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# Non-linear acoustic waves in discs

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**SUMMARY**

A pattern of trailing spiral shock waves can drive a gas inflow in a disc, and it is shown here how the inflow rate can be calculated from wave theory. A wave equation is derived and is used to calculate profiles and shock strengths for acoustic waves in discs, and hence to calculate net gas flow rates. In a steady state, the predicted dependence of the gas inflow rate on the wave trailing angle is in good agreement with the earlier numerical results of Spruit, except in the limit of very tightly wound waves where the present results should provide a better approximation. The finding of Spruit that one-armed steady shocks are not possible is also explained. The possible applicability of these results to astrophysical accretion flows, particularly in protostellar discs, is briefly discussed.

## 1 INTRODUCTION

A possible mechanism for driving an accretion flow in a disc is the propagation into the disc of externally generated shock waves which can remove angular momentum from the disc (Shu 1976; Sawada *et al.* 1987; Spruit 1987; Spruit *et al.* 1987). Numerical simulations by Sawada *et al.* (1986, 1987) and Rozyczka & Spruit (1989) suggest that a spiral shock pattern tidally excited by the companion star in a binary system can propagate deep into a disc and drive an accretion flow on to the central object. In a protostellar disc like the solar nebula surrounding the early Sun, similar but weaker waves can be excited by a Jupiter-like giant planet (Lin & Papaloizou 1986; Sekiya, Miyama & Hayashi 1988), and Larson (1989) has suggested that such waves might play an important role in dispersing protostellar discs and in driving accretion on to the central star.

The accretion rate produced by a pattern of stationary self-similar spiral shocks was calculated by Spruit (1987, hereafter S87) from a direct numerical solution of the equations of gas dynamics. In the present paper, we show that the results of S87 can be understood and extended using the theory of non-linear acoustic waves in discs. A review of the wave theory relevant to the accretion problem and a summary of some of the present results have already been given by Larson (1989, hereafter L89); here the derivation of these results is presented, with some improved approximations.

As in the studies quoted above, the vertical structure of the disc is neglected here. Lin, Papaloizou & Savonije (1990) have noted that if there is a vertical temperature decrease in the disc, acoustic waves will tend to be refracted away from the midplane, limiting the distance that they can propagate. However, such a vertical temperature gradient requires an internal heat source in the disc, whose existence is not clear. Even if the disc is internally heated by an effect such as turbulent viscosity, external radiative heating by a central star can cause the vertical temperature profile to become nearly isothermal (Watanabe, Nakagawa & Nakazawa 1989), in which case refraction would not be a serious obstacle to wave propagation.

## 2 WAVE-DRIVEN ACCRETION

If the average energy per unit mass associated with a linear (i.e. small-amplitude) wave is $E$, and if the phase velocity of the wave is $v$, then the average momentum per unit mass associated with the wave is $E/v$ (Pierce 1974). If a disc of surface density $\mu$ rotating with angular velocity $\Omega$ contains a wave pattern whose angular velocity is $\Omega_p$, the surface density of angular momentum associated with the wave is $\mu E/(\Omega_p-\Omega)$. For non-linear waves, similar expressions hold if $E$ is replaced by $2E_K$, where $E_K$ is the kinetic energy per unit mass associated with the wave (Lighthill 1978); in this paper, the symbol $E$ will henceforth be used for simplicity to denote the quantity $2E_K$. As discussed by L89, the wave crests in a differentially rotating disc tend to become wound into a trailing spiral pattern, and the wave then always transports angular momentum outward, regardless of whether it propagates inward or outward.

For linear waves, the group velocity derived from the dispersion relation for tightly wound acoustic waves (e.g. Binney & Tremaine 1987) is $v_g=c_s^2/v$, where $c_s=(\gamma P/\rho)^{1/2}$ is the sound speed. For a spiral wave pattern whose crests trail behind the radial direction by an angle $\theta$, the surface density of wave angular momentum at radius $r$ is $r\cos\theta\,\mu E/v$, where $v$ is taken as negative for inward-propagating waves, and the outward component of the group velocity $v_g$ is $\sin\theta\, c_s^2/v$.

Hence the rate at which angular momentum is transported outward across a circle of radius $r$ is

$$T=2\pi \sin\theta \cos\theta r^2 \mu c_s^2 E/v^2. \tag{1}$$

As noted by L89, the transport of angular momentum by acoustic wave motions exceeds the gravitational torque produced by the same pattern of trailing density fluctuations if the disc is gravitationally stable, i.e. if $Q>1$.

In the presence of wave dissipation, for example by the formation of shocks, the outward transfer of angular momentum associated with a trailing wave pattern will cause an inflow of gas in the disc. The special case of a steady inflow driven by a pattern of stationary self-similar spiral shocks was studied numerically by S87. In such a steady state, the outward transfer of angular momentum by the waves is balanced by the inward transfer of angular momentum by the flow, and the resulting inward mass flux is

$$F=T/r^2\Omega. \tag{2}$$

Following Shakura & Sunyaev (1973), we define an efficiency of angular momentum transport $\alpha$ such that the inward mass flux in a steady accretion flow is

$$F=2\pi\alpha\mu c_s^2/\Omega. \tag{3}$$

Equations (1–3) then imply that the value of $\alpha$ for a steady wave-driven flow is

$$\alpha=\sin\theta \cos\theta E/v^2. \tag{4}$$

(Note that this definition of $\alpha$ differs by a factor of $2\gamma/3$ from that used by S87 and L89.)

More generally, whether or not a steady state is present, the mass flux must depend on the rate at which wave energy and angular momentum are dissipated. This can be shown either by considering the change in the angular momentum of the disc caused by the transfer of wave-angular momentum to it, or by considering the change in the mechanical energy of the disc associated with the dissipation of wave energy. If $\Delta E$ is the wave energy per unit mass dissipated when the wave travels one wavelength and $m$ is the number of spiral arms in the wave pattern, the resulting value of $\alpha$ in a Keplerian disc is

$$\alpha=m\Delta E/\pi c_s^2. \tag{5}$$

For a steady accretion flow, equations (4) and (5) must of course agree, and this places an important constraint on the possible properties of such flows.

If the dissipation occurs in shocks, then $\Delta E$ is the amount of wave energy irreversibly converted into heat in each shock. For weak shocks, this is well approximated by

$$\Delta E \cong 2c_s^2(M_s^2-1)^3/3(\gamma+1)^2 M_s^4, \tag{6}$$

where $M_s$ is the Mach number of the shock. Hence, for shock-driven accretion flows,

$$\alpha \cong 2m(M_s^2-1)^3/3\pi(\gamma+1)^2 M_s^4. \tag{7}$$

Thus the accretion rate associated with a given wave pattern can be calculated if the Mach number of the associated shocks is known. Determination of the shock Mach number requires a calculation of the detailed form of the wave profile, which can be done using the wave equation derived below.

## 3 THE WAVE EQUATION

An accurate calculation of the wave profile requires a full solution to the equations of gas dynamics. However, useful approximate results which are more general than special solutions such as those of S87 and that yield more insight into the properties of waves in discs, can be obtained from a wave equation valid for tightly wound waves with velocity amplitudes which are small compared with the phase velocity. In deriving such a wave equation it is essential to retain terms of second order in small quantities, since non-linear effects are of crucial importance, but we shall assume that terms of third order in small quantities can be neglected.

The propagation of sound waves in discs is modified by the presence of the Coriolis force as an additional restoring force; consequently, the wave motions are two-dimensional and may be regarded as a combination of acoustic waves and epicyclic oscillations. For a wave propagating radially in an inviscid disc, the angular momentum of each fluid element is conserved; therefore its additional radial acceleration is just the usual epicyclic acceleration equal to minus $\kappa^2$ times the radial displacement, where $\kappa$ is the epicyclic frequency. The component of motion in the direction of propagation can then be treated as a one-dimensional motion in which each fluid element experiences an extra restoring force proportional to its displacement. This assumption is also valid for non-radial waves in a rigidly rotating disc, but not generally for non-radial waves in a differentially rotating disc. However, in a relatively cold disc like many astrophysical discs, waves will tend to become tightly wound by differential rotation and will then propagate nearly radially; thus the assumption of radial propagation should still often be a good approximation. For very open wave patterns, the acceleration due to the Coriolis effect is actually somewhat larger than $\kappa^2$ times the displacement in the direction of propagation, and this has the effect of increasing the phase velocity of the waves, but does not change their qualitative nature.

If $x$ is the spatial coordinate measured in the direction of wave propagation, $u$ is the component of fluid velocity in this direction, and $\xi$ is the $x$-displacement of a fluid particle from its rest location or epicycle centre, the one-dimensional Eulerian equations governing the motion of the fluid, including the epicyclic acceleration, are:

$$\partial\rho/\partial t+\partial(\rho u)/\partial x=0 \tag{8}$$

$$\partial u/\partial t+u\,\partial u/\partial x+c_s^2\,\partial(\ln\rho)/\partial x=-\kappa^2\xi \tag{9}$$

$$\partial\xi/\partial t+u\,\partial\xi/\partial x=u. \tag{10}$$

For a wave with an invariant profile propagating with the phase velocity $v$, all physical variables must depend only on $x-vt$; therefore we can write $\partial/\partial t=-v\partial/\partial x=-v d/dx$ and express equations (8–10) in terms of derivatives only with respect to $x$:

$$(v-u)\,d\rho/dx=\rho\,du/dx \tag{11}$$

$$(v-u)\,du/dx-c_s^2 d(\ln\rho)/dx=\kappa^2\xi \tag{12}$$

$$(v-u)\,d\xi/dx=-u. \tag{13}$$

Eliminating $d\rho/dx$ from equations (11) and (12), we obtain

$$(v-u)^2\,du/dx-c_s^2\,du/dx=(v-u)\,\kappa^2\xi. \tag{14}$$

Differentiating equation (14) with respect to $x$, and using equation (13) to eliminate $d\xi/dx$ and equation (14) again to eliminate $\xi$, we obtain the following wave equation for $u$:

$$(v-u)\,[(v-u)^2-c_s^2]\,d^2u/dx^2-[(v-u)^2+c_s^2]\,(du/dx)^2$$
$$-(v-u)\,(dc_s^2/dx)\,(du/dx)+(v-u)\,\kappa^2 u=0. \qquad (15)$$

To simplify this equation, we assume that during each wave cycle the gas behaves adiabatically with ratio of specific heats $\gamma$, and we employ the standard approximation

$$c_s \cong c_0+\tfrac{1}{2}(\gamma-1)\,c_0 u/v, \qquad (16)$$

where $c_0$ is the sound speed in the undisturbed gas. Equation (16) follows from $c_s \propto \rho^{(\gamma-1)/2}$ and the continuity equation (11), and is a good approximation even when $u/v$ is not very small and shocks of modest strength are present in the wave profile (Lighthill 1978). Multiplying equation (15) by $(1+u/v)$ and neglecting terms of third order in $u$, we then obtain

$$v(v^2-c_0^2)\,d^2u/dx^2-[2v^2+(\gamma-1)\,c_0^2]\,ud^2u/dx^2$$
$$-(v^2+\gamma c_0^2)\,(du/dx)^2+v\kappa^2u=0. \qquad (17)$$

Finally, we put this equation in dimensionless form by defining a wave Mach number $M=v/c_0$, a dimensionless velocity $U=u/v$, and a dimensionless $x$-coordinate $X=x/\lambda$, where $\lambda=2\pi v/n\kappa$ is the wavelength and $n=\omega/\kappa$ is the ratio of the wave frequency $\omega$ in the fluid frame to the epicyclic frequency $\kappa$; we then have

$$(M^2-1)\,d^2U/dX^2-(2M^2+\gamma-1)\,Ud^2U/dX^2$$
$$-(M^2+\gamma)\,(dU/dX)^2+(2\pi M/n)^2U=0. \qquad (18)$$

We see that, for a given $\gamma$, the wave form depends on the two dimensionless parameters $M$ and $n$, which are related, respectively, to the amplitude and frequency of the wave.

## 4 WAVE PROFILES

Some periodic solutions to equation (18), calculated for the typical case $n=2$ (appropriate for a stationary two-armed wave pattern) and for a series of values of $M$, are shown in Figs 1 and 2. Like water waves, waves of small amplitude are nearly sinusoidal, but the wave profile becomes progressively more sharply peaked at its crest as the amplitude increases (Fig. 1). Waves of very small amplitude satisfy the dispersion relation $\omega^2=c_s^2k^2+\kappa^2$, which for $n=2$ implies that the minimum value of $M$ for very weak waves is $(4/3)^{1/2}=1.155$. When the wave Mach number $M$ increases to a critical value of 1.182, at which point the maximum fluid velocity $u$ is 0.136 times the sound speed, a cusp appears at the wave crest where the relative velocity between the fluid and the wave is exactly sonic. If the wave amplitude is increased yet further, the wave 'breaks' at its crest and a discontinuity, i.e. a shock front, appears where the gas is suddenly decelerated from supersonic to subsonic motion relative to the wave (Fig. 2). The gas then re-accelerates smoothly through a sonic point back to the supersonic relative motion characteristic of most of the wave profile.

Interestingly, equation (18) has a simple analytic solution of parabolic form for large-amplitude waves containing a shock, but not for smaller amplitude waves with no shock. This analytic solution is

$$U=(M^2-1)/(2M^2+\gamma-1)$$
$$+[(M^2-1)/(M^2+\gamma)\,(2M^2+\gamma-1)]^{1/2}(2\pi M/n)\,X$$
$$+[(2\pi M/n)^2/2(4M^2+3\gamma-1)]\,X^2, \qquad (19)$$

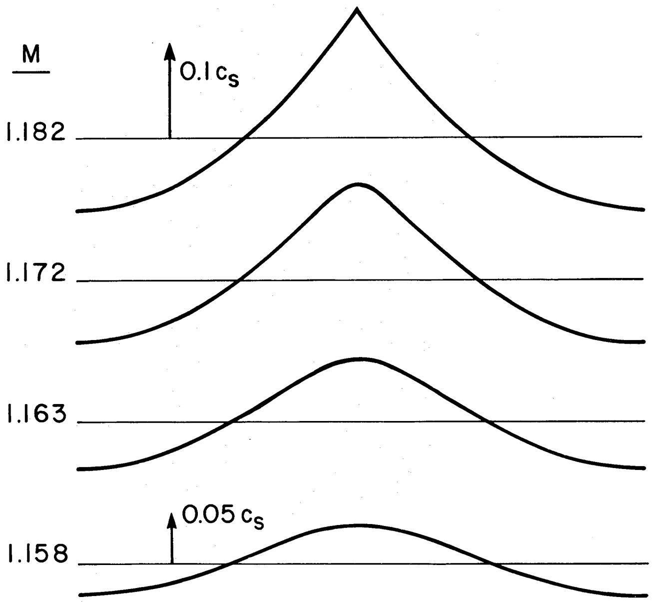


**Figure 1.** Wave profiles for periodic waves of small amplitude containing no shock, as calculated numerically from equation (18) for $n=2$ and $\gamma=5/3$. Profiles are shown for various values of the wave Mach number $M$, as indicated. The velocity perturbation $u$ is plotted as a function of the distance $x$ in the direction of propagation, and the vertical arrows indicate the scale in units of the sound speed $c_s$.

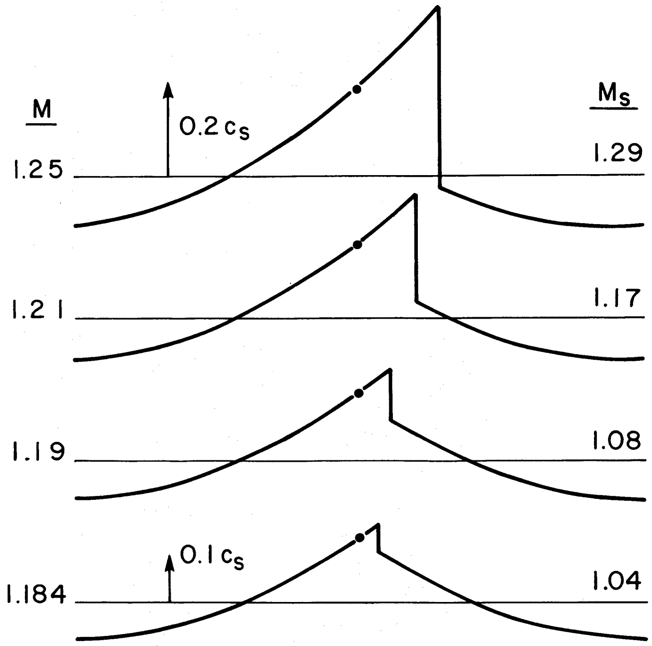


**Figure 2.** Wave profiles for periodic waves of large amplitude containing shocks, calculated from equations (19), (20), and (21) for $n=2$ and $\gamma=5/3$. The wave Mach number $M$ and the shock Mach number $M_s$ are indicated for each case. The velocity scale, reduced by a factor of 2 from Fig. 1, is indicated by the arrows. The dot on each profile indicates the location of the sonic point.

where $X$ is measured in the direction of propagation from the sonic point in the wave profile. From this analytic solution it can be shown that the critical value of $M$ for a wave with a cusp is insensitive to $\gamma$, varying only from 1.182 to 1.184 (for $n=2$) as $\gamma$ is decreased from 5/3 to 1. The critical wave amplitude is more sensitive to $\gamma$, the maximum velocity $u$ increasing from 0.136 to 0.170 times the sound speed as $\gamma$ decreases from 5/3 to 1. For solutions containing a shock, the location of the shock in the wave profile, and hence the strength of the shock, are fixed by the definition of the shock Mach number as

$$M_s = (v - u_1)/c_{s1}, \qquad (20)$$

where $u_1$ is the pre-shock fluid velocity and $c_{s1}$ the pre-shock sound speed, and by the relation between the shock Mach number and the velocity jump $\Delta u$ in the shock:

$$\Delta u = 2(M_s - M_s^{-1})\, c_{s1}/(\gamma + 1). \qquad (21)$$

In calculating $c_{s1}$, a relation similar to equation (16) has been used but with a slightly different coefficient for $u$ equal to $(M^2 + \gamma - M - 1)/M(M+1)$, chosen to ensure that the shock occurs exactly at the wave crest in the critical case.

A noteworthy feature of the analytic solution (19) is that, independently of $\gamma$, solutions exist only for values of $n$ greater than a minimum value given by

$$n_{\min} = \pi/8^{1/2} = 1.111. \qquad (22)$$

The existence of a minimum value of $n > 1$ precludes the existence of steady wave patterns with only a single shock, a result previously found empirically by S87 who could find no solutions with $n$ less than 1.15 for self-similar spiral shocks with $\theta = \pi/4$. The physical reason why steady wave patterns with $n = 1$ cannot exist is that the dispersion relation for linear waves implies that propagation can only occur if the wave frequency $\omega$ in the fluid frame exceeds the epicyclic frequency $\kappa$, i.e. if $n = \omega/\kappa > 1$. Non-linear waves have a larger Mach number $M$ than linear waves for a given $n$, or equivalently a larger value of $n$ for a given Mach number, so that the minimum value of $n$ is larger for non-linear waves than for linear waves. The impossibility of a steady one-armed shock pattern may explain why two-armed patterns are prevalent in numerical simulations, even when the waves are generated by a single source on one side of the disc.

## 5 SOME NUMERICAL RESULTS

For comparison with the numerical results of S87, we present first some results for the case $n = 2$ and $\gamma = 5/3$. In the limit of very weak shocks, i.e. very small values of $\Delta E$ and $\alpha$, the wave energy $E$ and phase velocity $v$ become asymptotically constant and equal to their values for the critical case, which are $4.05 \times 10^3 c_s^2$ and $1.182\, c_s$, respectively. The value of $\alpha$ is then given by equation (4) as

$$\alpha \simeq 0.0029 \cos\theta,$$

where $\sin\theta \simeq 1$ because in this limit the waves are very tightly wound with $\cos\theta \ll 1$.

For larger values of $\Delta E$ and $\alpha$, corresponding to larger values of $\cos\theta$, a useful expression for $\alpha$, which is derivable from equations (4) and (5), is

$$\alpha = (\pi/m)^{1/2} (\sin\theta \cos\theta)^{3/2} c_s E^{3/2}/v^3 \Delta E^{1/2},$$

since the ratio $E^{3/2}/v^3 \Delta E^{1/2}$ is found to become roughly constant for large $\cos\theta$. The approximate expression

$$\alpha \simeq 0.013 (\sin\theta \cos\theta)^{3/2}$$

reproduces the present results to an accuracy of 10 per cent for $\cos\theta > 0.08$ and $\alpha > 3 \times 10^{-4}$. It also agrees with the approximation given by S87,

$$\alpha \simeq 0.012 (\cos\theta)^{3/2},$$

to within 10 per cent over most of this range. However, it is not valid for very small values of $\cos\theta$ or $\alpha$. An interpolation formula that represents the present results over the entire range of values $0 < \cos\theta < 2^{-1/2}$ is

$$\alpha \simeq 0.010[(\cos\theta)^3 + 0.09(\cos\theta)^2]^{1/2}; \qquad (23)$$

it reproduces these results to within 3 per cent for all $\cos\theta < 0.55$, which is better than the intrinsic accuracy of the theory.

We thus see that the value of $\alpha$ for a steady wave-driven accretion flow depends fairly strongly on the angle $\theta$ by which the wave crests trail behind the radial direction. This angle depends in turn on the ratio of the phase velocity $v$ to the rotation speed $V$ of the disc; for a stationary wave pattern,

$$\cos\theta = v/V \sim 1.2\, c_s/V, \qquad (24)$$

where a typical value of $M$ has been substituted. Equations (23) and (24) then imply that

$$\alpha \sim 0.013[(c_s/V)^3 + 0.08(c_s/V)^2]^{1/2}, \qquad (25)$$

which shows that the effective value of $\alpha$ is strongly dependent on the ratio $c_s/V$ of the sound speed to the rotation speed in the disc.

As an example of a possible application of these results, it was noted by L89 that a steady wave-driven accretion flow in a protostellar disc would have an associated time-scale of the order of $10^6$ yr at $r = 1$ AU and $10^7$ yr at $r = 40$ AU. These time-scales are similar to the typical lifetimes of protostellar discs as inferred from observations. A very similar time-scale also applies to the removal of angular momentum from the inner solar nebula by tidal interaction with Jupiter, so that a Jupiter-like giant planet could in principle serve as a suitable wave generator for an accretion flow in a protostellar disc. In any event, numerical simulations clearly show that the tidal effect of Jupiter is capable of generating acoustic waves of significant amplitude in the neighbouring part of the solar nebula.

If the wave source is embedded in the disc, then the wave pattern is of course not stationary; a steady wave pattern will have the same angular velocity $\Omega_p$ as the wave source. The frequency ratio $n = m(\Omega - \Omega_p)/\kappa$ then becomes smaller than for a stationary wave pattern, and the effect of this is that the phase velocity and amplitude of waves with shocks are increased, so that the waves become more open and the resulting values of $\alpha$ become larger. The amplitude increases very strongly as the Lindblad resonance $n = 1$ is approached; in fact, equation (19) implies that the phase velocity and amplitude of waves with shocks become infinite when $n$ approaches the minimum value given by equation (22). Clearly the wave equation loses validity close to a resonance, because the assumption of a tightly wound wave pattern of modest amplitude is no longer satisfied; moreover, a strictly steady accretion flow is no longer possible. However, near-resonant effects can significantly increase wave amplitudes and gas flow rates even some distance from a resonance; for example, if $\alpha$ is calculated in the same way as above but for $n = 1.5$, the resulting values are about 3–4 times larger for a given $\cos\theta$ than the values found for $n = 2$.

This implies that at a radius of 0.4 times the radius $r_p$ of the wave source, the flow rate could be ~4 times larger than that given by the above equations; even at a radius of 0.25 $r_p$, it could be twice as large. Of course, these results are only indicative of the possible importance of near-resonant effects, since a steady flow would not exist in these circumstances.

## 6 CONCLUSIONS

Many types of disturbance might generate acoustic waves in discs, including the tidal influence of a companion star in a binary system, the effects of close encounters in a dense system of young stars, and the tidal effect of a giant planet in a protostellar disc. Differential rotation will tend to wind the wave crests into a trailing spiral pattern, and the waves will then transport angular momentum outward. In the presence of wave dissipation, for example by shocks, a gas inflow is produced in the disc interior to the wave source, and an outflow is produced in any part of the disc outside the source. As a mechanism of potentially general significance for driving accretion flows, acoustic waves have the advantage over turbulent viscosity that they can be generated in one or a few locations and then propagate far from their source, whereas turbulence requires a stirring mechanism to operate at all places and times in the disc. Also, wave effects are more readily calculable.

The results obtained in this paper show that in order for a shock front to be formed in a nearly stationary two-armed wave pattern, the wave amplitude must exceed about 0.14 times the sound speed. A two-armed wave pattern is probably the most important case in general because, as noted by L89, multi-armed wave patterns produce weaker effects, while as shown in this paper, a steady one-armed wave pattern is not possible in the absence of self-gravity.

The predicted accretion time-scale is independent of the mass of the disc, but depends importantly on its temperature: a hotter disc can support stronger, more open waves and hence more rapid accretion. This could lead to significant feedback effects if wave dissipation causes strong heating of the disc. The variation of wave amplitude with radius also depends on the surface density profile of the disc, as discussed by Spruit *et al.* (1987) and L89; thus a strictly steady accretion flow is possible only for a particular disc structure, which need not be realized in general. Time-dependent effects may then often be important, in which case the values of $\alpha$, calculated here for steady flows, may be valid only in a rough average sense. In fact, some of the numerical results of Sawada *et al.* (1986, 1987) do not show any approach to a steady state but instead exhibit complex time-dependent behaviour accompanied by large fluctuations in the accretion rate. It is conceivable that, in some circumstances, wave motions in discs could approach chaotic behaviour and begin to resemble turbulence. Many of the astronomical phenomena that have been attributed to accretion from discs, such as T Tauri activity, do in fact show strong time variability. It is also possible that, since wave-driven accretion is a forced rather than a spontaneous phenomenon, a disc could, in the absence of any disturbance, be completely quiescent and not exhibit any accretion.

Clearly much more work will be required before it can be established whether waves play an important role in driving astrophysical accretion flows. However, since it has now been shown both numerically and analytically that this is possible, at least in principle, and since waves can be generated in many ways, it seems likely that wave effects will find a place in the repertoire of possible mechanisms for driving accretion flows.

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## NOTE ADDED IN PROOF

The lower limit on the frequency ratio $n = m(\Omega - \Omega_p)/\kappa$, given in equation (22), does not preclude one-armed wave patterns if $\Omega_p < 0$ or if $\Omega_p > 2\Omega$; the latter case may apply, for example, to outward-propagating waves.