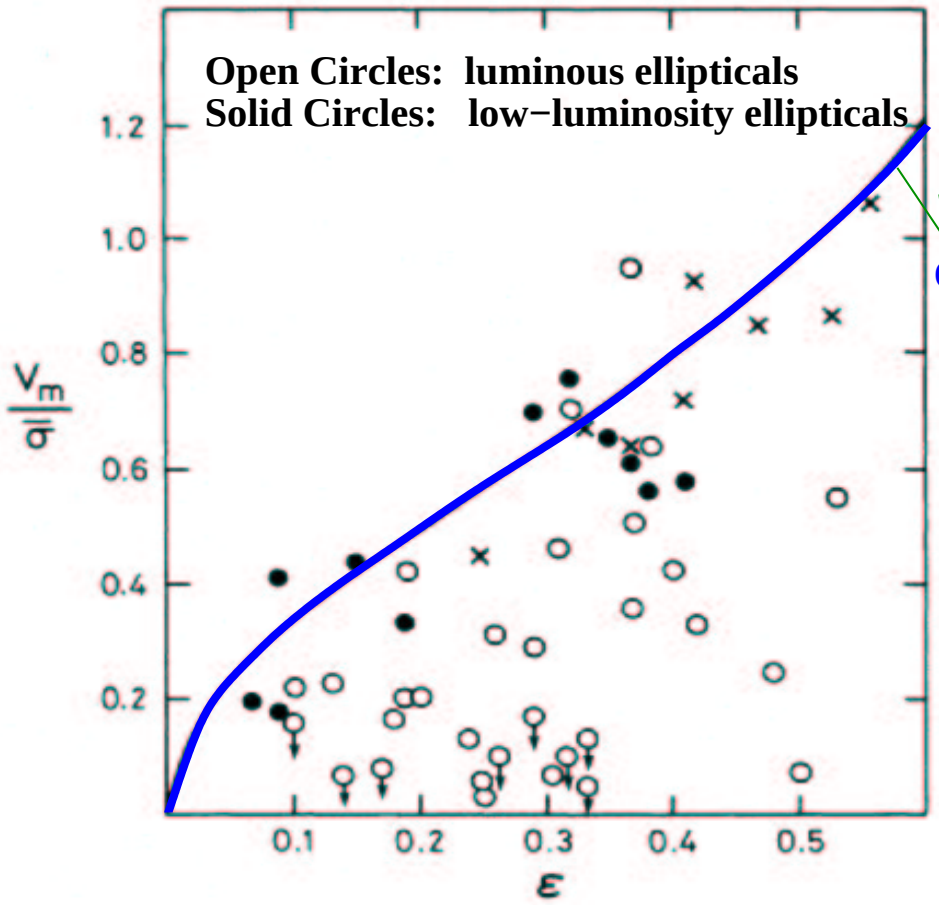
In this case the virial theorem gives

$$\boxed{\frac{\tilde{v}}{\tilde{\sigma}} \simeq \sqrt{2[(1-\delta)\,(c/a)^{-0.9} - 1]}}$$

This shows that observations of $\tilde{v}/\tilde{\sigma}$ and the ellipticity $\varepsilon = 1-(c/a)$ allow us to test whether elliptical galaxies are supported by rotation or by anisotropic presssure.

A potential problem is that we can not directly measure $\tilde{v}$ nor $\tilde{\sigma}$. Rather, we measure properties that are projected along the line-of-sight. Furthermore, in general we don't see a system edge-on but under some unknown inclination angle i. Note that i also affects the measured v and σ. As shown in Binney & Tremaine, the overall effect is to move a point on the oblate rotator line mainly along that line.

Flattening of Oblate Spheroids V



Oblate Isotropic Rotator

(from: Davies et al. 1983)

Observations reveal a dichotomy: luminous ellipticals are supported by anisotropic pressure, while fainter ellipticals (and bulges) are consistent with being oblate, isotropic rotators.

NOTE: If luminous ellipticals are anisotropic, there is no good reason why they should be axisymmetric: massive ellipticals are triaxial

The Jeans Theorem I

RECALL: An integral of motion is a function $I(\vec{x},\vec{v})$ of the phase-space coordinates that is constant along all orbits, i.e.,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\partial I}{\partial v_i}\frac{\mathrm{d}v_i}{\mathrm{d}t} = \vec{v}\cdot\vec{\nabla}I - \vec{\nabla}\Phi\cdot\frac{\partial I}{\partial \vec{v}} = 0$$

Compare this to the CBE for a steady-state (static) system:

$$\vec{v}\cdot\vec{\nabla}f - \vec{\nabla}\Phi\cdot\frac{\partial f}{\partial \vec{v}} = 0$$

Thus the condition for I to be an integral of motion is identical with the condition for I to be a steady-state solution of the CBE. Hence:

Jeans Theorem Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of motion. Any function of these integrals is a steady-state solution of the CBE.

PROOF: Let f be any function of the n integrals of motion $I_1, I_2, \ldots I_n$ then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{k=1}^{n}\frac{\partial f}{\partial I_k}\frac{\mathrm{d}I_k}{\mathrm{d}t} = 0$$

which proofs that f satisfies the CBE.

The Jeans Theorem II

More useful than the Jeans Theorem is the Strong Jeans Theorem, which is due to Lynden-Bell (1962).

Strong Jeans Theorem **The DF of a steady-state system in which almost all orbits are regular can be written as a function of the independent isolating integrals of motion, or of the action-integrals.**

Note that a regular orbit in a system with n degrees of freedom is uniquely, and completely, specified by the values of the n isolating integrals of motion in involution. Thus the DF can be thought of as a function that expresses the probability for finding a star on each of the phase-space tori.

We first consider an application of the Jeans Theorem to Spherical Systems As we have seen, any orbit in a spherical potential admits four isolating integrals of motion: E, L_x, L_y, L_z.

Therefore, according to the Strong Jeans Theorem, the DF of any† steady-state spherical system can be expressed as $f = f(E, \vec{L})$.

---

† **except for point masses and uniform spheres, which have five isolating integrals of motion**

Jeans Theorem & Spherical Systems

If the system is spherically symmetric in all its properties, then $f = f(E, L^2)$ rather than $f = f(E, \vec{L})$: ie., the DF can only depend on the magnitude of the angular momentum vector, not on its direction.

Contrary to what one might naively expect, this is not true in general. In fact, as beautifully illustrated by Lynden-Bell (1960), a spherical system can rotate without being oblate.

Consider a spherical system with $f(E, \vec{L}) = f(E, -\vec{L})$. In such a system, for each star S on a orbit $\mathcal{O}$, there is exactly one star on the same orbit $\mathcal{O}$ but counterrotating with respect to S. Consequently, this system is perfectly spherically symmetric in all its properties.

Now consider all stars in the $z = 0$-plane, and revert the sense of all those stars with $L_z < 0$. Clearly this does not influence $\rho(r)$, but it does give the system a net sense of rotation around the z-axis.

Thus, although a system with $f = f(E, L^2)$ is not the most general case, systems with $f = f(E, \vec{L})$ are rarely considered in galactic dynamics.

Isotropic Spherical Models I

An even simpler case to consider is the one in which $f = f(E)$.

Since $E = \Phi(\vec{r}) + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]$ we have that

$$\langle v_r^2 \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_r^2 \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

$$\langle v_\theta^2 \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_\theta^2 \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

$$\langle v_\phi^2 \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_\phi^2 \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

Since these equations differ only in the labelling of one of the variables of integration, it is immediately evident that $\langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$.

Assuming that $f = f(E)$ is identical to assuming that the system is isotropic

Note that from

$$\langle v_i \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_i \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

it is also immediately evident that $\langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0$. Thus, similar as for a system with $f = f(E, L^2)$ a system with $f = f(E)$ has no net sense of rotation.

Isotropic Spherical Models II

In what follows we define the relative potential $\Psi \equiv -\Phi + \Phi_0$ and relative energy $\mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2$. In general one chooses Φ_0 such that $f > 0$ for $\mathcal{E} > 0$ and $f = 0$ for $\mathcal{E} \leq 0$

Now consider a self-consistent, spherically symmetric system with $f = f(\mathcal{E})$. Here self-consistent means that the potential is due to the system itself, i.e.,

$$\boxed{\nabla^2\Psi = -4\pi G\rho = -4\pi G \int f(\mathcal{E}) \mathrm{d}^3\vec{v}}$$

(note the minus sign in the Poisson equation), which can be written as

$$\boxed{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = -16\pi^2 G \int_0^\Psi f(\mathcal{E})\sqrt{2(\Psi-\mathcal{E})}\,\mathrm{d}\mathcal{E}}$$

Note: Here we have chosen Φ_0 so that $\Psi(r \to \infty) = 0$. In systems with infinite total mass, such as the logarithmic potential or the isothermal sphere, the system is more conveniently normalized such that $\Psi(r \to \infty) = -\infty$. In that case $\int_0^\Psi \mathrm{d}\mathcal{E}$ needs to be replace by $\int_{-\infty}^\Psi \mathrm{d}\mathcal{E}$.

Isotropic Spherical Models III

This relation may be regarded either as non-linear equation for $\Psi(r)$ given $f(\mathcal{E})$, or as linear equation for $f(\mathcal{E})$ given $\Psi(r)$.

"from ρ to f"

As an example, consider a stellar-dynamical system with a DF

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma_0^2)^{3/2}} \exp\left(\frac{\Psi - \frac{1}{2}v^2}{\sigma_0^2}\right)$$

The corresponding density is

$$\rho(\Psi) = 4\pi \int_0^{\sqrt{2\Psi}} f(\mathcal{E}) v^2 \mathrm{d}v = 4\pi \int_0^{\Psi} f(\mathcal{E}) \sqrt{2(\Psi - \mathcal{E})}\,\mathrm{d}\mathcal{E} = \rho_1 \mathrm{e}^{\Psi/\sigma_0^2}$$

The Poisson equation reads

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = -4\pi G \rho_1 \mathrm{e}^{\Psi/\sigma_0^2} \quad \Rightarrow \quad \frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\frac{4\pi G \rho_1}{r^2} \int_0^r r^2 \mathrm{e}^{\Psi/\sigma_0^2}\,\mathrm{d}r$$

Inspection shows that the solution for $\Psi(r)$ and the corresponding $\rho(r)$ are

$$\Psi(r) = -2\sigma_0^2 \ln r \qquad \rho(r) = \frac{\sigma_0^2}{2\pi G r^2}$$

which is the potential-density pair of a singular isothermal sphere.

Isotropic Spherical Models IV

Note that the DF of the singular isothermal sphere implies that

$$f(v) \propto \mathrm{e}^{-\frac{v^2}{2\sigma_0^2}}$$

which is identical to a Maxwell-Boltzmann distribution, if we set $\sigma_0^2 = \frac{k_B T}{m}$. Therefore, the structure of a singular isothermal sphere is identical to that of an isothermal self-gravitating sphere of gas.

The isothermal nature of this system becomes evident if we consider the Jeans equation. For a system with $f = f(\mathcal{E})$ there is only one non-trivial Jeans equation:

$$\frac{1}{\rho}\frac{\mathrm{d}\rho\sigma^2}{\mathrm{d}r} = \frac{\mathrm{d}\Psi}{\mathrm{d}r}$$

where $\sigma^2 \equiv \langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$.

Substituting the expressions for ρ and Ψ this yields

$$\boxed{\sigma^2(r) = \sigma_0^2}$$

thus the local velocity dispersion, which is related to the "temperature", is independent of r.

Eddington's Formula

"from f to ρ"

Using that Ψ is a monotonic function of r, so that ρ can be regarded as a function of Ψ, we have

$$\rho(\Psi) = \int f \mathrm{d}^3\vec{v} = 4\pi \int_0^\Psi f(\mathcal{E})\sqrt{2(\Psi - \mathcal{E})}\mathrm{d}\mathcal{E}$$

differentiating both sides with respect to Ψ yields

$$\frac{1}{\sqrt{8}\pi}\frac{\mathrm{d}\rho}{\mathrm{d}\Psi} = \int_0^\Psi \frac{f(\mathcal{E})\,\mathrm{d}\mathcal{E}}{\sqrt{\Psi - \mathcal{E}}}$$

which is an Abel integral equation, whose solution is

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}}\int_0^\mathcal{E}\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E} - \Psi}}$$

This is called Eddington's formula, which may also be written in the form

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^\mathcal{E}\frac{\mathrm{d}^2\rho}{\mathrm{d}\Psi^2}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E} - \Psi}} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\right)_{\Psi=0}\right]$$

Eddington's Formula

Given a spherically symmetric density distribution, which can be written as $\rho = \rho(\Psi)$ (which is not always possible), Eddington's formula yields a corresponding DF $f = f(\mathcal{E})$.

Note, however, that there is no guarantee that the solution for $f(\mathcal{E})$ satisfies the physical requirement that $f \geq 0$ for all $\mathcal{E}$.

Using Eddington's formula

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \int_0^{\mathcal{E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi} \frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}$$

we see that the requirement $f(\mathcal{E}) \geq 0$ is identical to the the requirement that the function

$$\int_0^{\mathcal{E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi} \frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}$$

is an increasing function of $\mathcal{E}$.

If a density distribution $\rho(r)$ does not satisfy this requirement, then the model obtained by setting the anisotropy parameter $\beta = 0$ [i.e., by assuming that $f = f(\mathcal{E})$] and solving the Jeans Equations is unphysical.

Anisotropic Spherical Models I

In the more general case, spherical systems (with spherical symmetry in all their properties) have $f = f(E, L^2)$.

These models are anisotropic, in that $\langle v_r^2 \rangle \neq \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$.

Anisotropic spherical models are non-unique: many different $f(E, L^2)$ can correspond to a given $\rho(r)$ and $\Psi(r)$. These models differ, though, in their dynamic properties. No equivalent of the Edddington Formula thus exists, that allows to compute $f(E, L^2)$ given $\rho(r)$.

Additional assumptions need to be made. For example, Kent & Gunn (1982) discussed models with $f(E, L^2) = g(E) L^{-2\beta}$, which have a constant anisotropy, i.e., $\beta(r) = \beta$.

An other example are the so called Osipkov-Merritt models (Osipkov 1979; Merritt 1985) were the assumption is made that $f(E, L^2) = f(Q)$ with

$$Q = \mathcal{E} - \frac{L^2}{2r_a^2}$$

Here r_a is the so-called anisotropy radius.

Anisotropic Spherical Models II

The usefulness of the Osipkov-Merritt models becomes apparent from

$$\rho_Q(r) \equiv \left(1+\frac{r^2}{r_a^2}\right)\rho(r) = 4\pi\int_0^\Psi f(Q)\sqrt{2(\Psi-Q)}\,\mathrm{d}Q$$

Thus $[\rho_Q(r), f(Q)]$ are similarly related as $[\rho(r), f(\mathcal{E})]$ so that we may use Eddington's formula to write

$$f(Q) = \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}Q}\int_0^Q \frac{\mathrm{d}\rho_Q}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{Q-\Psi}}$$

For Osipkov-Merritt models one can show that

$$\beta(r) = \frac{r^2}{r^2+r_a^2}$$

Thus, these models are isotropic for $r \ll r_a$, become radially anisotropic at around r_a, and become competely radial at large r.

Since purely radial orbits contribute density at the center, models with constant density cores can only have DFs of the Osipkov-Merritt form, i.e., $f = f(E, L^2) = f(Q)$, for sufficiently large r_a. Alternatively, if r_a is relatively small, the (self-consistent) $\rho(r)$ needs to have a central cusp.

Anisotropic Spherical Models III

Next we consider the family of Quasi-Separable DFs (Gerhard 1991):

$$f(\mathcal{E}, L^2) = g(\mathcal{E})\, h(x) \qquad\qquad x = \frac{L}{L_0 + L_c(\mathcal{E})}$$

with L_0 a constant, and $L_c(\mathcal{E})$ the angular momentum of the circular orbit with energy $\mathcal{E}$.

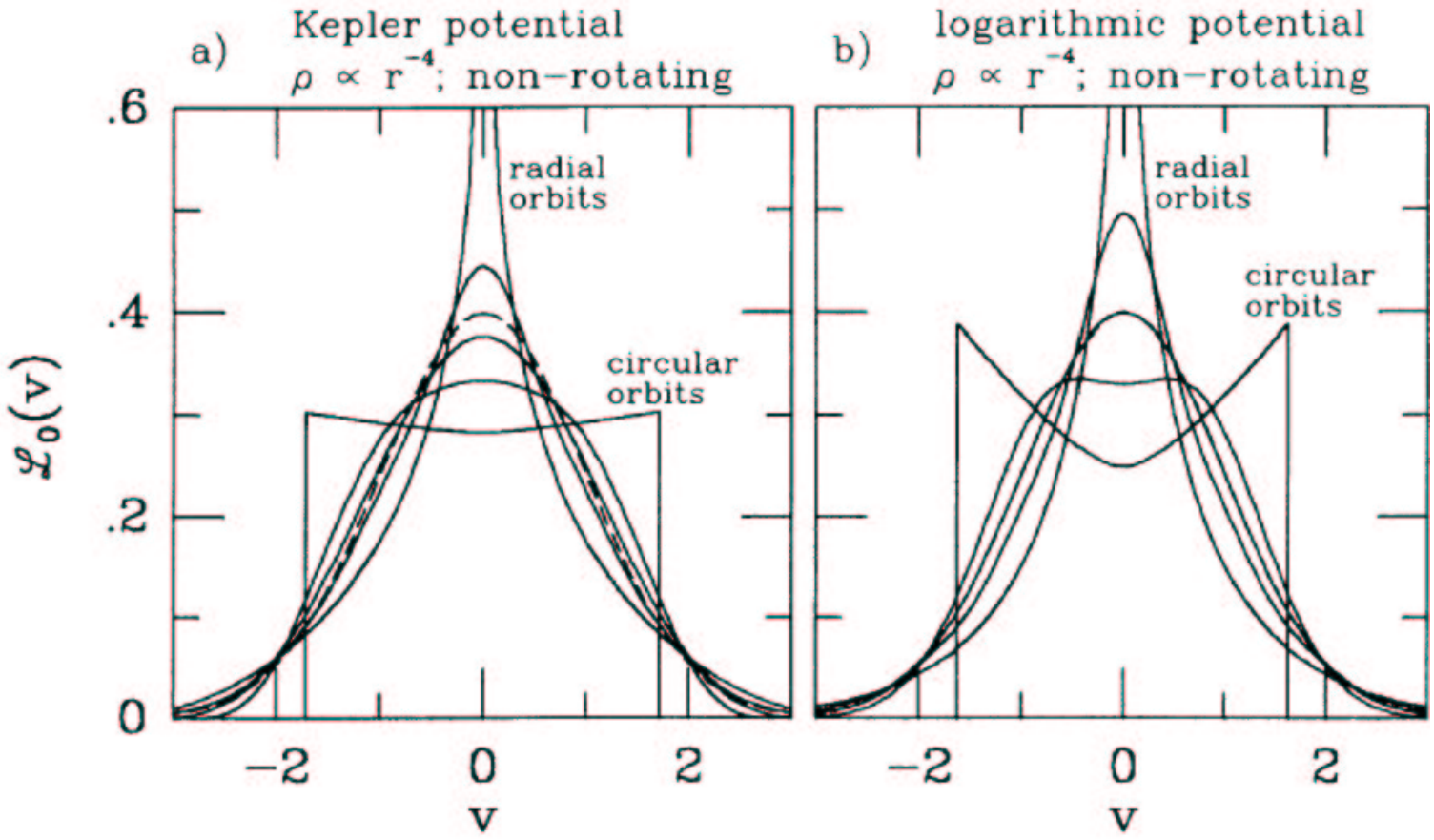
Here $g(\mathcal{E})$ controls the distribution of stars between energy surfaces, while the circularity function $h(x)$ describes the distribution of stars over orbits of different angular momenta on surfaces of constant $\mathcal{E}$.

Depending on the choise for $h(x)$ one can construct models with different anisotropies. If $h(x)$ decreases with increasing x, the model will be radially anisotropic, and vice versa.

Once a choise for $h(x)$ is made, one can (numerically) obtain $g(\mathcal{E})$ for a given $\rho(r)$.

All these various models are useful to explore how different orbital anisotropies impact on observables, such as the line-of-sight velocity distributions (LOSVDs).

Anisotropic Spherical Models IV



Velocity profiles, $\mathcal{L}(v)$, for the outer parts of spherical $f(E, L^2)$ models. Results are shown for $\beta = \infty$ (circular orbits), -1, 0 (isotropic model), 0.5, and 1 (radial orbits). The unit of velocity is the velocity dispersion, which is different for each curve.

(from: van der Marel & Franx 1993)

Velocity profiles are not expected to be Gaussian

Spherical Models: Summary I

In its most general form, the DF of a static, spherically symmetric model has the form $f = f(E, \vec{L})$. From the symmetry of individual orbits one can see that one always has to have

$$\langle v_r \rangle = \langle v_\theta \rangle = 0 \qquad \langle v_r v_\phi \rangle = \langle v_r v_\theta \rangle = \langle v_\theta v_\phi \rangle = 0$$

This leaves four unknowns: $\langle v_\phi \rangle$, $\langle v_r^2 \rangle$, $\langle v_\theta^2 \rangle$, and $\langle v_\phi^2 \rangle$

If one makes the assumption that the system is spherically symmetric in all its properties then $f(E, \vec{L}) \rightarrow f(E, L^2)$ and

$$\langle v_\phi \rangle = 0 \qquad \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$$

In this case the only non-trivial Jeans equation is

$$\frac{1}{\rho}\frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + 2\frac{\beta\langle v_r^2\rangle}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

with the anisotropy parameter defined by

$$\beta(r) = 1 - \frac{\langle v_r^2\rangle(r)}{\langle v_r^2\rangle(r)}$$

Spherical Models: Summary II

Many different models, with different orbital anisotropies, can correspond to the same density distribution. Examples of models are:

- $f(E, L^2) = f(E)$ — **isotropic model, i.e., $\beta = 0$**
- $f(E, L^2) = g(E)\delta(L)$ — **radial orbits only, i.e. $\beta = 1$**
- $f(E, L^2) = g(E)\delta[L - L_c(E)]$ — **circular orbits only, i.e., $\beta = -\infty$**
- $f(E, L^2) = g(E)L^{-2\beta}$ — **constant anisotropy, i.e. $\beta(r) = \beta$**
- $f(E, L^2) = g(E)h(L)$ — **anisotropy depends on circularity function h**
- $f(E, L^2) = f(E + L^2/2r_a^2)$ — **center isotropic, outside radial**

Suppose I have measured the surface brightness profile $\Sigma(R)$ and the line-of-sight velocity dispersion $\sigma_p^2(R)$. Depending on the assumption regarding $\beta(r)$ these data imply very different mass distributions $M(r)$. One can (partially) break this mass-anisotropy degeneracy by using information regarding the LOSVD shapes.

Axisymmetric Models

Next we consider **axisymmetric** systems. If we only consider systems for which most orbits are regular, then the **strong Jeans Theorem** states that, in the most general case, $f = f(E, L_z, I_3)$.

From the symmetries of the individual orbits, it is evident that in this case

$$\langle v_R \rangle = \langle v_z \rangle = 0 \qquad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0$$

Note that, in this case, $\langle v_R v_z \rangle \neq 0$, which is immediately evident when considering a **thin tube orbit**. In other words, in general the **velocity ellipsoid** is not aligned with (R, ϕ, z).

Thus, in a three-integral model with $f = f(E, L_z, I_3)$ the **stress tensor** contains four unknowns: $\langle v_R^2 \rangle$, $\langle v_\phi^2 \rangle$, $\langle v_z^2 \rangle$, and $\langle v_R v_z \rangle$.

In this case there are two non-trivial Jeans Equations:

$$\frac{\partial(\rho\langle v_R^2\rangle)}{\partial R} + \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z} + \rho\left[\frac{\langle v_R^2\rangle - \langle v_\phi^2\rangle}{R} + \frac{\partial\Phi}{\partial R}\right] = 0$$
$$\frac{\partial(\rho\langle v_R v_z\rangle)}{\partial R} + \frac{\partial(\rho\langle v_z^2\rangle)}{\partial z} + \rho\left[\frac{\langle v_R v_z\rangle}{R} + \frac{\partial\Phi}{\partial z}\right] = 0$$

which clearly doesn't suffice to solve for the four unknowns.

$f(E, L_z)$ Models I

To make progress, one therefore often makes the additional assumption that the DF has the two-integral form $f = f(E, L_z)$.

In this case we have that $\rho = \int_{v^2<2\Psi} f(\Psi - \frac{1}{2}v^2, Rv_\phi)\mathrm{d}^3\vec{v}$

where we have imposed the usual condition $f = 0$ for $\mathcal{E} < 0$.

Let $\vec{v}_m$ be the meridional component of $\vec{v}$ and define the cylindrical coordinates (v_m, v_ϕ, ψ) in velocity space, with

$$v_R = v_m \cos\psi\,, \qquad v_z = v_m \sin\psi$$

then

$$\rho = \int_0^{2\pi} \mathrm{d}\psi \int_0^{\sqrt{2\Psi}} v_m \mathrm{d}v_m \int_{v_\phi^2<(2\Psi - v_m^2)} f[\Psi - \tfrac{1}{2}(v_m^2 + v_\phi^2), Rv_\phi]\mathrm{d}v_\phi$$

Note that one can see now that

$$\langle v_R v_z \rangle = \tfrac{1}{\rho}\int_{v^2<2\Psi} v_R v_z f(\Psi - \tfrac{1}{2}v^2, Rv_\phi)\mathrm{d}^3\vec{v} = 0$$

which follows from the fact that $\int_0^{2\pi} \sin\psi\, \cos\psi\, \mathrm{d}\psi = 0$. Thus, models with $f = f(E, L_z)$ have their velocity ellipsoid aligned with (R, ϕ, z).

$f(E, L_z)$ Models II

In addition, since $\int_0^{2\pi} \sin^2\psi \mathrm{d}\psi = \int_0^{2\pi} \cos^2\psi \mathrm{d}\psi = \pi$ we have that

$$\langle v_R^2 \rangle = \langle v_z^2 \rangle = \pi \int_0^{\sqrt{2\Psi}} v_m^3 \mathrm{d}v_m \int_{v_\phi^2 < (2\Psi - v_m^2)} f[\Psi - \tfrac{1}{2}(v_m^2 + v_\phi^2), R v_\phi] \mathrm{d}v_\phi$$

Thus, we have that

$$f = f(E, L_z) \quad \Longrightarrow \quad \langle v_R^2 \rangle = \langle v_z^2 \rangle \text{ and } \langle v_R v_z \rangle = 0$$

Now we have two unknowns left, $\langle v_R^2 \rangle$ and $\langle v_\phi^2 \rangle$, and the Jeans equations reduce to

$$\frac{\partial(\rho \langle v_R^2 \rangle)}{\partial R} + \rho \left[\frac{\langle v_R^2 \rangle - \langle v_\phi^2 \rangle}{R} + \frac{\partial \Phi}{\partial R} \right] = 0$$
$$\frac{\partial(\rho \langle v_z^2 \rangle)}{\partial z} + \rho \frac{\partial \Phi}{\partial z} = 0$$

which can be solved. Note, however, that the Jeans equations provide no information regarding how $\langle v_\phi^2 \rangle$ splits in streaming and random motions.

In practice one often follows Satoh (1980), and writes that $\langle v_\phi \rangle^2 = k \left[\langle v_\phi^2 \rangle - \langle v_R^2 \rangle \right]$. Here k is a free parameter, and the model is isotropic for $k = 1$.

$f(E, L_z)$ Models III

Now let us return to our expression for the density ρ. If we change the variables of integration from (v_m, v_ϕ) to $(\mathcal{E}, \Psi)$, we obtain

$$\begin{aligned}
\rho &= \frac{2\pi}{R}\int_0^\Psi \mathrm{d}\mathcal{E}\int_{L_z^2<2(\Psi-\mathcal{E})R^2} f(\mathcal{E}, L_z)\mathrm{d}L_z \\
&= \frac{2\pi}{R}\int_0^\Psi \mathrm{d}\mathcal{E}\int_0^{R\sqrt{2(\Psi-\mathcal{E})}} \left[f(\mathcal{E}, L_z) + f(\mathcal{E}, -L_z)\right]\mathrm{d}L_z \\
&= \frac{4\pi}{R}\int_0^\Psi \mathrm{d}\mathcal{E}\int_0^{R\sqrt{2(\Psi-\mathcal{E})}} f_+(\mathcal{E}, L_z)\mathrm{d}L_z
\end{aligned}$$

where we have defined f_+ as the part of the DF that is even in L_z, i.e.,

$$\begin{aligned}
f(\mathcal{E}, L_z) &= f_+(\mathcal{E}, L_z) + f_-(\mathcal{E}, L_z) \\
f_\pm(\mathcal{E}, L_z) &\equiv \tfrac{1}{2}\left[f(\mathcal{E}, L_z) \pm f(\mathcal{E}, -L_z)\right]
\end{aligned}$$

We thus see that the density depends only on the even part of the DF (i.e., the density contributed by a star does not depend on its sense of rotation). This also implies that there are infinitely many DFs $f(E, L_z)$ that correspond to exactly the same $\rho(R, z)$, namely all those that only differ in $f_-(\mathcal{E}, L_z)$.

$f(E, L_z)$ Models IV

Thus, given a density distribution $\rho(R, z)$, one can in principle obtain a unique $f_+(\mathcal{E}, L_z)$. In practice, the computation of $f_+(\mathcal{E}, L_z)$ was considered very difficult.

It was thought that one needs to (i) be able to express ρ explicitely as a function of R and Ψ, and (ii) perform a complicated integral transform.

However, this situation changed drastically when Hunter & Qian (1993) derived an axisymmetric analogue of Eddington's Formula based on a complex contour integral, which does not require explicit knowledge of $\rho(R, \Psi)$.

In addition, Evans (1993, 1994) has found a large and useful family of models for which all relevant calculations can be done analytically. This is the family of power-law models, which we already encoutered in the exersizes.

So for a wide range of $\rho(R, z)$, we can compute the unique, corresponding $f_+(\mathcal{E}, L_z)$. But what about the odd part of the DF, $f_-(\mathcal{E}, L_z)$?

While $f_-(\mathcal{E}, L_z)$ has no influence on the density distribution, it specifies the asymmetry between clockwise and counter-clockwise orbits. Hence, it is responsible for the mean streaming velocity.

$f(E, L_z)$ Models V

In fact, it is straightforward to show that

$$\langle v_\phi \rangle = \frac{4\pi}{\rho R^2} \int_0^{\Psi} \mathrm{d}\mathcal{E} \int_0^{R\sqrt{2(\Psi-\mathcal{E})}} f_-(\mathcal{E}, L_z)\, L_z\, \mathrm{d}L_z$$

In principle, if we were to know $\langle v_\phi \rangle(R, z)$, we could solve for $f_-(\mathcal{E}, L_z)$ using the Hunter & Qian complex contour integral method, just like we can recover $\rho(R, z)$ from $f_+(\mathcal{E}, L_z)$.

In practice, the observationally accessible quantities are $\Sigma(x, y)$ and $v_{\rm los}(x, y)$. Unless the system is seen edge-on, one can not uniquely deproject these for $\rho(R, z)$ and $\langle v_\phi \rangle(R, z)$.

$f(E, L_z)$ Models VI

Most $f(E, L_z)$-modelling therefore uses the following methodology:

(1a) Assume functional form for $\nu(R, z)$ and value for inclination angle i. Project ν along line-of-sight and adjust free parameters by fitting $\Sigma(x, y)$.

(1b) Assume value for i and deproject $\Sigma(x, y)$ using some assumptions. Examples are the Richardson-Lucy algorithm (Binney, Davies & Illingworth 1990) and the Multi-Gaussian-Expansion method (Emsellem, Monnet & Bacon 1994). Warning, unless $i = 90^o$ these are not unique!

(2) Assume mass-to-light ratio, $\Upsilon(R, z)$, and compute $\Psi(R, z)$ from $\rho(R, z) = \Upsilon(R, z)\nu(R, z)$ using the Poisson equation.

(3) Solve Jeans Equations for $\langle v_R^2 \rangle(R, z)$ and $\langle v_\phi^2 \rangle(R, z)$.

(4) Make assumptions regarding split of $\langle v_\phi^2 \rangle$ in streaming motion and random motion (i.e., pick a value for the Satoh k-parameter).

(5) Project model and compare resulting $v_{\rm los}(x, y)$ and $\sigma_{\rm los}(x, y)$ to data obtained from spectroscopy.

(6) Repeat analysis to constrain i, k and $\Upsilon(R, z)$.

For examples, see Binney, Davies & Illingworth (1990), van der Marel (1991), Cretton & van den Bosch (1999)

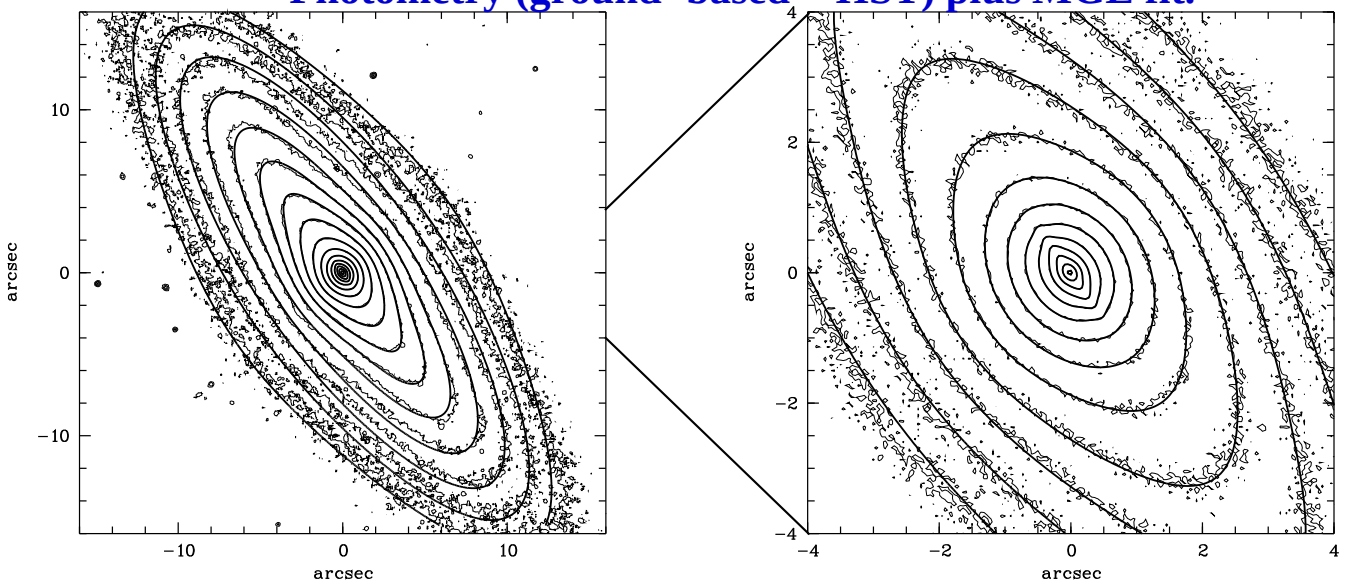
$f(E, L_z)$ Models VII

The results of $f(E, L_z)$ modeling can be summarized as follows

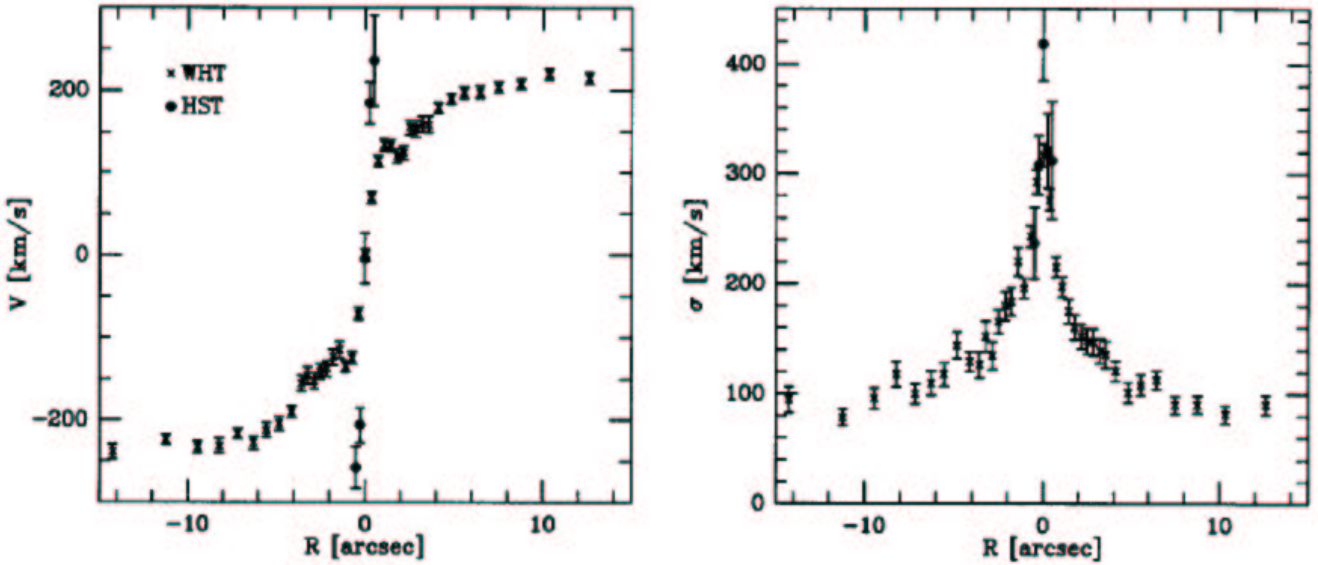
- **Because $\sigma_R = \sigma_z$, flattening of oblate models must come from large ϕ-motions; f must be biased towards high-L_z orbits.**
- **Prolate models require a deficit of high L_z orbits. Strongly elongated, prolate systems with $f = f(E, L_z) > 0$ do not exist.**
- **Isotropic oblate models, i.e. those with $k = 1$, in general give poor fits to the data.**
- **Anisotropic models fit some galaxies, but not all; those must have $f(E, L_z, I_3)$ or be triaxial.**
- **There is a degeneracy between the mass-to-light ratio and the anisotropy (e.g., Binney & Mammon 1982). Can be broken by using higher-order Jeans Equations plus observational constraints on LOSVD shape (e.g., skewness & kurtosis).**
- **Several studies have used these models to argue in favor of massive black holes. However, if $f(E, L_z)$ model can only fit data with BH, this is still no proof: need to show that $f \geq 0$, and that no $f(E, L_z, I_3)$ models without BH can fit data equally well.**

Example: NGC 4342

Photometry (ground−based + HST) plus MGE fit.

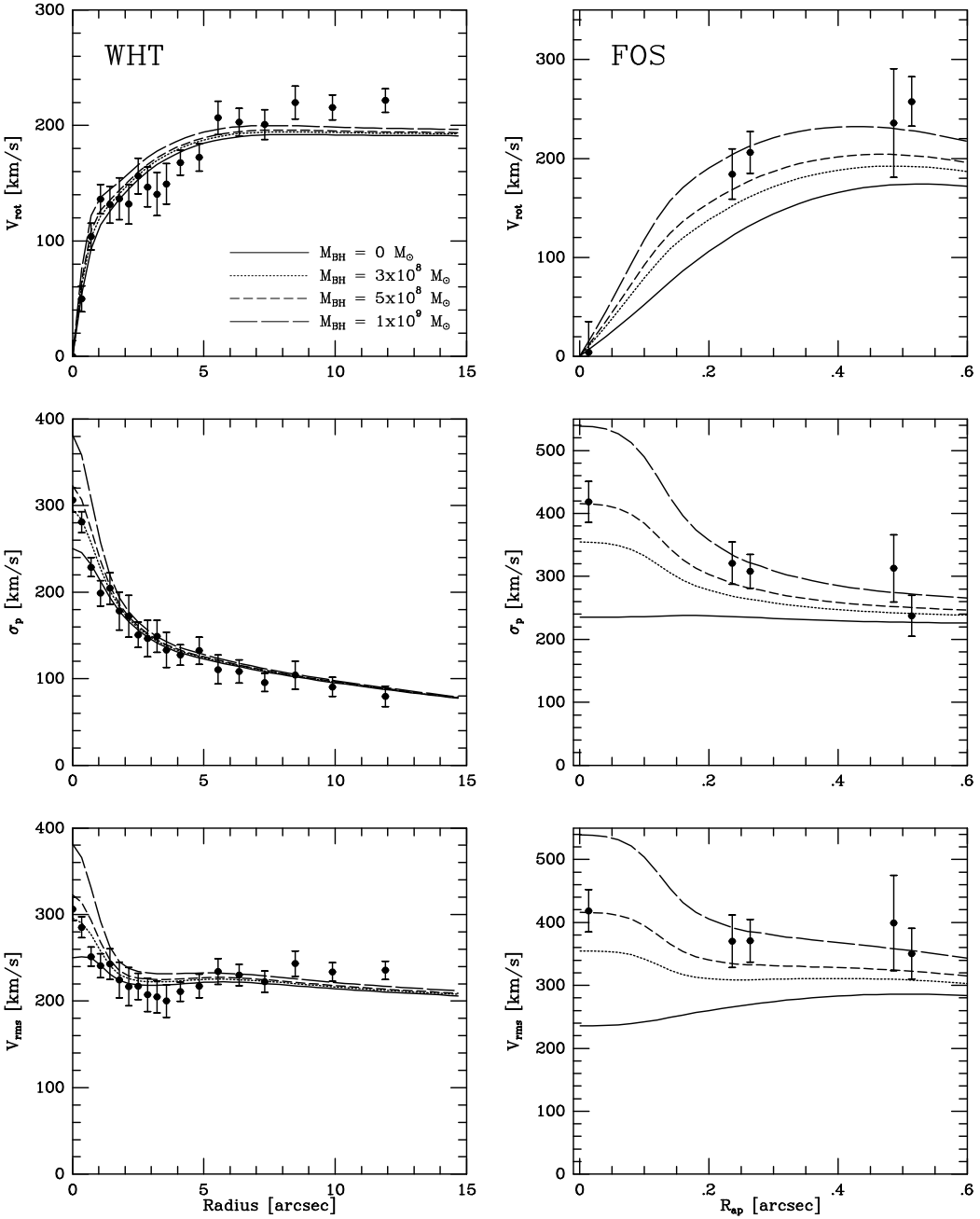


Rotation velocities and velocity dispersions along major axis (WHT + HST)



(from: Cretton & van den Bosch 1999)

Example: NGC 4342



Jeans Models

$$k = 1, i = 90^o$$

σ_p in center requires BH

Note: $v_{\rm rms}$ is not well fitted

This is independent of the assumed value for k, which only determines how $v_{\rm rms}$ splits in $v_{\rm rot}$ and σ_p

$$\Rightarrow f = f(E, L_z, I_3)$$

(Cretton & van den Bosch 1999)

Three-Integral Models

In our discussion on orbits we have seen that most orbits in realistic, axisymmetric galaxy potentials are regular, quasi-periodic, and confined to the surfaces of invariant tori.

Therefore, most orbits admit three isolating integrals of motion in involution. According to strong Jeans Theorem, we thus expect $f = f(E, L_z, I_3)$.

In this case the two non-trivial Jeans equations have four unknowns and can not be solved. In addition, there is no equivalent of Eddington's formula to obtain "f from ρ".

In the past, axisymmetric three integral models have been constructed using

- special, separable potentials (Stäckel potentials), for which I_3 is known exactly (Bishop 1986; de Zeeuw & Hunter 1990)
- approximate third integrals (Petrou 1983). The most detailed work along this direction is due to Dehnen & Gerhard (1993), who evaluated the approximate I_3 from resonant perturbation theory.
- orbit superposition techniques. These are based on integrating large numbers of orbits, and then to find the combination of orbits that best matches the data (Schwarzschild 1979; Richstone 1984; Cretton et al. 1999). This method has recently received much attention.

Three-Integral Models

Schwarzschild's orbit superposition technique for modelling axisymmetric galaxies is based on the following steps:

(1) Use techniques described above to obtain model for 3D light distribution $\nu(R,z)$, under an assumed value for the inclination angle i.

(2) Assume a value for the stellar mass-to-light ratio Υ, and compute the corresponding potential $\Psi(R,z)$ from the Poisson equation. To this potential one may add that of a central BH and/or a dark matter halo.

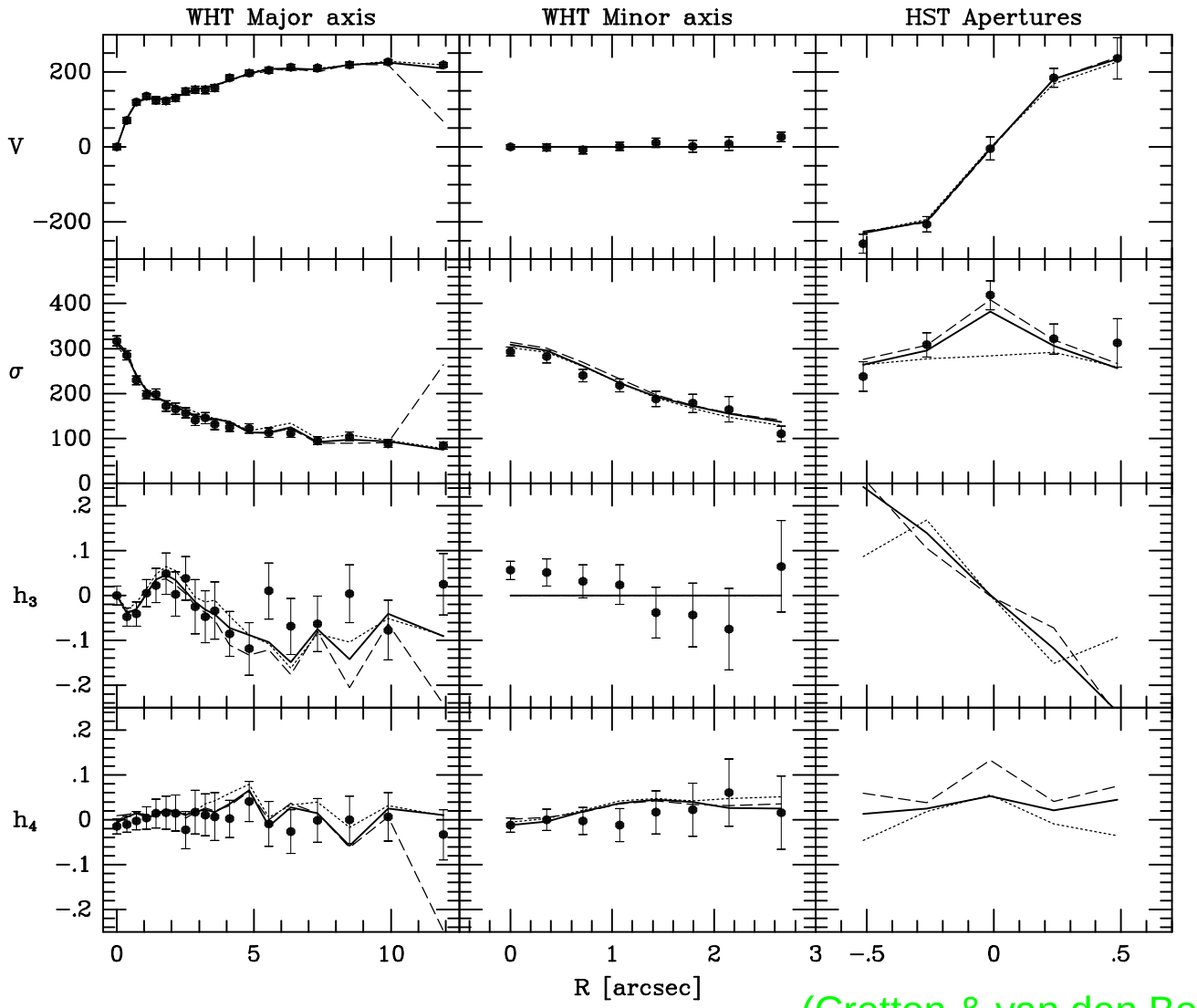
(3) Integrate a large sample of orbits in the total potential. Make sure to cover full allowed ranges of E, L_z, and I_3: sampling (E,L_z)-space is trivial, while I_3 is sampled by location of turning point on zero-velocity curve.

(4) Compute for each orbit k it's contribution a_{kj} to each observable j, such as the value of the velocity profile $\mathcal{L}(v_j)$ at the projected location (x_j,y_j).

(5) Find the orbital weights $w_k \geq 0$ that minimize the quantity $\sum_j \left[\mathcal{L}(x_j,y_j,v_j) - \sum_k w_k a_{kj}\right]^2$. Since the number of orbits is typically much larger than the number of observational constraints, this is typically done using a Non-Negative Least Squares algorithm.

(6) Repeat entire analysis for different i, Υ, $M_{\rm BH}$, etc. in order to constrain these free parameters.

Example: NGC 4342



(Cretton & van den Bosch 1999)

Models without BH are now clearly ruled out. Note though, that unlike the 2I Jeans models, the 3I models can accurately fit the ground-based $V(R)$ and $\sigma(R)$. This owes to the larger amount of freedom of these models.

Triaxial Systems I

The dynamics of triaxial systems is much more complicated than that of axisymmetric or spherical systems. The main reasons are the lack of symmetry, and the presence of four main orbit families (as opposed to one).

Motivation for considering Triaxial Systems

- **Slow rotation of (massive) ellipticals (Bertola & Capaccioli 1975; Illingworth 1977) implies that they are inconsistent with isotropic oblate rotators. They are supported by anisotropic pressure, and, as argued by Binney (1978), are therefore more likely to be triaxial then axisymmetric.**

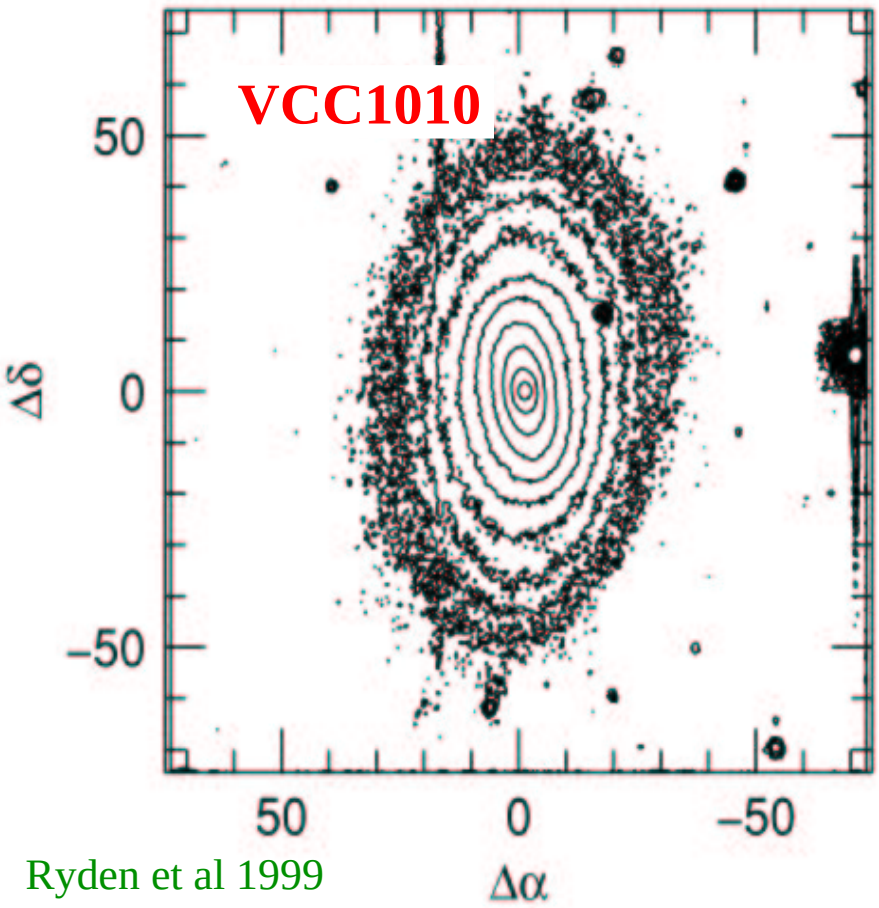
- **Some ellipticals reveal zero-velocity curves that are misaligned with principal axes. This implies triaxiality, as the presence of both long- and short-axis tubes means that the total angular momentum vector may point anywhere in the plane containing both the long and the short axes (Franx, Illingworth & de Zeeuw 1991).**

- **Some galaxies reveal isophotal twists, in which the position angle of the isophotes changes with radius. This has a most natural explanation if these systems are triaxial with axis ratios that vary with radius (e.g., Stark 1977)**

- **Numerical simulations show that collapsed haloes are often triaxial (van Albada 1982; Warren et al. 1992)**

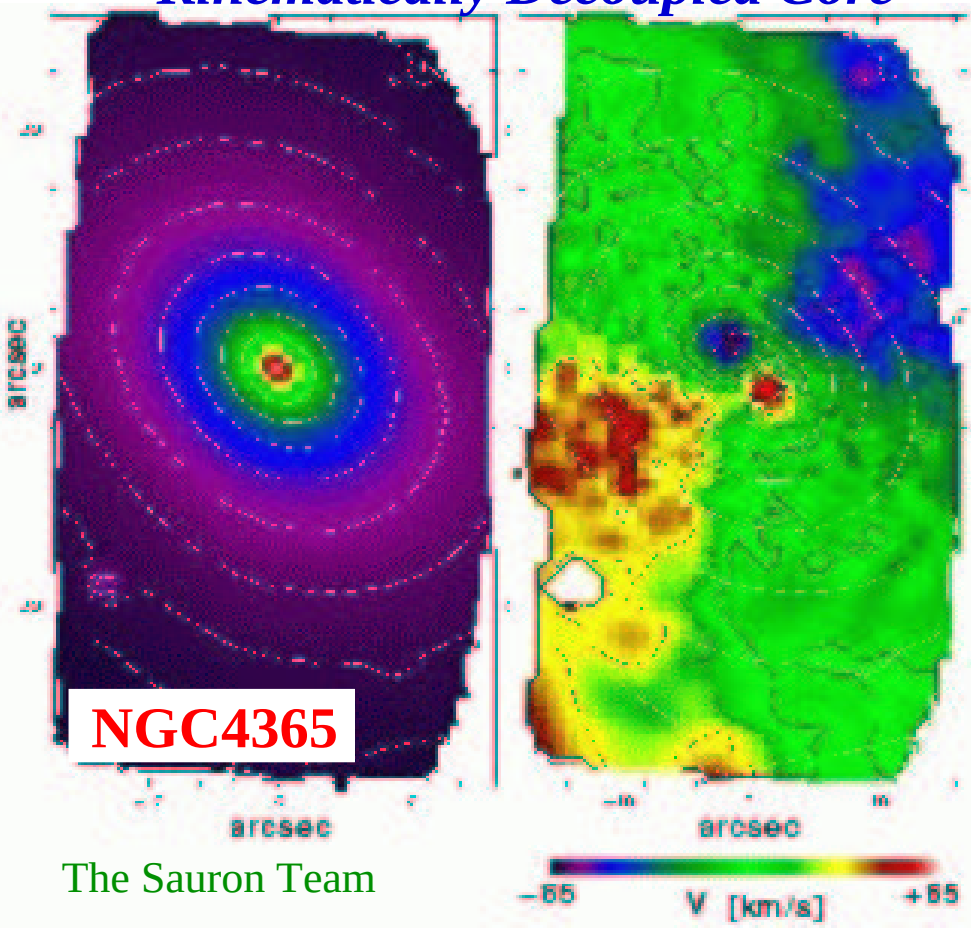
Observational Hints for Triaxiallity

Isophotal Twist



Ryden et al 1999

Kinematically Decoupled Core



The Sauron Team

Triaxial Systems II

But, case for triaxiality may not be that strong:

- **N-body simulations that include a dissipative component often reveal evolution towards axisymmetry (Udry 1993; Dubinsky 1994)**
- **Pressence of central BH tends to drive system towards axisymmetry (Norman, May & van Albada 1985; Merritt & Quinlan 1998). In both cases, this is due to fact that box orbits become stochastic in steep potential.**
- **In realistic systems with steep central cusp large fraction of phase-space is occupied by stochastic orbits. Precludes stationary triaxial solutions (Schwarzschild 1993; Merritt 1997).**
- **Anistropic pressure in axisymmetric systems can also explain slow-rotators if system has sufficient amount of counter-rotation. Some axisymmetric systems like this are known (e.g., NGC 4550; Rix et al. 1992).**
- **Low luminosity ellipticals, in general, lack isophotal twists, are strongly cusped, and are rotationally supported: they are perfectly consistent with being axisymmetric (Bender et al. 1989; Ferrarese et al. 1994; Faber et al. 1997).**

Self-Consistent Triaxial Models

In triaxial systems that are close to separable (i.e., most orbits are regular), the strong Jeans Theorem implies that $f = f(E, I_2, I_3)$.

The seminal work of Schwarzschild (1979, 1982) showed that self-consistent models of triaxial systems exist, both with stationary figures and with slowly tumbling figures. To this extent he used orbit superposition techniques.

In particular, Schwarzschild's work has shown that many different orbital configurations, i.e., many different $f(E, I_2, I_3)$, can produce the same density distribution. However, these can have very different kinematical structures (Statler 1987, 1991).

The orbit superposition technique only works well when most orbits are regular (in separable potentials, or in potentials with large cores).

In more realistic systems with central cusps, large fraction of phase-space is occupied by stochastic orbits. These are difficult to deal with in orbit-superposition techniques: depends on rate of stochastic diffusion.

Most work has focussed on class of separable (Stäckel) potentials (Kuzmin 1973; de Zeeuw 1985) and on scale-free potentials (Gerhard & Binney 1985). Probably these are not very realistic, but they provide useful insight

Kinematics I

We now focus shortly on how to extract kinematic information from spectra of galaxies.

The spectrum at a given location (x, y) of the projected galaxy is the sum of the spectra of all stars along the line-of-sight (los) Doppler shifted according to their velocity along the los

For a source with los-velocity v the observed and emitted frequencies are related according to

$$\boxed{\nu_{\rm obs} = (1-\beta)\gamma\nu_{\rm em}}$$

with $\beta \equiv v/c$ and $\gamma \equiv (1-\beta^2)^{-1/2}$. To lowest order in v/c we have that $\gamma = 1$ and thus

$$\nu_{\rm obs} = \left(1 - \frac{v}{c}\right)\nu_{\rm em}$$

Using that the wavelength $\lambda = c/\nu$ we obtain, again to lowest order in v/c that

$$\boxed{\lambda_{\rm obs} = \left(1 + \frac{v}{c}\right)\lambda_{\rm em}}$$

Kinematics II

If we now define $x \equiv \ln\lambda$ then

$$\Delta x = \ln\left(\frac{\lambda_{\rm obs}}{\lambda_{\rm em}}\right) = \ln\left(1 + \frac{\Delta v}{c}\right) \simeq \frac{\Delta v}{c}$$

where we have used the first term of the series expansion of $\ln(1+x)$

Let $S(x)$ correspond to an observed spectrum, rebinned linearly in x. Then we may write that

$$\boxed{S(x) = T(x) \otimes B(x)}$$

Here $T(x)$ is the template spectrum, which is luminosity weighted spectrum of all the various stars along the los, $B(x)$ is the broadening function, which gives the probability distribution of the los velocities of all these stars, and $\otimes$ denotes convolution.

Since convolution in real space corresponds to multiplication in Fourier space we have that

$$\boxed{\hat{S}(k) = \hat{T}(k) \cdot \hat{B}(k)}$$

where $\hat{F}$ indicates the Fourier Transform of F, and k is the wavenumber.

Kinematics III

This gives us immediately an unparameterized method to obtain the broadening function from the observed spectrum

$$B(x) = \widehat{\left[\frac{\hat{S}}{\hat{T}}\right]}$$

This method has the advantage that no assumption is made regarding the functional form of $B(x)$. However, the disadvantage is that one needs to adopt some noise filtering which introduces correlations between different points in the $B(x)$ estimate. This complicates comparison with models.

The alternative is to use a parameterized method, by assuming a functional form $\tilde{B}(x)$ for the broadening function which has n free parameters. The best-fit parameters are obtained by minimizing

$$\chi^2 = \int \left[S(x) - \tilde{B}(x) \otimes T(x)\right]^2 \mathrm{d}x$$

Rather than talking about the broadening function $B(x)$ one often prefers to talk about the velocity profile $\mathcal{L}(v) \equiv B(v/c)$, also called the LOSVD

Kinematics IV

A typical functional form to assume for $\mathcal{L}(v)$ is a simple Gaussian

$$\mathcal{L}(v) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}w^2}$$

where $w \equiv (v - V)/\sigma$. Note that this parameterization has two free parameters: V and σ.

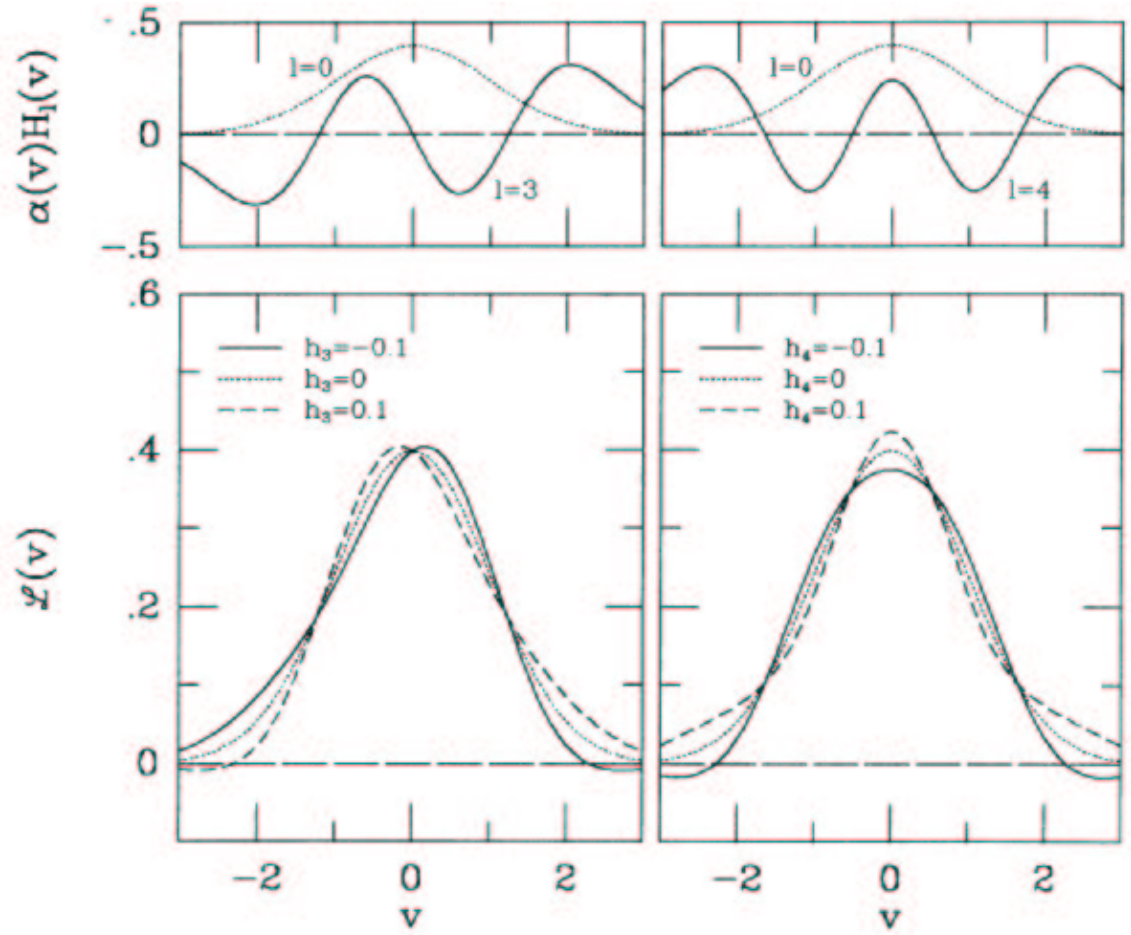
However, $\mathcal{L}(v)$ is generally not Gaussian, and a more general parameterization is required. Van der Marel & Franx (1993) and Gerhard (1993) have suggested using the Gauss-Hermite series

$$\mathcal{L}(v) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}w^2} \left[1 + \sum_{j=3}^{N} h_j \, H_j(w)\right]$$

where $H_l(x)$ are Hermite Polynomials of degree l, and h_j are additional free parameters that describe the deviation of $\mathcal{L}(v)$ from a pure Gaussian (i.e., $h_j = 0$ for a Gaussian LOSVD).

Typically one truncates the series expansion at $N = 4$ so that the LOSVD has four free parameters: V, σ, h_3, and h_4. Their best-fit values are determined by minimizing the χ^2 defined above.

Kinematics V



(from: van der Marel & Franx 1993)

Note that the odd Gauss-hermite coefficients (h_3, h_5, etc.) characterize anti-symmetric deviations of $\mathcal{L}(v)$ from a Gaussian, while the even coefficients (h_4, h_6, etc.) characterize the symmetric deviations.

The LOSVD shape contains useful information to break the degeneracy between mass and anisotropy

The Relaxation Puzzle

Relaxation: the process by which a physical system acquires equilibrium or returns to equilibium after a disturbance.

Often, but not always, relaxation erases the system's 'knowledge' of its initial conditions.

In the first lecture we defined the two-body relaxation time as the time required for a particle's kinetic energy to change by its own amount due to long-range collisions. We found that

$$t_{\rm relax} = \frac{N}{10 \ln N} t_{\rm cross}$$

For galaxies and DM haloes one always finds that $t_{\rm relax} \gg t_{\rm Hubble}$. As first pointed out by Zwicky (1939), how is it possible that galaxies appear relaxed?

In particular, if galaxies did not form near equilibrium, two-body relaxation certainly did not have the time to get them there. This would imply that galaxies all form in (virial) equilibrium, which seems very contrived.

The solution to this puzzle is that alternative relaxation mechanisms exist.

Relaxation Mechanisms

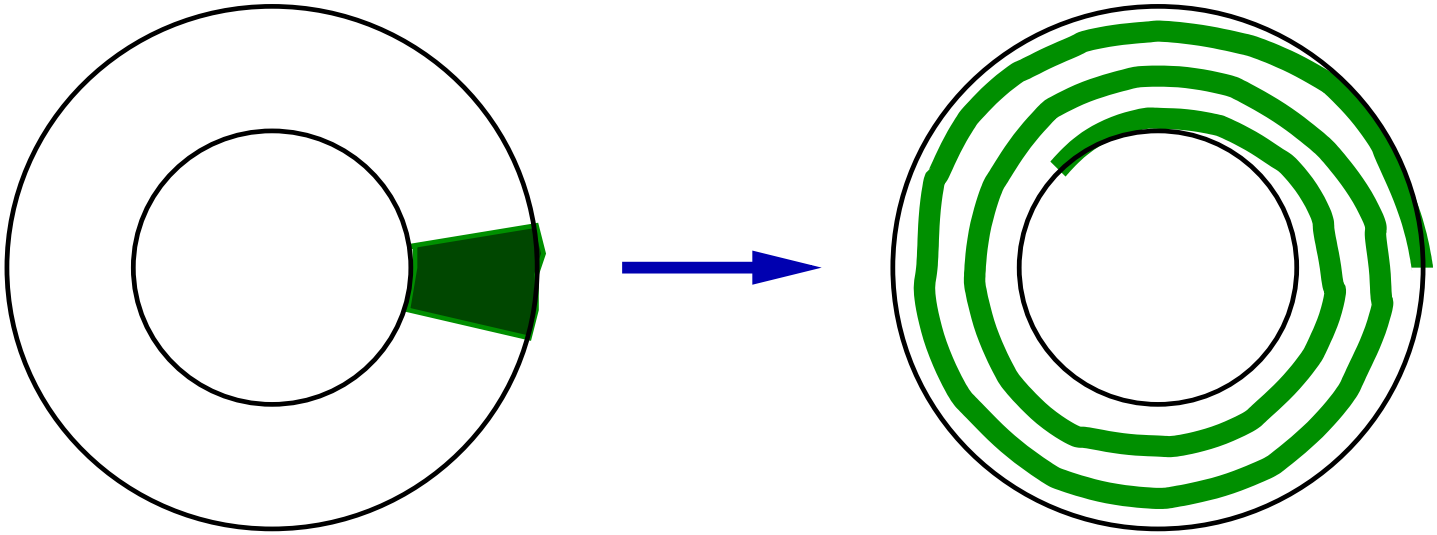
In gravitational N-body systems the four most important relaxation mechanisms are:

- **Phase Mixing The spreading of neighboring points in phase-space due to the difference in frequencies between neighboring tori.**
- **Chaotic Mixing The spreading of neighboring points in phase-space due to the chaotic nature of their orbits.**
- **Violent Relaxation The change of energy of individual particles due to the change of the overall potential.**
- **Landau Damping The damping (and decay) of perturbations in the density and/or potential.**

We will discuss each of these in turn.

As we will see later, Violent Relation and Landau Damping are basically specific examples of Phase Mixing!!.

Phase Mixing I



Consider circular motion in a disk with $V_c(R) = V_0 = \text{constant}$. The frequency of a circular orbit at radius R is then

$$\omega = \frac{1}{T} = \frac{V_0}{2\pi R}$$

Thus, points in the disk that are initially close will separate according to

$$\Delta\phi(t) = \Delta\left(\frac{V_0}{R}t\right) = 2\pi\Delta\omega t$$

We thus see that the timescale on which the points are mixed over their accessible volume in phase-space is of the order of

$$\boxed{t_{\rm mix} \simeq \frac{1}{\Delta\omega}}$$

Phase Mixing II

In this example the phase-mixing occurs purely in 'real space'. In the more general case, however, the phase-mixing occurs in phase-space.

Phase mixing is the simplest mechanism that causes relaxation in gravitational N-body systems. Because the frequencies of regular motion on adjacent invariant tori are generally similar but different, two points on adjacent tori that are initially close together in phase-space, will seperate linearly with time. However, two points on the same torus do not phase-mix; their separation remains invariant.

Note that phase-mixing decreases the coarse-grained DF around a point, by mixing in 'vacuum' (i.e., unpopulated regions of phase-space). Nevertheless, as assured by the CBE, the flow in phase-space of a collisionless system is perfectly incompressible: unlike the coarse-grained DF, the fine-grained DF is not influenced by phase-mixing and is perfectly conserved.

Although phase-mixing is a relaxation mechanism, in that it drives the system towards a state in which the phase-space density is more and more uniform, it does not cause any loss of information: the system preserves all knowledge of the initial conditions.

In other words, in an integrable, Hamiltonian system phase mixing is a time-reversible relaxation mechanism!

The Lyapunov Exponent I

The Lyapunov exponent of a dynamical system is a measure that determines, for a given point in phase space, how quickly trajectories that begin in this point diverge over time.

Actually, for each point in a $2N$-dimensional phase space (N is the number of degrees of freedom) there are $2N$ Lyaponov exponents λ_i, but it is common to just refer to the largest one.

Consider a small $2N$-dimensional sphere with radius r centered on a phase-space point $\vec{x}$. Different points on the sphere's surface evolve differently with time, causing the sphere to deform into a $2N$-dimensional ellipsoid with principal axes $L_i(t)$.

The Lyapunov exponents for $\vec{x}$ are defined as

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{\mathrm{d}L_i(t)}{\mathrm{d}r}\right)$$

In a collisionless system

$$\sum_{i=1}^{2N} \lambda_i = 0$$

which expresses the incompressibility of the flow (conservation of volume).

The Lyapunov Exponent II

If the trajectory through $\vec{x}$ is regular then $\lambda_i = 0$ for $i = 1, ..., 2N$.

On the other hand, if the trajectory is stochastic then $\lambda \equiv \max \lambda_i > 0$.

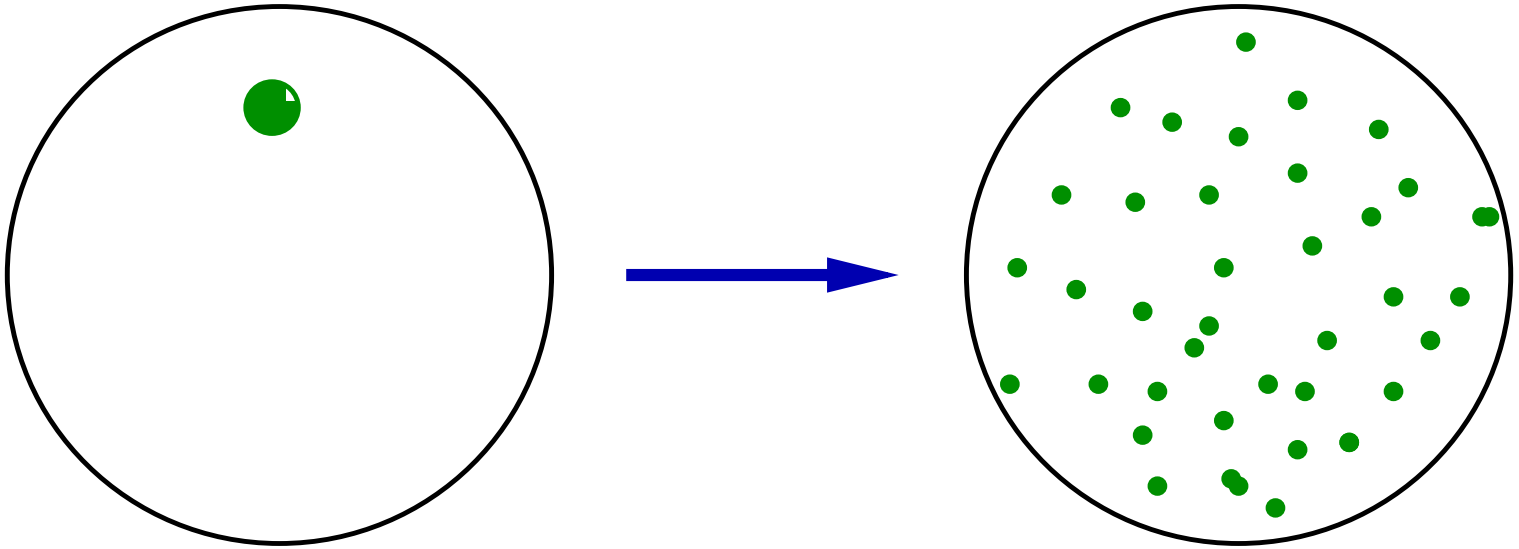
Note that a positive Lyapunov exponent implies that the neighboring trajectories diverge exponentially:

$$\delta\Gamma(t) \propto \mathrm{e}^{\lambda t}$$

The inverse of the largest Lyapunov exponent is called the Lyapunov time, and defines the characteristic e-folding time.

For a chaotic (stochastic) orbit the Lyapunov time is finite, while it is infinite for regular orbits.

Chaotic Mixing I



In the parts of phase-space that are not filled with regular, but with stochastic orbits, mixing occurs naturally due to the chaotic behavior of the obits.

Chaos implies a sensitivity to initial conditions: two stars intially close together separate exponentially with time.

After a sufficiently long time, the group of stars that were initially close together will have spread over the entire accessible phase-space (ie., the Arnold web). As for phase-mixing, chaotic mixing thus smooths out (i.e., relaxes) the coarse-grained DF, but leaves the fine-grained DF invariant.

Chaotic Mixing II

Unlike for phase-mixing, chaotic mixing is irreversible, in the sense that an infinitely precise fine-tuning of the phase-space coordinates is required to undo its effects.

Chaotic mixing occurs on the Lyapunov timescale.

However, the mixing rate of chaotic ensembles typically falls below the Lyapunov rate once the trajectories separate, because stochastic orbits are often confined over long periods of time to restricted parts of phase-space.

The time scale on which the orbits are uniformly spread over their accessible phase-space then becomes dependent on the efficiency of Arnold diffusion.

All in all, the effective rates of phase mixing and chaotic mixing might therefore be comparable in real galaxies.

For a nice review, see Merritt (1999)

Violent Relaxation I

Since $E = \frac{1}{2}v^2 + \Phi$ and $\Phi = \Phi(\vec{x}, t)$ we can write that

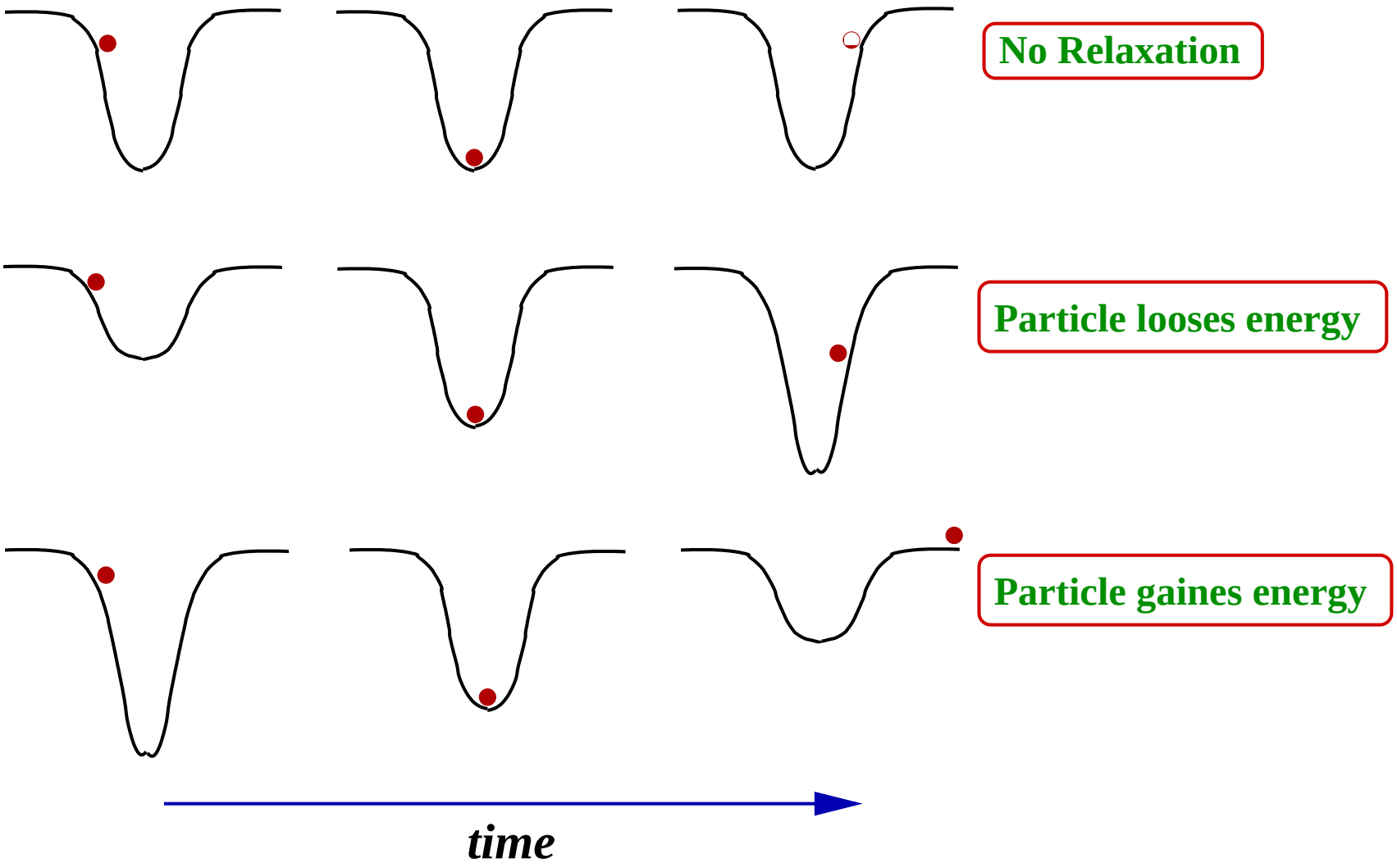
$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\partial E}{\partial \vec{v}} \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{\partial E}{\partial \Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
&= -\vec{v} \cdot \vec{\nabla}\Phi + \frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
&= -\vec{v} \cdot \vec{\nabla}\Phi + \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \\
&= -\vec{v} \cdot \vec{\nabla}\Phi + \frac{\partial \Phi}{\partial t} + \vec{v} \cdot \vec{\nabla}\Phi \\
&= \frac{\partial \Phi}{\partial t}
\end{aligned}$$

Thus we see that the only way in which a particle's energy can change in a collisionless system, is by having a time-dependent potential.

An interesting case to consider is the collapse of a dark matter halo, or that of a galaxy. In this case the potential will vary as function of time, and the particles thus change their energy

Exactly how a particle's energy changes depends in a complex way on the particle's initial position and energy: particles can both gain or loose energy (see fig. on next page). Overall, however, the effect is to widen the range of energies.

Violent Relaxation II



Violent Relaxation III

A few remarks regarding Violent Relaxation:

- **During the collapse of a collisionless system the CBE is still valid, i.e., the fine-grained DF does not evolve ($\mathrm{d}f/\mathrm{d}t = 0$). However, unlike for a 'steady-state' system, $\partial f/\partial t \neq 0$.**

- **A time-varying potential does not guarantee violent relaxation. One can construct oscillating models that exhibit no tendency to relax: although the energies of the individual particles change as function of time, the relative distribution of energies is invariant (cf. Sridhar 1989).**

Although fairly artificial, this demonstrates that violent relaxation requires both a time-varying potential and mixing to occur simultaneous.

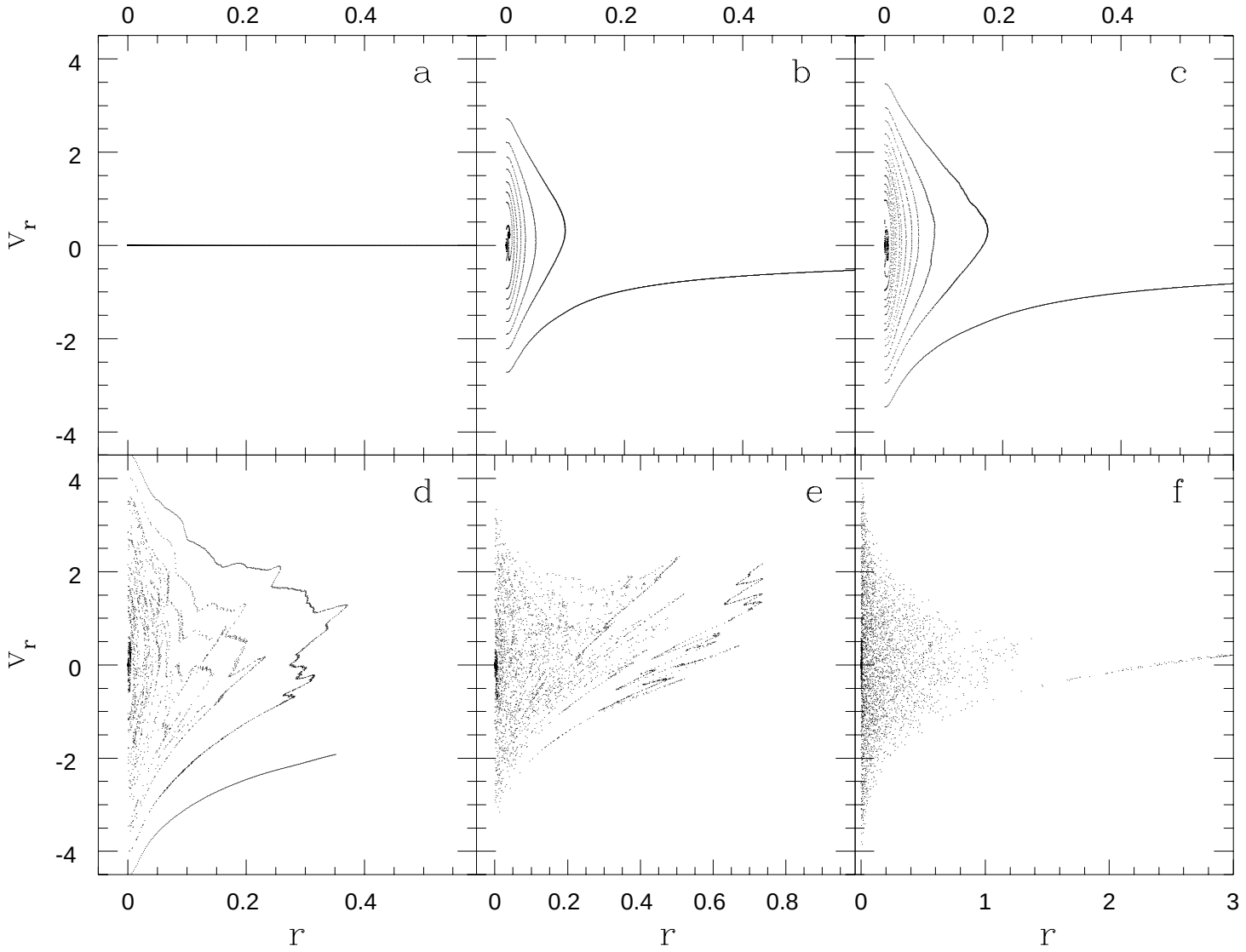
- **The time-scale for violent relaxation is**

$$t_{\rm vr} = \left\langle \frac{(\mathrm{d}E/\mathrm{d}t)^2}{E^2} \right\rangle^{-1/2} = \left\langle \frac{(\partial\Phi/\partial t)^2}{E^2} \right\rangle^{-1/2} = \tfrac{3}{4}\langle \dot{\Phi}^2/\Phi^2 \rangle^{-1/2}$$

where the last step follows from the time-dependent virial theorem (see Lynden-Bell 1967).

Note that $t_{\rm vr}$ is thus of the order of the time-scale on which the potential changes by its own amount, which is basically the collapse time. $\triangleright$ the relaxation mechanism is very fast. Hence the name 'violent relaxation'

Violent Relaxation IV



(from: Henriksen & Widrow 1997)

Collapse of a spherical system with $\rho_{\rm init} \propto r^{-3/2}$

Violent Relaxation V

In the collapse simulation of Henriksen & Widrow (1997), phase-mixing is the dominant relaxation mechanism during the initial phases of the collapse. After some time, there is a transition to a more 'chaotic' flow: due to the time-varying potential particles on neighboring phase-space streams start to mix rapidly (i.e., violent relaxation kicks in).

Violent relaxation leads to efficient fine-grained mixing of the DF, and erases the system's memory of its initial conditions.

For comparison: phase-mixing only leads to a relaxation of the coarse-grained DF and is reversible.

Another important aspect of violent relaxation is the fact that it changes a particle's energy in a way that is independent of the particle's mass. Thus violent relaxation drives the system to a relaxed state that is very different from the one promoted by collional relaxation, where momentum conservation in collisions causes equipartition of energy (ie. mass segregation).

Landau Damping I

In 1946 Landau showed that waves in a collsionless plasma can be damped, despite the fact that there is no dissipation.

This damping mechanism, called Landau Damping, also operates in gravitational, collisionless systems, and is thus another relaxation mechanism.

The physical reason for this collisionless damping arises from the detailed interaction of the wave with the orbits of the background particles which are not part of the wave (i.e., particle-wave interactions).

To gain insight, it is useful to start by considering a fluid. Perturbation analysis of the fluid shows that if the perturbation has a wavelength $\lambda < \lambda_J$, with λ_J the Jeans length, then the perturbation is stable, and the wave propagates with a phase velocity

$$v_p = c\sqrt{1 - \lambda^2/\lambda_J^2}$$

with c the sound speed. Note that larger waves move slower, which owes to the fact that self-gravity becomes more and more important.

When $\lambda = \lambda_J$ the wave no longer propagates. Rather, the perturbation is unstable: self-gravity overpowers the pressure, causing the perturbation to grow.

Landau Damping II

One can apply a similar perturbation analysis to collisionless, gravitational systems (see B&T, Section 5.1). This yields a similar Jeans criterion, but with the velocity dispersion of the stars, σ playing the role of the sound speed.

Once again, perturbations with $\lambda > \lambda_J$ are unstable and cause the perturbation to grow.

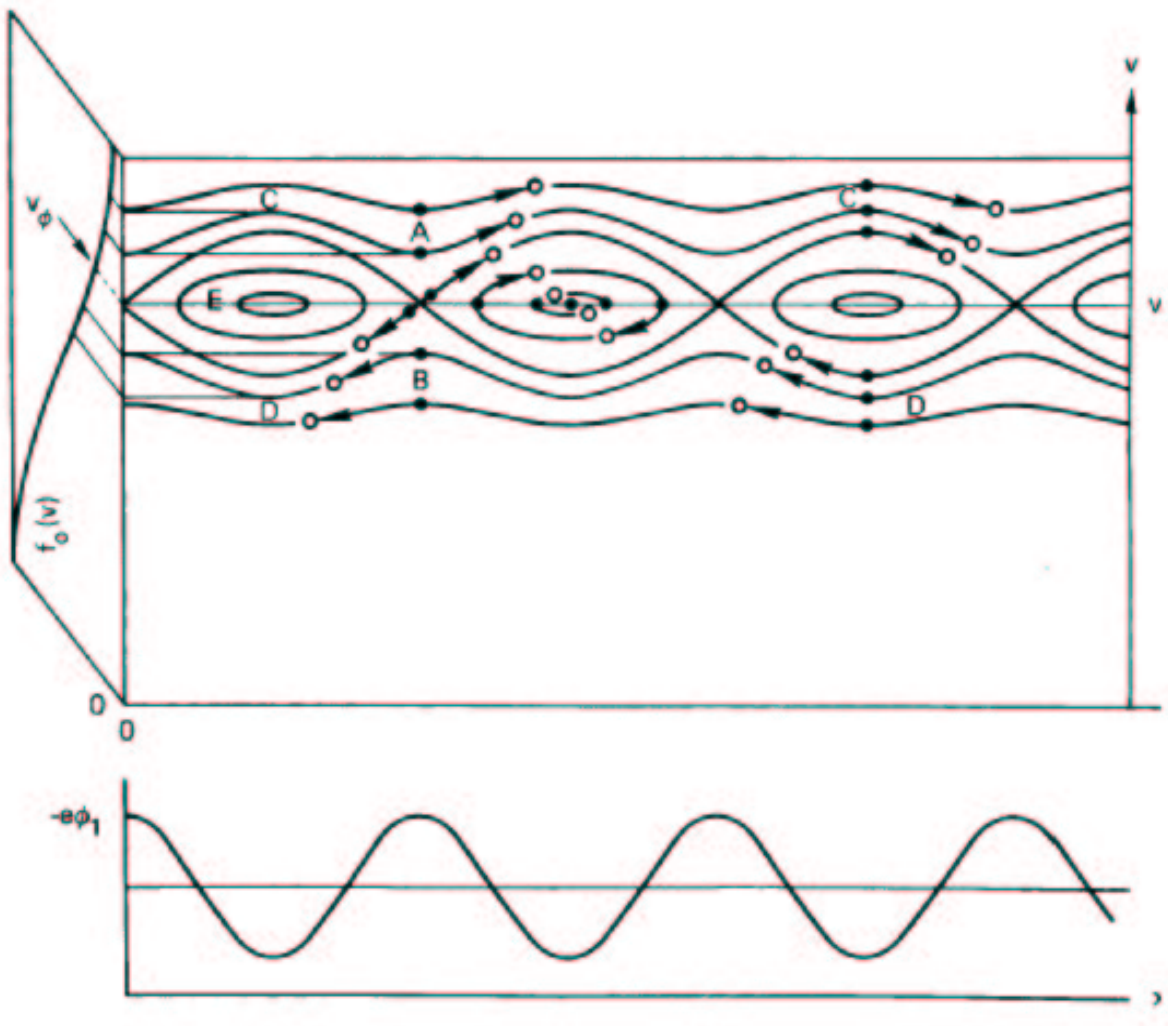
For $\lambda < \lambda_J$, however, the situation is somewhat different. While the fluid supports gravity-modified sound waves that are stable, the equivalent waves in gravitational systems experience Landau Damping.

Consider a density wave with $\lambda < \lambda_J$. While for a fluid the phase velocity $v_p < c$, in a gravitational system we have that $v_p < \sigma$.

Stars that move faster than the wave (i.e., with $v > v_p$) loose energy and tend to be captured by the wave: they tend to amplify the wave. Inversely, stars with $v < v_p$ gain energy and thus tend to damp the wave.

If, for simplicity, we assume a Gaussian distribution of velocities, centered on $v = 0$ and with a velocity dispersion of σ, we see immediately that there will be more stars with $v < v_p$ than with $v > v_p$. Consequently, the net effect will be to damp the wave.

Landau Damping III



Particles initially at A and D gain energy during the first quarter cycle of the wave (i.e., their net velocity increases), while those at B and C loose energy.

Since there are more particles at A than at C and more at D than at B (see distribution at left), the particles experience a net gain. Consequently, the wave has to experience a net loss.

The End-State of Relaxation I

The various relaxation mechanism discussed will drive the system to an equilibrium configuration. However, there are many different equilibrium configurations for a collisionless system of mass M and energy E. So to which of these configurations does a system settle?

This is a very complicated problem which is still largely unsolved.

One might expect the system to evolve to a most probable state.

This actually happens in a collisional gas, where the collisions cause a rapid exchange of energy between the particles. The most probable distribution is obtained by maximizing the entropy, which results in the Maxwell-Boltzmann velocity distribution.

If one applies the same logic to collisionless systems, and (somewhat naively) defines the systems entropy by

$$S = -\int f \ln f \, \mathrm{d}^3\vec{x} \mathrm{d}^3\vec{v}$$

then one finds (not surprisingly) that the functional form of f which maximizes S subject to given values of the system's mass and energy is that of a singular isothermal sphere, for which the DF again has a Maxwellian velocity distribution (see Lynden-Bell 1967).

The End-State of Relaxation II

However, a singular isothermal sphere has infinite mass, and is thus inconsistent with our constraint. Thus, no DF that is compatible with finite M and E maximizes S!

In fact, one can show that one can always increase the system's entropy (as defined above) by increasing the systems central concentration (see Tremaine et al. 1986).

Thus, violent relaxation will drive the system towards its equilibrium state by making the system more concentrated, but can never reach it. In other words, there is no end state. This seems unsatisfactory.

There are two possible 'solutions', both of which are probably correct:

(1) The relaxation is not complete.

(2) The expression for the entropy is not appropriate.

As an example of the latter, since gravitational systems do not obey extensive statistics, a more appropriate definition for the entropy may be

$$S_q = -\int f^q \ln_q f \, \mathrm{d}^3\vec{x}\mathrm{d}^3\vec{v}$$

where $\ln_q(x) = (x^{1-q} - 1)/(1-q)$ with $q \neq 1$ (see Hansen et al. 2004). Note that for $q = 1$ one recovers the normal Boltzmann-Gibbs entropy S.

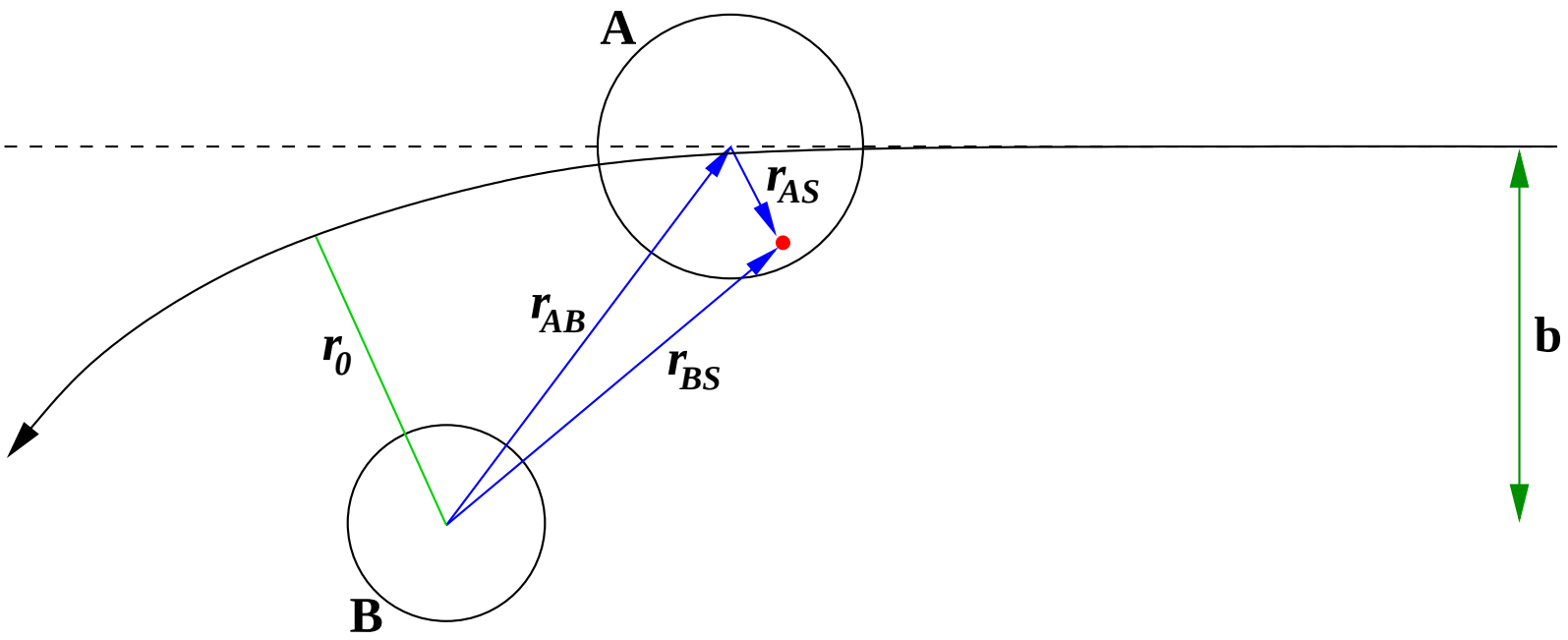
The End-State of Relaxation III

Despite the uncertainties regarding the definition of entropy, it has become clear that relaxation is in general not complete. There are various reasons for this:

- **The time-scales for phase-mixing and chaotic mixing may not be small enough compared to the Hubble time.**
- **Violent relaxation is only efficient as long as the potential fluctuates. This requires global, coherent modes, which, however, as strongly damped by Landau Damping. This occurs roughly on the collapse time-scale, which is also the time-scale on which violent relaxation works. Thus it is not too suprising if it is not complete.**
- **This is strongly supported by numerical simulations, which show that the end-state of violent relaxation depends on the clumpiness of the initial conditions (van Albada 1982). This illustrates that the final state is not completely governed by statistical mechanics alone, but also by the details of the collapse.**

Much work is still required before we have a proper understanding of why dark matter haloes and galaxies have the (semi)-equilibrium states they have.

Collisions & Encounters I



Let A encounter B with an initial velocity v_∞ and an impact parameter b.

A star S (red dot) in A gains energy wrt the center of A due to the fact that the center of A and S feel a different gravitational force due to B.

Let $\vec{v}$ be the velocity of S wrt A then

$$\frac{\mathrm{d}E_S}{\mathrm{d}t} = \vec{v} \cdot \vec{g}[\vec{r}_{BS}(t)] \equiv \vec{v} \cdot \left(-\vec{\nabla}\Phi_B[\vec{r}_{AB}(t)] - \vec{\nabla}\Phi_B[\vec{r}_{BS}(t)] \right)$$

We define $\vec{r}_0$ as the position vector $\vec{r}_{AB}$ of closest approach, which occurs at time t_0.

Collisions & Encounters II

If we increase v_∞ then $|\vec{r}_0| \longrightarrow b$ and the energy increase

$$\Delta E_S(t_0) \equiv \int_0^{t_0} \vec{v} \cdot \vec{g}[\vec{r}_{BS}(t)]\,\mathrm{d}t$$

dimishes, simply because t_0 becomes smaller. Thus, for a larger impact velocity v_∞ the star S withdraws less energy from the relative orbit between A and B.

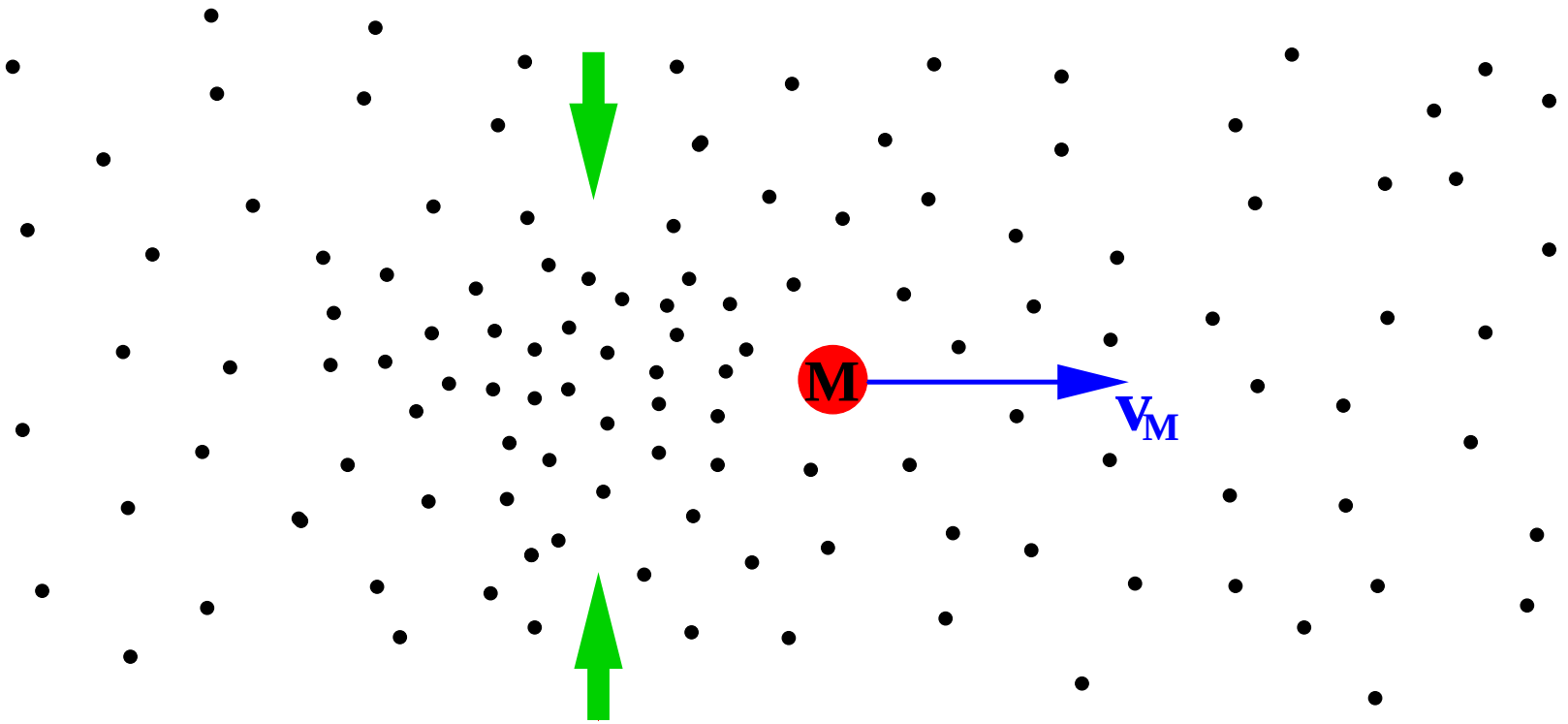
This implies that we can define a critical velocity $v_{\rm crit}$, such that for $v_\infty > v_{\rm crit}$ galaxy A reaches $\vec{r}_0$ with sufficient energy to escape to infinity. If, on the other hand, $v_\infty < v_{\rm crit}$ then systems A and B will merge.

If $v_\infty \gg v_{\rm crit}$ then we can use the impulse approximation to analytically calculate the effect of the encounter.

In most cases of astrophysical interest, however, $v_\infty \lesssim v_{\rm crit}$ and we have to resort to numerical simulations to compute the outcome of the encounter. However, in the special case where $M_A \ll M_B$ or $M_A \gg M_B$ we can describe the evolution with dynamical friction, for which analytical estimates are available.

Dynamical Friction I

Consider the motion of a system with mass M through a medium consisting of many individual 'particles' of mass $m \ll M$. As an example, think of a satellite galaxy moving through the dark matter halo of its parent galaxy.



Due to gravitational focussing M creates an overdensity of particles behind its path (the wake). The backreaction of this wake on M is called dynamical friction and causes M to slow down. Consequently, energy is transferred from the massive to the less massive bodies: dynamical friction is a manifestation of mass segregation.

Dynamical Friction II

Assuming, for simplicity, a uniform density medium with an isotropic velocity distribution $f(v_m)$ of the particles $m \ll M$, then

$$\boxed{\vec{F}_{\rm df} = M\frac{{\rm d}\vec{v}_M}{{\rm d}t} = -\frac{4\pi G^2 M^2}{v_M^2}\ln\Lambda\,\rho(<v_M)}$$

with $\ln\Lambda$ the Coulomb logarithm and

$$\rho(<v_M) = 4\pi\int_0^{v_M} f(v_m)v_m^2{\rm d}v_m$$

the mass density of background particles with velocities $v_m < v_M$.

The derivation of this equation (see B&T Sect. 7.1) is due to Chandrasekhar (1943), and one therefore often speaks of Chandrasekhar dynamical friction.

Note that $\vec{F}_{\rm df} \propto M^2$: the amount of material that is deflected (i.e., the 'mass' of the wake) is proportional to M and the gravitational force that this wake exerts on M is proportional to M times its own mass.

Note that $\vec{F}_{\rm df} \propto v_M^{-2}$ in the limit of large v_M, but $\vec{F}_{\rm df} \propto v_M$ in the limit of small v_M [i.e., for sufficiently small v_M one may replace $f(v_m)$ with $f(0)$].

Note that $\vec{F}_{\rm df}$ is independent of m!

The Coulomb Logarithm

One has that $\Lambda = b_{\rm max}/b_{\rm min}$ with $b_{\rm min}$ and $b_{\rm max}$ the minimum and maximum impact parameters for which encounters can be considered effective:

Encounters with $b > b_{\rm max}$ don't cause a significant deflection, and these therefore do not contribute significantly to the wake. Encounters with $b < b_{\rm min}$ cause a very strong deflection so that these also do not contribute to the wake.

We can estimate $b_{\rm min}$ as the impact parameter that corresponds to a close encounter (see first lecture), and thus $b_{\rm min} \simeq \frac{GM}{\langle v^2 \rangle}$ with $\langle v^2 \rangle^{1/2}$ the rms velocity of the background particles.

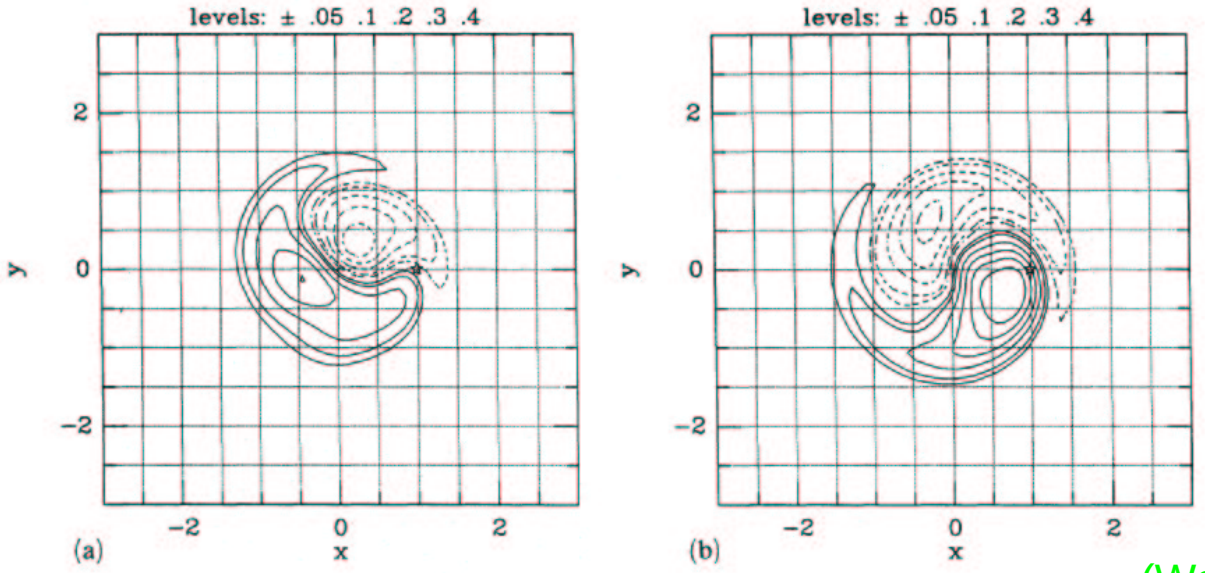
The maximum impact parameter, $b_{\rm max}$, is much harder to estimate (see White 1976), and one typically simply takes $b_{\rm max} \simeq L$ with L the size of the system.

Typical values that one encounters for the Coulomb Logarithm are $3 \lesssim \ln\Lambda \lesssim 30$.

Dynamical Friction: Local vs. Global

Note that Chandrasekhar Dynamical Friction is a purely local phenomenon: The dynamical friction force $\vec{F}_{\rm df}$ depends only on the local density $\rho(<v)$, and the backreaction owes to a local phenomenon, namely wake-creation due to gravitational focussing.

However, a system A can also experience dynamical friction due to a system B when it is located outside of B (Lin & Tremaine 1983). Clearly, this friction can not arise from a wake. Instead, it arises from torques between A and stars/particles in B that are in resonance with A (Tremaine & Weinberg 1984).



(Weinberg 1989)

The extent to which dynamical friction is a local (wake) versus a global (resonant-coupling) effect is still being debated.

Orbital Decay I

Consider a singular, isothermal sphere with density and potential given by

$$\rho(r) = \frac{V_c^2}{4\pi G r^2} \qquad \Phi(r) = V_c^2 \ln r$$

If we further assume that this sphere has, at each point, an isotropic and Maxwellian velocity distribution, then

$$f(v_m) = \frac{\rho(r)}{(2\pi\sigma^2)^{3/2}} \exp\left[-\frac{v_m^2}{2\sigma^2}\right]$$

with $\sigma = V_c/\sqrt{2}$. Now consider a test-particle of mass M moving on a circular orbit (i.e., $v_M = V_c$) through this sphere. The Chandrasekhar dynamical friction that this particle experiences is

$$F_{\rm df} = -\frac{4\pi \ln\Lambda G^2 M^2 \rho(r)}{V_c^2}\left[\mathrm{erf}(1) - \frac{2}{\sqrt{\pi}}\mathrm{e}^{-1}\right] \simeq -0.428\,\ln\Lambda\,\frac{GM^2}{r^2}$$

The test-particle has an angular momentum $L = r v_M$, which it looses, due to dynamical friction, at a rate

$$\frac{\mathrm{d}L}{\mathrm{d}t} = r\frac{\partial v_M}{\partial t} = r\frac{F_{\rm df}}{M} = -0.428\,\ln\Lambda\,\frac{GM}{r}$$

Due to this angular momentum loss the test-particle moves to a smaller radius, while it continues on circular orbits with $v_M = V_C$.

Orbital Decay II

The rate at which the radius changes follows from

$$V_c \frac{\mathrm{d}r}{\mathrm{d}t} = -0.428 \ln\Lambda \frac{GM}{r}$$

Solving this differential equation subject to the initial condition $r(0) = r_i$ one finds that the test-particle reaches the center after a time

$$\boxed{t_{\rm df} = \frac{1.17}{\ln\Lambda} \frac{r_i^2 V_c}{GM}}$$

As an example, consider the LMC. Assume for simplicity that the LMC moves on a circular orbit at $r_i = 50\,\mathrm{kpc}$, that the mass of the LMC is $M = 2 \times 10^{10}\,\mathrm{M}_\odot$, and that the MW can be approximated as a singular isothermal sphere with $V_c = 220\,\mathrm{km\,s^{-1}}$ and with a radius of $r = 200\,\mathrm{kpc}$.

We then find that the LMC will reach the center of the MW halo after a time $t_{\rm df} \simeq \frac{7.26}{\ln\Lambda}\,\mathrm{Gyr}$. Using the approximation for Λ discussed before we find that $\ln\Lambda \simeq 6$, and thus $t_{\rm df} \simeq 1.2\,\mathrm{Gyr}$.

Orbital Decay III

The derivation on the previous pages was for a circular orbit. We now focus on the orbital decay of an eccentric orbit, whose eccentricity is defined as

$$e = \frac{r_+ - r_-}{r_+ + r_-}$$

with r_+ and r_- the apo- and pericenter, respectively.

For simplicity, we once again focus on a singular isothermal sphere, for which the radius of a circular orbit with energy E is given by

$$r_c(E) = \frac{1}{\sqrt{e}} \exp\left(\frac{E}{V_c^2}\right)$$

We can express the angular momentum of an eccentric orbit in terms of the orbit's circularity

$$\eta \equiv \frac{L}{L_c(E)} = \frac{L}{r_c(E) V_c}$$

The circularity η is uniquely related to the orbital eccentricity e, with $\mathrm{d}e/\mathrm{d}\eta < 0$:

Circular orbit: $\eta = 1$ and $e = 0$

Radial orbit: $\eta = 0$ and $e = 1$

We now investigate how dynamical friction influences the orbit's evolution.

Orbital Decay IV

Dynamical friction transfers both energy and angular momentum from the test-particle to the particle's that make up the halo. Let's examin how this influences the orbit's eccentricity

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\mathrm{d}e}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t}$$

Using the definition of the orbital circularity we obtain

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{L}{L_c(E)}\right) = \frac{1}{L_c(E)}\frac{\mathrm{d}L}{\mathrm{d}t} - \frac{L}{L_c^2(E)}\frac{\partial L_c(E)}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} = \eta\left[\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}t} - \frac{1}{V_c^2}\frac{\mathrm{d}E}{\mathrm{d}t}\right]$$

where we have used that $L_c(E) = r_c(E)V_c$. Using that $L = rv_\perp$, with $v_\perp$ the velocity in the direction perpendicular to the radial vector, we find that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}t} \qquad\qquad \frac{\mathrm{d}L}{\mathrm{d}t} = r\frac{\mathrm{d}v_\perp}{\mathrm{d}t} = \frac{L}{v}\frac{\mathrm{d}v}{\mathrm{d}t}$$

Combining all the above we finally find that

$$\boxed{\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\eta}{v}\frac{\mathrm{d}e}{\mathrm{d}\eta}\left[1 - \left(\frac{v}{V_c}\right)^2\right]\frac{\mathrm{d}v}{\mathrm{d}t}}$$

where $\mathrm{d}v/\mathrm{d}t = F_{\rm df}/M < 0$ **(see van den Bosch et al. 1999).**

Orbital Decay V

At pericenter we have that $v > V_c$. Since $\eta > 0$, $\frac{\mathrm{d}e}{\mathrm{d}\eta} < 0$, and $\frac{\mathrm{d}v}{\mathrm{d}t} < 0$ we thus have that $\frac{\mathrm{d}e}{\mathrm{d}t} < 0$; the eccentricity decreases and the orbit becomes more circular.

However, at apocenter $v < V_c$ and therefore $\frac{\mathrm{d}e}{\mathrm{d}t} > 0$: the orbit becomes more eccentric during an apocentric passage.

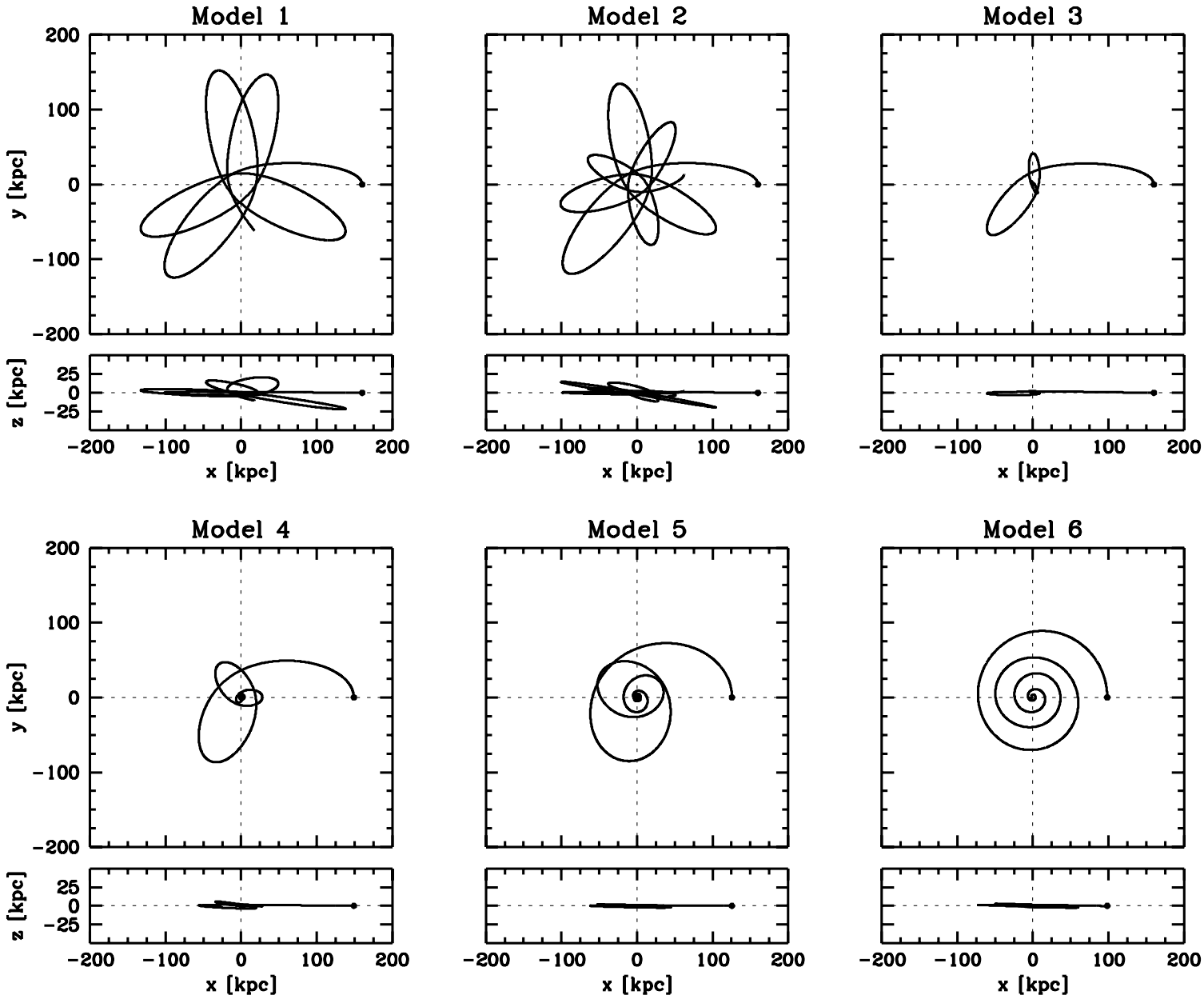
The overall effect of dynamical friction on the orbit's eccentricity, integrated over an entire orbit, can not be obtained from inspection: numerical simulations are required.

For realistic density distributions one finds that $\frac{\mathrm{d}e}{\mathrm{d}t} \sim 0$: contrary to what is often claimed in the literature, dynamical friction does (in general) not lead to circularization of the orbit (see van den Bosch et al. 1999).

As an example of an orbit that circularizes, consider a space-ship on an eccentric orbit around the Earth. It only experiences a friction, due to the Earth's atmosphere, during a pericentric passage, and this causes the 'grazing' orbit of the space-ship to circularize.

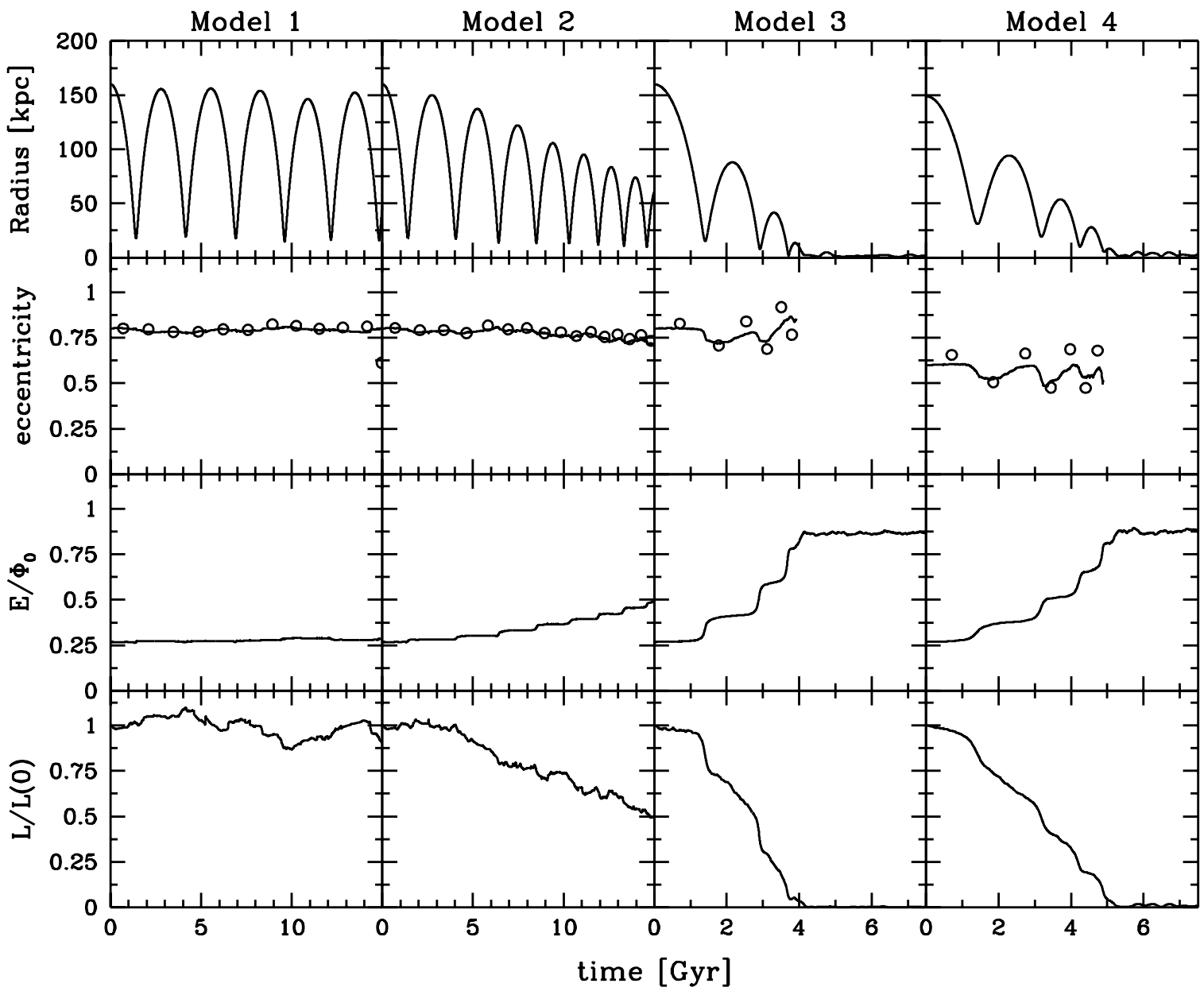
Numerical simulations have shown that $t_{\rm df} \propto \eta^{0.53}$.

Orbital Decay VI



van den Bosch et al. (1999)

Orbital Decay VII



van den Bosch et al. (1999)

The Impulse Approximation I

There are two kinds of encounters between collisionless systems that can be treated analytically:

- **Encounters of very unequal mass ▷ Dynamical Friction**
- **Encounters of very high speed ▷ Impulse Approximation**

As we have seen before, when v_∞ becomes larger, the effect of the encounter diminishes. Therefore, for sufficiently large v_∞ we can treat the encounter as a perturbation.

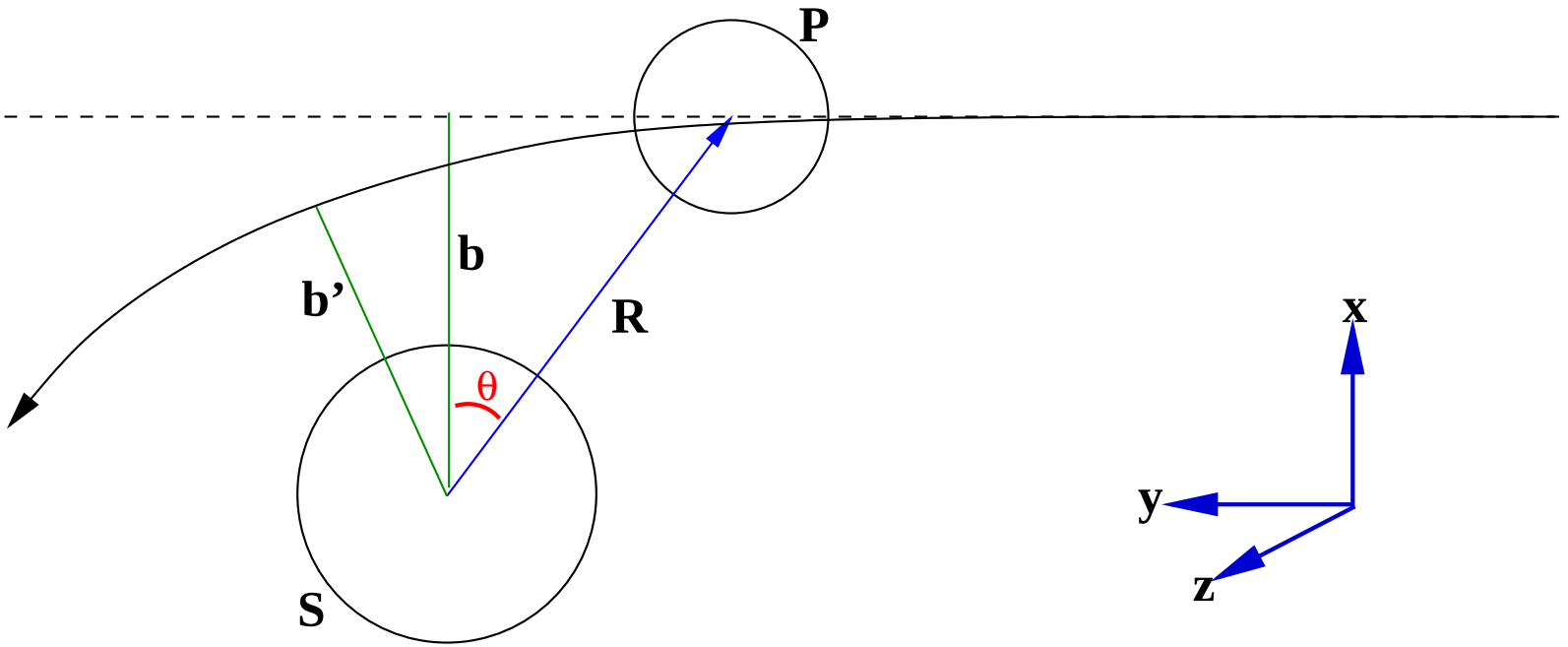
The crucial assumption of the impulse approximation is that the tidal forces due to the perturber act on a timescale $\ll$ orbital time scale of the perturbed stars, so that we may consider the star stationary during the encounter.

- ▷ **No resonant effects**
- ▷ **Instantaneous change in velocity of each star**
- ▷ **Magnitude of $\Delta\vec{v}$ depends on location of star but not on its velocity**
- ▷ **If the encounter speed is sufficiently large then perturber moves in straight line with $v_p(t) = v_\infty \vec{e}_y \equiv v_p \vec{e}_y$ and $\vec{R}(t) = (b, v_p t, 0)$.**

Note that the equations for $v_p(t)$ and $\vec{R}(t)$ define the coordinate system that we will adopt in what follows.

The Impulse Approximation II

Consider a system P, which we call the perturber, encountering another system S with an impact parameter b and an initial velocity v_∞. Let $\vec{R}(t)$ be the position vector of P from S and $v_p(t)$ the velocity of P wrt S.



In the large-v_∞ limit we have the $b' \simeq b$ and $v_p(t) \simeq v_\infty \vec{e}_y \equiv v_p \vec{e}_y$ so that $\vec{R}(t) = (b, v_p t, 0)$.

A star in S experiences a gravitational force due to P given by

$$\vec{a}_*(t) = \frac{GM_p f(R)\vec{R}}{R^3}$$

with $f(R)$ the fraction of P's mass that falls within R.

The Impulse Approximation III

We consider the case with $b > \max[R_p, R_s]$ with R_p and R_s the sizes of P and S, respectively.

In this distant encounter approximation we have that $f(R) = 1$, and

$$\begin{aligned}
\Delta\vec{v}_* &= \int_{-\infty}^{\infty} \vec{a}(t)\mathrm{d}t \\
&= GM_p \int_{-\infty}^{\infty} \frac{(b, v_p t, 0)\mathrm{d}t}{(b^2 + v_p^2 t^2)^{3/2}} \\
&= \frac{GM_p}{v_p} \left(\int_{-\infty}^{\infty} \frac{b\ \mathrm{d}s}{(s^2+b^2)^{3/2}}, \int_{-\infty}^{\infty} \frac{s\ \mathrm{d}s}{(s^2+b^2)^{3/2}}, , 0 \right) \\
&= \frac{GM_p}{v_p} \left(\frac{2}{b}, 0, 0\right) \\
&= \frac{2GM_p}{v_p b} \vec{e}_x
\end{aligned}$$

The ratio M_p/v_p is called the collision strength. In impulse approximation the mass and velocity of the perturber only enter through this ratio.

We can split this $\Delta\vec{v}_*$ in two components: the component $\Delta\vec{v}_S$ which describes change in center of mass velocity of S, and the component $\Delta\vec{v}$ wrt the systematic velocity of S.

The Impulse Approximation IV

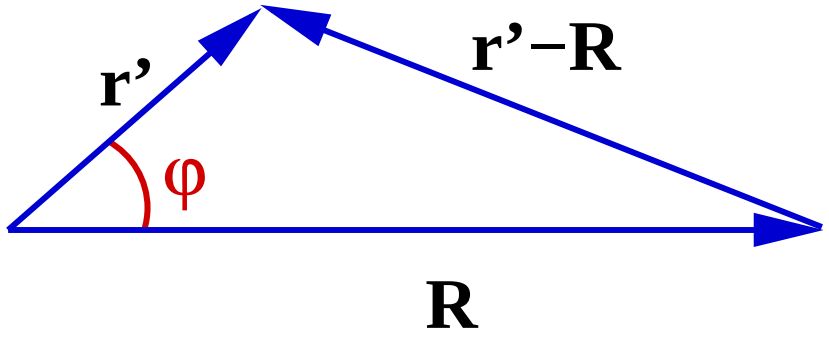
Since we are interested in how P modifies the internal structure of S, we are only interested in $\Delta\vec{v}$.

Note that $\Delta\vec{v}$ arises due to the tidal forces on S, which arise from the fact that the gravitational attraction of P is not uniform over S.

We define a rotating coordinate frame (x', y', z') centered on the center of S, and with the x'-axis pointing towards the instantaneous location of P.

Let $\vec{r}'$ be the position vector of a star in S, and $\vec{R} = R\vec{e}_{x'}$ the position vector of P.

The potential at $\vec{r}'$ due to P is $\Phi(\vec{r}') = -\frac{GM_p}{|\vec{r}' - \vec{R}|}$.



From the above figure one can see that

$$|\vec{r}' - \vec{R}|^2 = (R - r'\cos\phi)^2 + r'^2\sin^2\phi = R^2 - 2rR\cos\phi + r'^2$$

The Impulse Approximation V

Using the series expansion $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \ldots$ **this yields**

$$\frac{1}{|\vec{r}' - \vec{R}|} = \frac{1}{R}\left[1 - \frac{1}{2}\left(-2\frac{r'}{R}\cos\phi + \frac{r'^2}{R^2}\right) + \frac{3}{8}\left(-2\frac{r'}{R}\cos\phi + \frac{r'^2}{R^2}\right)^2 + \ldots\right]$$

which allows us to write

$$\Phi(\vec{r}') = -\frac{GM_p}{R} - \frac{GM_p r'}{R^2}\cos\phi - \frac{GM_p r'^2}{R^3}\left(\frac{3}{2}\cos^2\phi - \frac{1}{2}\right) - \ldots$$

The first term is a constant and foes not yield any force.

The second term yields a uniform acceleration $\frac{GM_p}{R^2}\vec{e}_{x'}$ **directed towards** P**. This is the term that causes the center of mass of** S **to change its velocity, and is not of interest to us.**

In the tidal approximation one considers the third term:

$$\Phi_3(\vec{r}') = -\frac{GM_p}{R^3}\left(\frac{3}{2}r'^2\cos^2\phi - \frac{1}{2}r'^2\right)$$

Using that $r'\cos\phi = x'$ **and that** $r'^2 = x'^2 + y'^2 + z'^2$ **we obtain**

$$\Phi_3(x', y', z') = -\frac{GM_p}{2R^3}\left(2x'^2 - y'^2 - z'^2\right)$$

The Impulse Approximation VI

The above allows us to write the tidal forces on S as

$$F_{x'} = \frac{2GM_p x'}{R^3} \qquad F_{y'} = -\frac{GM_p y'}{R^3} \qquad F_{z'} = -\frac{GM_p z'}{R^3}$$

These are related to the tidal forces in the (x, y, z) coordinate system:

$$\begin{aligned} F_x &= F_{x'}\cos\theta - F_{y'}\sin\theta \\ F_y &= F_{x'}\sin\theta + F_{y'}\cos\theta \\ F_z &= F_{z'} \end{aligned}$$

while (x', y', z') are related to (x, y, z) according to

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}$$

so that we obtain, after some algebra

$$\begin{aligned} F_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} &= \frac{GM_p}{R^3}\left[x\left(2 - 3\sin^2\theta\right) + 3\,y\,\sin\theta\cos\theta\right] \\ F_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{GM_p}{R^3}\left[y\left(2 - 3\cos^2\theta\right) + 3\,x\,\sin\theta\cos\theta\right] \\ F_z = \frac{\mathrm{d}v_z}{\mathrm{d}t} &= -\frac{GM_p z}{R^3} \end{aligned}$$

The Impulse Approximation VII

Integrating these equations over time yields the cumulative velocity changes wrt the center of S. Using that $\vec{R}(t) = (b, v_p t, 0)$, and thus $\cos\theta = b/R$ and $\sin\theta = v_p t/R$ we obtain

$$\Delta v_x = \frac{2GM_p x}{v_p b^2} \qquad \Delta v_y = 0 \qquad \Delta v_z = -\frac{2GM_p z}{v_p b^2}$$

We thus have that $\Delta\vec{v} = \frac{2GM_p}{v_p b^2}(x, 0, -z)$, and

$$\Delta E = \tfrac{1}{2}\left(\vec{v} + \Delta\vec{v}\right)^2 + \Phi(\vec{r}') - \tfrac{1}{2}\vec{v}^2 - \Phi(\vec{r}') = \vec{v}\cdot\Delta\vec{v} + \tfrac{1}{2}(\Delta v)^2$$

Note that, in the impulse approximation, the potential energy does not change during the encounter.

We are interested in computing $\Delta E_{\rm tot}$ which is obtained by integrating ΔE over the entire system S.

First we note that the integral of the first term of ΔE typically is equal to zero, by symmetry. Therefore

$$\begin{aligned}
\Delta E_{\rm tot} &= \tfrac{1}{2}\int \rho(\vec{r}')|\Delta\vec{v}|^2 {\rm d}^3\vec{r}' \\
&= \frac{2G^2M_p^2}{v_p^2 b^4}\int \rho(\vec{r}')(x^2+z^2){\rm d}^3\vec{r}' \\
&= \frac{2G^2M_p^2}{v_p^2 b^4} M_s\langle x^2+z^2\rangle
\end{aligned}$$

The Impulse Approximation VIII

Assuming spherical symmetry for S, so that $\langle x^2 + z^2 \rangle = \frac{2}{3}\langle x^2 + y^2 + z^2 \rangle = \frac{2}{3}\langle r^2 \rangle$ we finally obtain

$$\Delta E_{\rm tot} = \frac{4}{3} G^2 M_s \left(\frac{M_p}{v_p}\right)^2 \frac{\langle r^2 \rangle}{b^4}$$

As shown by Aguilar & White (1985), this derivation, which is originally due to Spitzer (1958), is surprisingly accurate for encounters with $b \gtrsim 5 {\rm max}[r_p, r_s]$ (with r_p and r_s the median radii of P and S), even for relatively slow collisions with $v_\infty \simeq \langle v_s^2 \rangle^{1/2}$.

The above shows that fast encounters pump energy into the systems involved. This energy originates from the kinetic energy associated with the orbit of P wrt S. Note that $\Delta E_{\rm tot} \propto b^{-4}$, so that close encounters are far more important that distant encounters.

As soon as the amount of energy pumped into S becomes comparable to its binding energy, the system S will become tidally disrupted.

Some stars can be accelerated to velocities that exceed the local escape velocity $\triangleright$ encounters, even those that do not lead to tidal disruption, may cause mass loss of S. In this case, the first terms of ΔE is not zero, and the above impulse approximation has to be handled with care.

Return to Equilibrium

As we have seen, a fast encounter transfers orbital energy to the two systems involved in the encounter, whose kinetic energy has subsequently increased.

After the encounter the systems are therefore no longer in virial equilibrium. The systems now need to readjust themselves to find a new virial equilibrium. Interestingly, this process changes the internal kinetic energy more than did the encounter itself.

Let the initial kinetic and total energies of a system be T_0 and E_0, respectively. According to the virial theorem we have that $E_0 = -T_0$.

Due to the encounter $T_0 \rightarrow T_0 + \delta T$, and thus also $E_0 \rightarrow E_0 + \delta T$.

Applying the virial theorem we obtain that after the relaxation the new kinetic energy is

$$T_1 = -E_1 = -(E_0 + \delta T) = T_0 - \delta T$$

Thus, the relaxation process decreases the kinetic energy by $2\delta T$ from $T_0 + \delta T$ to $T_0 - \delta T_0$.

Similarly, the gravitational energy becomes less negative:

$$W_1 = 2E_1 = 2E_0 + 2\delta T = W_0 + 2\delta T$$

Since the gravitational radius $r_g = GM^2/|W|$ the system will expand!

Heat Capacity of Gravitating Systems

As we have seen on the previous page, by pumping energy ('heat') into the system, it has actually grown 'colder'. This is a consequence of the negative heat capacity of gravitational systems.

By analogy with an ideal gas we defined the temperature of a self-gravitating system as

$$\tfrac{1}{2}m\langle v^2\rangle = \tfrac{3}{2}k_B T$$

Unlike an isothermal gas, the temperature in a self-gravitating system is typically a function of position. Therefore, we define the mean temperature as

$$\langle T\rangle \equiv \tfrac{1}{M}\int \rho(\vec{x})T(\vec{x})\mathrm{d}^3\vec{x}$$

and the total kinetic energy of a system of N particles is then $K = \tfrac{3}{2}Nk_B\langle T\rangle$. According to the virial theorem we thus have that $E = -\tfrac{3}{2}Nk_B\langle T\rangle$.

This allows us to define the heat capacity of the system as

$$\boxed{C \equiv \frac{\mathrm{d}E}{\mathrm{d}\langle T\rangle} = -\tfrac{3}{2}Nk_B}$$

which is thus negative: by losing energy the system becomes hotter!

Heat Capacity of Gravitating Systems

Note that all systems in which the dominant forces are gravitational have a negative heat capacity. This includes the Sun, where the stability of nuclear burning is a consequence of $C < 0$: If the reaction rates become 'too high', the excess energy input into the core makes the core expand and cool. This makes the reaction rates drop, bringing the system back to equilibrium.

The negative specific heat also results in fascinating phenomena in stellar-dynamical systems.

Consider a central density cusp. If the cusp is sufficiently steep one has that $\sigma(r)$ increases with radius: the center is colder than its surroundings.

Two-body interactions tend towards thermal equilibrium, which means that they transport heat from outside to inside.

Since $C < 0 \quad \Rightarrow \quad \sigma_0 \downarrow$, i.e., the center becomes colder!

As a consequence $\vec{\nabla} T \uparrow$, and the heat flow becomes larger.

This leads to run-away instability, known as Gravothermal Catastrophe.

Thus, if radial temperature gradient exists, and two-body relaxation time is sufficiently short (e.g., in globular clusters), the system can undergo core collapse.