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Gravitational torques and star formation

Richard B. Larson *Yale University Observatory, Box 6666, New Haven, Connecticut 06511, USA*

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Summary. Any irregularities that occur in the structure of a collapsing cloud will produce non-radial gravitational forces and, if such irregularities are sheared by differential rotation into trailing spiral features of large amplitude, the associated gravitational torques will transfer angular momentum outward on an orbital time-scale. Even if no redistribution of angular momentum occurs during the collapse and a flattened disc-like configuration is formed, such a disc will be a fragile structure at best, and with realistic initial conditions it will be highly unstable to the development of trailing spiral density enhancements. Thus it seems inescapable that in a realistic collapsing cloud there will sooner or later occur rapid outward transfer of angular momentum by gravitational torques. The only stable outcome of the collapse will then be a system in which most of the mass is in a central star or binary system, and less than half remains in a disc around the star; less than one-quarter can remain in a disc with the size of our Solar System.

1 Introduction

Perhaps the most outstanding problem in the theory of star formation is the 'angular momentum problem': how can a star containing at most only a small amount of angular momentum form from a typical protostellar cloud containing, at least initially, orders of magnitude more angular momentum (Mestel 1965)? Clearly angular momentum must be lost or redistributed during the star formation process, but no mechanism has yet been demonstrated to be capable of doing the entire job. The low rotational velocities observed in nearby protostellar clouds (Myers & Benson 1983) suggest that significant loss of angular momentum occurs at a very early stage, but most collapsing clouds nevertheless retain more angular momentum than can be accommodated within a single star, as is shown by the common occurrence of binary systems (Abt 1978). Thus, some mechanism is still needed to redistribute angular momentum in collapsing clouds and transfer it away from individual pre-stellar condensations.

Calculations of the collapse of a rotating cloud show that, even if angular momentum is conserved for each fluid element, a small region at the centre or in an annulus near the centre can collapse indefinitely to very high densities (Bodenheimer 1981). However, only a minute fraction of the cloud's mass can ever attain stellar densities in this way, and the rest is left behind in an extended, rotationally supported envelope or disc. If material from this envelope is to be incorporated into a forming star, it must somehow lose most of its angular momentum. Thus the fraction of a collapsing cloud that ends up in a star or stellar system depends on the amount of angular momentum transferred outward during the collapse; a high efficiency of star formation requires very efficient redistribution of angular momentum. Observations suggest that the efficiency of star formation in collapsing protostellar clouds is in fact high; for example, in the Taurus clouds, small apparently protostellar clumps with masses of only a few solar masses are associated with T Tauri stars of about one solar mass (Myers & Benson 1983). Moreover, the young stars in Taurus typically have ages between 10^5 and 10^6 yr that are comparable to the collapse times of protostellar clouds (Cohen & Kuhi 1979); thus, redistribution of angular momentum must be not only efficient but rapid, occurring on a time-scale not significantly exceeding the free-fall time.

A number of mechanisms have been suggested for transferring angular momentum in collapsing clouds. In particular, magnetic torques can play an important role in braking the rotation of protostars during early stages of the collapse (Mouschovias 1978). However, when the central density reaches a sufficiently high (but uncertain) value, ambipolar diffusion is expected to decouple the magnetic field from the gas, and magnetic torques can then no longer brake the rotation. As discussed by Mouschovias, this process can plausibly result in angular momenta of the same order as those of wide binary systems, but the problem of redistributing angular momentum on the scale of individual pre-stellar condensations remains.

Another suggestion is that turbulent viscosity can transfer angular momentum, either during the initial dynamical collapse (Tscharnuter 1981) or subsequently in a disc formed as a result of the collapse (Lynden-Bell & Pringle 1974; Cameron 1978). However, such an assumed turbulent viscosity is *ad hoc,* since no mechanism is known that could sustain strong turbulence throughout a protostellar cloud or disc. Even the concept of turbulent viscosity must be applied with caution, because turbulence refers only to fluctuations in a flow and has no independent existence, unlike the molecular motions responsible for normal viscosity (Tennekes & Lumley 1972).

Finally, it has been noted that if a binary or multiple system forms in a collapsing cloud and most of the initial angular momentum goes into orbital motion (Larson 1972; Bodenheimer 1978), then tidal torques could help to transfer residual spin angular momentum from the individual pre-stellar condensations into the orbital motion (Larson 1977). Such tidal coupling may well be important for close binaries, but it is less clear how it could play a role for wide binaries. In any case, the formation of single stars is still not accounted for; even if single stars originate as escapers from multiple systems (Larson 1972), they are most likely to come from very loose multiple systems where tidal effects are weak.

Tidal torques provide an example of how purely gravitational forces can act to transfer angular momentum in a non-axisymmetric mass distribution. This paper will explore the possibility that gravitational torques occur very generally in collapsing clouds and play a major role in redistributing angular momentum during star formation. Although it has sometimes been suggested that gravitational torques might be important in discs (e.g. Cameron 1978; Goldreich & Tremaine 1980; Cassen *et al.* 1981), most discussions of the angular momentum problem have assumed axial symmetry for collapsing protostars and so have neglected any possible role of non-radial gravitational forces. In general, non-radial gravitational forces should be present because there is no reason why a collapsing cloud should be precisely axisymmetric. Non-axisymmetric features or inhomogeneities will generally tend to wind up into trailing spiral patterns, which will produce gravitational torques that transfer angular momentum outward (Section 2). Even if axial symmetry is approached in a disc, instabilities in the disc will almost certainly destroy the axial symmetry and generate trailing spiral structure (Section 3). Thus, it is almost unavoidable that in a protostellar cloud or pre-stellar disc there will be gravitational torques that transfer angular momentum outward, causing the system to become more centrally condensed.

Since gravitational torques probably occur in a wide variety of rotating self-gravitating systems, they may be of importance for the formation and evolution of many types of condensed objects or systems in astronomy. This paper will focus on applications to star formation, but similar considerations may also apply to the development of centrally condensed structures in galaxies and to the formation and growth of compact objects such as black holes.

2 Gravitational torques

Since molecular clouds are generally very irregular in shape, a protostellar cloud will almost certainly be non-axisymmetric when it begins to collapse. If the self-gravity of the cloud is sufficiently predominant over internal pressure, any asymmetries or inhomogeneities of large enough size will be amplified during the collapse. Because of the differential rotation that soon develops as the cloud becomes centrally condensed, non-axisymmetric features will tend to wind up to form trailing spiral patterns. Such spiral patterns were found in some of Larson's (1978) particle simulations of three-dimensional collapse; even though the number of particles was small, trailing spiral features could be discerned at some stage in about half of the cases calculated (see, for example, fig. 2 of Larson 1978). Such trailing spiral patterns generate gravitational torques that act to transfer angular momentum outward (Julian & Toomre 1966; Lynden-Bell & Kalnajs 1972).

Gravitational torques can also be induced by the formation of condensations in a collapsing cloud; for example, two condensations orbiting around each other will create trailing spiral-shaped wakes that exert a gravitational drag on them, causing them to lose angular momentum and fall closer together. This phenomenon, known as 'dynamical friction' in stellar dynamics, plays an important role in the merging of galaxies (White 1978). It was also found to occur in some of Larson's (1978) simulations of collapsing gas clouds, in which nearby condensations sometimes merged because of gravitational drag. More striking demonstrations of this effect in the gas-dynamical case are provided by the calculations of Gingold & Monaghan (1981) and Boss (1982), in which two or three orbiting condensations generate trailing spiral-shaped wakes and rapidly fall together because of the resulting loss of angular momentum.

It thus appears that gravitational transfer of angular momentum, possibly enhanced by the formation of bound orbiting condensations, can be of major importance even during the early dynamical stages of collapse, before any equilibrium disc-like structure has had a chance to form. Quantitative predictions of the magnitude and outcome of such effects will require further three-dimensional collapse calculations; the existing three-dimensional calculations have not primarily addressed such questions, and have yielded the results mentioned above only as by-products of attempts to understand fragmentation.

Gravitational torques can also be of major importance in any disc that forms as a result of the collapse of a rotating cloud. This case is more amenable to analysis, and will be considered in more detail in this paper. Self-gravitating discs are very prone to instability (Ostriker & Peebles 1973) and numerical models of stellar discs typically develop a strong transient 'barred spiral' structure that transfers angular momentum outward, causing the disc to become more centrally condensed (Hohl 1975). Similar instabilities are expected to occur also in gas discs (Bardeen 1975). The presence of a large mass concentration at the centre of a disc, such as an already formed star, may suppress the bar-forming instability (Ostriker 1972), but local instabilities can still occur if the disc is cold enough, leading to the rapid growth of trailing spiral density enhancements (Goldreich & Lynden-Bell 1965). Such trailing spiral features have been found in numerical simulations of cold gas discs with central objects (Cassen *et al.* 1981). Again, the expected effect is to transfer angular momentum outward and cause the disc to become more centrally condensed.

The magnitude of the gravitational torque associated with a spiral pattern in a disc can be estimated from a formula given by Lynden-Bell & Kalnajs (1972). For a spiral pattern with m arms, wavenumber k, radial wavenumber k_r, and gravitational potential amplitude S, the torque exerted on material outside radius r by material inside this radius is

$$T = (m/4)(k_r/k)(rS^2/G) = (m/4)\sin\gamma\,(rS^2/G), \tag{1}$$

where γ is the angle by which the arms trail beind a radial orientation. For a pattern of short wavelength in an infinitely thin disc, $S = -(2\pi G/k)\,\delta\mu$ where $\delta\mu$ is the amplitude of the perturbation in the surface density μ; we can then write

$$T = (\pi^2/m)\sin\gamma\cos^2\gamma\, r^3 G\,(\delta\mu)^2, \tag{2}$$

where the tangential wavelength $2\pi(k\cos\gamma)^{-1}$ has been set equal to m^{-1} times the circumference $2\pi r$. If the angular momentum of the disc interior to radius r is approximated as $\pi r^3\mu V$ where μ is the surface density and V the rotational velocity at radius r, and if a time-scale for outward transfer of angular momentum is defined as $\tau = \pi r^3\mu V/T$, we obtain

$$\tau = m\mu V/[\pi\sin\gamma\cos^2\gamma\; G(\delta\mu)^2] \sim m\mu V/[G(\delta\mu)^2]. \tag{3}$$

This can also be written as

$$\tau \sim m(M/M_d)(\delta\mu/\mu)^{-2}P \tag{4}$$

where $M \cong V^2 r/G$ is the total mass interior to radius r, $M_d \cong 2\pi r^2\mu$ is the disc mass interior to r, and $P = 2\pi r/V$ is the orbital period at radius r.

Equation (4) shows that if most of the mass is in the disc, so that $M/M_d \sim 1$, and if a spiral disturbance of large amplitude and wavelength is present, so that $\delta\mu/\mu \sim 1$ and $m \sim 1$, the time-scale for redistribution of angular momentum is of the same order as the orbital period. This prediction is in good agreement with the results of numerical experiments on the dynamics of discs. Equation (4) also suggests that a similar time-scale will apply whenever strong spiral density perturbations are present, even if this occurs prior to the formation of a well-defined disc in a collapsing cloud. Again, this appears to be roughly consistent with numerical results; for example, simulations of galaxy mergers show that merging typically occurs within an orbital period (White 1978; Villumsen 1982) and very similar results have been found for the merging of gaseous condensations (Gingold & Monaghan 1981). Thus, whenever large deviations from axial symmetry are present, outward transfer of angular momentum and central condensation of the system are likely to occur on an orbital time-scale.

Equation (4) can also be applied to any accretion disc that may form around a young star. Models of protostellar accretion discs have been developed in which it is assumed that an inflow is driven by turbulent viscosity (Lynden-Bell & Pringle 1974; Cameron 1978; Lin & Papaloizou 1980), but it is also possible that gravitational torques could drive an accretion flow, especially if the disc is gravitationally unstable and large density fluctuations occur. Even if only small density fluctuations occur, gravitational effects can still be important over many orbital periods, and might compete with or dominate over viscous effects. For example, if $M/M_d = 10$ and $m = 1-10$, the time-scale predicted by equation (4) is $\sim (10\text{–}100)(\delta\mu/\mu)^{-2}$ orbital periods, so that even if $\delta\mu/\mu$ is as small as 10^{-2}, gravitational torques will be important on a time-scale of $\sim 10^5\text{–}10^6$ orbital periods, comparable (at $r \sim 1$ AU) to the collapse time of a protostellar cloud. This suggests that gravitational torques can be important in protostellar accretion discs if such discs typically have non-axisymmetric density fluctuations of order 10^{-2} or greater.

3 Gravitational instabilities in discs

Extensive recent work, as summarized by Toomre (1981), has shown that much of the complex behaviour of unstable discs, even large-scale or 'global' responses, can be understood on the basis of the local stability analysis for gas discs first carried out by Goldreich & Lynden-Bell (1965). An important result found by Goldreich & Lynden-Bell and emphasized by Toomre is that the growth of density perturbations is enhanced by the shear present in a differentially rotating disc. As a result, density enhancements in a marginally stable disc can be strongly amplified as they wind up, forming trailing spiral patterns of large amplitude; this 'swing amplification' effect can occur even in discs that are stable to axisymmetric modes. The resulting spiral patterns are transient and the total growth in density is finite, although it may be very large, since amplification occurs only during the limited period when rapid swinging is taking place. Thus, the phenomenon is not a true instability in the sense of unlimited growth, and it is necessary to consider the amount of amplification that can take place as a function of the properties of the disc.

For an infinitely thin isothermal disc with epicyclic frequency κ and sound speed c, the dispersion relation for axisymmetric modes with frequency ω and wavenumber k is

$$\omega^2 = \kappa^2 - 2\pi G\mu k + c^2k^2 \tag{5}$$

(Goldreich & Ward 1973). Instability occurs, i.e. ω^2 becomes negative, for some values of k if the surface density μ exceeds a critical value μ_c given by

$$\mu_c = c\kappa/\pi G. \tag{6}$$

The parameter Q often used in discussions of disc stability is equal to $\mu_c/\mu = c\kappa/\pi G\mu$; in this paper we shall use the density ratio $\mu/\mu_c = Q^{-1}$ as a measure of how much swing amplification can occur. The most unstable wavenumber, i.e. the value of k for which ω^2 is a minimum, is equal to H^{-1}, where $H = c^2/\pi G\mu$ is the scale height of a self-gravitating isothermal gas sheet with surface density μ and sound speed c.

For shearing disturbances, the growth rate is goverend by an equation like equation (5) with κ^2 replaced by

$$\tilde{\kappa}^2(\gamma) = \kappa^2 - 8A\Omega\cos^2\gamma + 12A^2\cos^4\gamma \tag{7}$$

(Goldreich & Tremaine 1978; Toomre 1981); here A is the Oort constant measuring the local shear rate, Ω is the angular velocity of rotation, and γ is the (time-dependent) angle by which a spiral density enhancement trails from a radial orientation. Generally, $\tilde{\kappa}^2(\gamma)$ is smaller than κ^2; this is because the effective epicyclic frequency is reduced in a shearing reference frame, since the shear and the epicyclic motion are in the same sense. As an example, fig. 4 of Toomre (1981) shows $\tilde{\kappa}^2(\gamma)$ for the case $V =$ constant, and it can be seen that $\tilde{\kappa}^2$ is approximately equal to $\frac{1}{2}\kappa^2$ for $|\gamma| < 65°$, the range of angles in which most of the growth occurs. Thus, as a rough criterion, we might expect significant growth of spiral density enhancements if $\mu > 2^{-1/2}\mu_c$.

More quantitatively, we wish to estimate the maximum swing amplification factor that can be attained as a function of μ/μ_c. This can be done roughly by plotting ω^2 versus γ for various values of μ/μ_c and the tangential wavenumber $k \cos\gamma$, and comparing with fig. 5 of Toomre (1981) which gives an analogous plot for a stellar disc. From this comparison it appears that a gas disc with $Q = 1.3$ and an optimally chosen wavelength will yield approximately the same maximum amplification as a stellar disc with $Q = 1.5$. Adopting this approximate correspondence between gas and stellar discs, we can then use Toomre's fig. 7, which summarizes the results of extensive computations, to estimate the maximum growth factor as a function of Q or μ/μ_c. We estimate that the maximum growth factor is of order 3 for $\mu/\mu_c = 0.6$, 15 for $\mu/\mu_c = 0.8$ and 100 for $\mu/\mu_c = 1.0$.

All of the above results are valid only for an infinitely thin disc, and they require significant correction to allow for the reduction in gravity forces in a disc of realistic thickness. As a first approximation, the reduction factor for forces in the central plane of a self-gravitating isothermal disc has been calculated numerically as a function of kH, and the axisymmetric stability analysis has been repeated. The result is that axisymmetric instability occurs for $\mu > 1.43\,\mu_c$ and the most unstable wavenumber satisfies $kH = 0.53$. The actual thickness effect will be somewhat greater because the forces are reduced by a larger factor outside the central plane of the disc. The analysis of Goldreich & Lynden-Bell (1965), which used a different approximation for discs of finite thickness, predicts instability to axisymmetric modes if

$$\mu > 0.47\, c\kappa/G = 1.48\,\mu_c, \tag{8}$$

in reasonable agreement with this expectation.

Finally, we wish to estimate the thickness effect for shearing, swing-amplified modes. Since these are generally characterized by values of kH similar to the axisymmetric modes, we shall assume that a similar thickness correction is required. In particular, we shall assume that for shearing modes, the surface density of a disc of finite thickness must be 1.5 times larger than that of a thin disc to yield the same amount of swing amplification. The estimated maximum amplification factors are then of order 3 for $\mu/\mu_c = 0.9$, 15 for $\mu/\mu_c = 1.2$ and 100 for $\mu/\mu_c = 1.5$. Thus, when the effects of swing amplification and finite thickness are both properly taken into account, they approximately compensate and the simple criterion $\mu > \mu_c$ remains a good predictor of instability in gas discs, in the sense of significant growth of spiral density perturbations. The growth factor increases rapidly with μ/μ_c, being insignificant for $\mu/\mu_c < 0.5$ and very large for $\mu/\mu_c > 1.5$. For comparison, some sample calculations carried out by Goldreich & Lynden-Bell (1965) for discs that are marginally stable to axisymmetric modes showed somewhat larger amplification factors than estimated here, ranging from ~ 100 to ~ 1000 for varying assumptions about the rotation curve. Thus the present estimates of growth factors may, if anything, be conservative.

4 Stability of isothermal discs

During the early stages of collapse, a protostellar cloud is expected to remain approximately isothermal, and any disc that forms as a result of the collapse will also be nearly isothermal, except for the densest central regions. The initial distribution of angular momentum will almost certainly be altered by gravitational torques during the collapse, but as a limiting case we shall suppose that the disc has the same distribution of angular momentum as the initial cloud; if it can be shown that any such disc is unstable, a strong case will be made for the occurrence of gravitational torques at some stage in the collapse.

Recent calculations of the collapse of an axisymmetric rotating cloud by Norman, Wilson

& Barton (1980) and by Hayashi, Narita & Miyama (1982) have found that the collapse produces a flattened, centrally condensed disc-like configuration whose central density increases indefinitely. The density in this disc varies approximately as r^{-2}, the surface density varies as r^{-1} and the rotational velocity is approximately constant. Analytic models of isothermal discs with exactly these properties have been found by Toomre (1982) and by Hayashi *et al.* (1982), and the latter authors suggest that these models represent the final equilibrium state of a collapsing cloud that has constant mass per unit specific angular momentum dM/dh. These isothermal disc models generalize the Mestel (1963) disc, which also has $\mu \propto r^{-1}$ and $V =$ constant, to the case of finite temperature and thickness.

A remarkable property of the isothermal discs is that the density contours all have flattened toroidal shapes and they all pass through the centre, leaving no central hole. The inability of numerical codes to resolve this highly singular situation at the centre may have been responsible for the lack of agreement between different numerical codes as to whether the maximum density occurs at the centre or in a ring. We shall assume that the centrally condensed solution is correct, although it has not been demonstrated that this is a unique solution.

The isothermal disc models form a one-parameter family depending on V/c, the ratio of rotational velocity to sound speed; the parameter n of Toomre is equal to $V^2/2c^2$, and the parameter γ of Hayashi *et al.* is equal to $1+n$. Hayashi *et al.* show that discs with $n < 1.41$ are unstable to radial collapse, so that radially stable discs exist only for $n > 1.41$. The stability of such discs to shearing disturbances can be evaluated using the analytic expressions for μ and μ_c given by the above authors; from these we obtain

$$\mu/\mu_c = (\pi/4)\,[(n+1)^{1/2} \sin \pi/2\,(n+1)]^{-1}. \tag{9}$$

This expression increases monotonically with n and is equal to 0.83 for $n = 1.41$, 1.0 for $n =$ 2.8 and 1.2 for $n = 4.6$. Thus it appears that the entire family of isothermal discs is at best only marginally stable to non-axisymmetric disturbances, and is susceptible to significant swing amplification; even with the minimum value of n for radial stability, swing amplification by at least a factor of ~ 2 can occur, according to the estimates of Section 3. Instability to axisymmetric modes occurs, according to equation (8), for $n > 7.7$. For comparison, Hayashi *et al.* (1982) found from a numerical study of the growth of axisymmetric disturbances that axisymmetric instability occurs for $n \gtrsim 6$, corresponding to $\mu/\mu_c \gtrsim 1.33$.

To determine the value of n appropriate for a disc formed by the collapse of a rotating cloud, it is convenient to relate this parameter to the mass per unit specific angular momentum dM/dh, since this quantity is by assumption conserved during the collapse. Using $dM/dh = 2\pi r\mu/V$, we obtain

$$(G/c)\,dM/dh = \pi\,[(2n)^{1/2} \sin \pi/2(n+1)]^{-1}. \tag{10}$$

The value of $(G/c)\,dM/dh$ near the rotation axis of a uniform, rigidly rotating, spherical cloud can be expressed in terms of the parameters $\alpha = 5c^2R/2GM$ and $\beta = \omega^2R^3/3GM$ giving the ratios of the thermal and rotational energies to the gravitational energy; the result is

$$(G/c)\,dM/dh = (15/8)^{1/2}(\alpha\beta)^{-1/2}. \tag{11}$$

An upper limit on the product $\alpha\beta$, and hence a lower limit on $(G/c)\,dM/dh$, is set by the requirement that the cloud be able to collapse. According to the extensive numerical study of Boss & Haber (1982), collapse with a fixed boundary can occur if $\alpha + 2.0\beta < 1.10$, and collapse with a constant boundary pressure can occur if $\alpha + 3.8\beta < 1.52$. If $\alpha + x\beta < y$, it can be shown that $\alpha\beta < y^2/4x$; hence we deduce that, in both cases, a necessary condition for collapse to occur is $\alpha\beta < 0.15$. Using equations (9)–(11), we then predict that $n > 3.7$ and

$\mu/\mu_c > 1.1$ for at least the inner part of any disc formed by the collapse of a rotating cloud. Such a disc will be susceptible to strong swing amplification by at least a factor of ~ 8, according to the estimates of Section 3. Thus, density perturbations as small as ~ 10 per cent or less in amplitude can lead to strong non-axisymmetric distortions and gravitational torques. Even smaller perturbations can alter the structure of the disc on a longer time-scale, as noted in Section 2.

In fact, most protostellar clouds appear to rotate sufficiently slowly that the actual value of $\alpha\beta$ is considerably smaller than the maximum value for collapse found above; any disc resulting from the collapse will then be even more unstable than inferred above. For example, a survey by Myers & Benson (1983) detected few rotational velocities exceeding $0.1\ \mathrm{km\,s^{-1}}$ in the dense, apparently protostellar cores of many dark clouds; the resulting median upper limit on β is 0.04. Thus, even if α is as large as 1, $\alpha\beta$ has a median observed upper limit of 0.04. The corresponding lower limits on n and μ/μ_c in any resulting disc are $n > 21$ and $\mu/\mu_c > 2.4$. Such a disc would be highly unstable and would experience rapid redistribution of angular momentum by gravitational torques, even before all of the cloud material has been added to the disc. We recall that these results have been derived for the unlikely limiting case in which no redistribution of angular momentum occurs during the initial collapse; in reality, significant outward transfer of angular momentum probably occurs even prior to the formation of a disc. We thus arrive at the main conclusion of this paper, which is that *rapid outward transfer of angular momentum by gravitational torques appears unavoidable at some stage in the collapse of a rotating cloud with realistic initial conditions.*

The main assumption on which this conclusion depends is that the collapsing cloud and any disc formed in it remain isothermal at all stages. In reality, the central part of the disc will attain a high enough density that it becomes opaque and its temperature rises significantly. For a disc formed from a cloud with $\beta = 0.04$, this will occur only in the innermost few per cent of the mass, and the stability of the rest of the disc will be unaffected because it depends only on a local stability criterion. For smaller values of β, a fraction varying as β^{-3} of the mass of the disc will become opaque and non-isothermal. The resulting higher temperature will tend to make the opaque region more stable. As an example of what might happen with a small enough β, Cameron & Pine (1973) have constructed models of discs that are relatively compact and have high opacity, and they assume that these discs are adiabatic rather than isothermal. These disc models are marginally stable and have $\mu \sim \mu_c$. However, a major conclusion of Cameron & Pine is that the time-scale for radiative cooling of these discs is very short, only a few years in the inner parts, so that such a disc would soon cool and μ_c would be reduced, making the disc unstable. Thus it appears that the above conclusion about the inevitable occurrence of instabilities and gravitational torques is not altered even if the material in a collapsing cloud or protostellar disc becomes opaque and no longer isothermal.

5 Remnant discs and the efficiency of star formation

The previous sections have shown that the endpoint of the collapse of a protostellar cloud cannot be a disc with the same distribution of angular momentum as the initial cloud, because such a disc would be unstable and therefore would almost certainly never form in the first place. Gravitational transfer of angular momentum must rapidly lead to a more centrally condensed configuration, and material will continue to either condense into a central core or be dispersed to large radii until the amount of matter left orbiting around the core is small enough that a stable disc can form. The core will also continue to evolve rapidly until it becomes sufficiently hot to be largely pressure supported and sufficiently opaque to retain its heat for many rotations; at this point it can be called a star. In this section we consider the possible properties of any remnant discs of protostellar material around newly-formed stars. The structure of such discs cannot be predicted without a detailed knowledge of the formation process, but an upper limit on the surface density and mass of such a disc can be set from the requirement that the disc must be stable against the growth of non-axisymmetric perturbations.

In order to ensure reasonable stability, the surface density of a disc must be significantly smaller than the critical density $\mu_c = c\kappa/\pi G$, since a disc with $\mu = \mu_c$ is still subject to substantial swing amplification by a factor of ~ 5, according to Section 3. Moreover, a disc with a central star may be slightly less stable than a purely self-gravitating disc with the same μ/μ_c, since the scale height and hence the thickness effect are smaller. We shall assume, somewhat arbitrarily, that adequate stability against shearing disturbances is ensured if $\mu < 0.5\,\mu_c$. Since the disc is heated by the central star, its temperature depends on the luminosity of the star and on distance from the star. To consider a specific example, we assume that the central star has the properties of the present Sun, and that the temperature in the disc is given by $T = 300\,r(\mathrm{AU})^{-1/2}$ K. Since it will turn out that the disc mass in the region of the present Solar System is less than a solar mass, we assume as a first approximation that all of the mass is in the central star and that the rotation law is Keplerian. With these assumptions, we can calculate μ_c as a function of r, and hence the maximum stable disc mass $M_d(r)$ interior to radius r, assuming $\mu < 0.5\,\mu_c$. The result is

$$M_d(r) < 0.14\,r(\mathrm{AU})^{1/4} M_\odot. \tag{12}$$

This maximum disc mass would be increased if the central star were more luminous than the present Sun, but this is a small effect since the stable disc mass varies only as $L^{1/8}$.

According to equation (12), the largest mass that could have existed in a stable 'solar nebula' with the size of the present Solar System ($r = 40$ AU) is about $1/3\,M_\odot$. This prediction agrees crudely with the results of the numerical experiments of Cassen *et al.* (1981), which were based on different assumptions about the structure of the disc. For example, for a disc with $\mu \propto r^{-1}$ and $T = 100$ K at all radii, Cassen *et al.* find that the maximum stable disc mass is between 1/3 and $1\,M_\odot$ and that a disc as massive as $1\,M_\odot$ develops strong spiral structure.

The maximum possible mass of a disc around a young star increases with the radius of the disc, although this dependence is weak. An upper limit on the possible radius of a protostellar disc can be inferred from the observed upper limits on the rotational velocities of protostellar clouds, which imply $\beta < 0.04$ for most clouds (Section 4). If a cloud with an initial radius of 0.1 pc and $\beta < 0.04$ collapses with conservation of angular momentum to form a disc, the resulting disc radius is $r < 2500$ AU. From equation (12), the maximum possible disc mass is then predicted to be $M_d(r) < 1.0\,M_\odot$. Although the assumptions underlying equation (12) begin to break down at such large radii, this is not likely to alter the conclusion that the mass of the disc cannot exceed that of the central star. This result also does not depend significantly on the mass of the central star, since at a given radius the epicyclic frequency κ varies as $M^{1/2}$ and the sound speed c varies as $L^{1/8}$; thus the stable disc mass, which varies as the product of these factors, is nearly proportional to the stellar mass.

The above results have implications for the efficiency of star formation on the scale of individual collapsing protostars. If a protostellar cloud collapses to a radius less than a few thousand AU, most of the mass that remains in such a region must soon end up in a central star or stellar system, and less than half can remain behind in a disc; in other words, the efficiency of star formation in such a region must exceed 50 per cent. In a region with the size of the Solar System, the efficiency of star formation must exceed 75 per cent. The prediction that the efficiency of star formation in collapsing protostellar clouds must be greater than 50 per cent is consistent with studies of star formation in regions such as the Taurus dark clouds, where typical cloud cores have masses of $\sim 2 M_\odot$, and typical young stars, many of which may be in binary systems, have masses of $\sim 0.5-1.0 M_\odot$.

7 Conclusions

Departures from axial symmetry will almost certainly occur in a collapsing cloud and cause redistribution of angular momentum by gravitational torques. Any irregularities will tend to be wound up by differential rotation into trailing spiral shapes, and if a trailing spiral pattern of large amplitude is produced, angular momentum will be transferred outward on an orbital time-scale. If bound condensations form, gravitational drag will reduce their orbital angular momentum and cause them to spiral closer together, and tidal interactions will tend to reduce their spin angular momentum.

Even if no redistribution of angular momentum occurs during early stages and the cloud collapses to a disc, any such disc will be susceptible to strong amplification of shearing disturbances. The amplification factor depends on the mass per unit specific angular momentum in the disc, and if this quantity is as high as is observed in the cores of dark clouds, the disc will be highly unstable, leading to rapid outward transfer of angular momentum. Thus a typical protostellar cloud cannot collapse with detailed conservation of angular momentum, and must rapidly become increasingly centrally condensed. Even after the collapse has ceased to be isothermal, radiative cooling will continue to be effective in destabilizing any flattened disc-like configuration that may form.

The only possible stable outcome of the collapse is a system in which most of the mass that is not dispersed is in a central star or binary system, and no more than half remains in a disc around the star; no more than one-quarter of the mass can remain in a disc with the size of our Solar System. If such a disc possesses or acquires more mass, gravitational instabilities will cause rapid restructuring of the disc until most of the mass is in the central star or stellar system. Thus, even without other effects such as magnetic or viscous torques, purely gravitational effects will ensure a high efficiency of star formation in collapsing protostellar clouds.

The slow observed rotation of the cores of dark clouds suggests that magnetic torques have already caused significant loss of angular momentum at an early stage of star formation. Thus, the solution of the angular momentum problem probably involves both magnetic and gravitational torques, magnetic torques acting during early stages and gravitational torques becoming dominant after the collapse is well under way. During the later stages of collapse, it appears unavoidable that gravitational torques will cause most of the mass to condense into a star or stellar system within a time not much greater than the free-fall time. Thus, the basic qualitative outcome of the collapse will not be very different from the case of spherical collapse, although the details must be far more complex and remain to be elucidated.

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