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THE EVOLUTION OF STAR CLUSTERS

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SUMMARY

The method for computing the evolution of star clusters which was developed in a previous paper (Larson 1970) is here applied to calculating the evolution of a number of different models of star clusters. As in Paper I, the calculations refer only to the special case of equal masses, the stars all being assumed to have a mass of $1 M_\odot$. Three types of systems are considered, and their evolution is described in some detail: (1) a globular cluster of mass $2 \times 10^5 M_\odot$, (2) a galactic cluster of mass $100 M_\odot$, and (3) a dense galactic nucleus of mass $10^8 M_\odot$. In general, the results are in fair agreement with the predictions of classical relaxation theory, except that in some respects the classical picture is seen to be over-simplified. Some of the principal conclusions of the project are (1) the rate of evolution and the stellar escape rate are quite sensitive to the structure of the system; (2) the velocity distribution always tends to become strongly anisotropic in the outer part of the system; and (3) the existence of a tidal limit has a strong effect on the escape rate, and may increase it by a large factor compared to an isolated cluster.

1. INTRODUCTION

In a previous paper (Larson 1970, hereafter referred to as Paper I) a method was described for computing the evolution of a star cluster, using a fluid-dynamical approach based on moment equations derived from the Boltzmann and Fokker–Planck equations. In Paper I the method was applied to a rather artificial example consisting of a system of particles enclosed in a rigid spherical container with perfectly reflecting walls. In the present paper we apply the method to some more realistic models of stellar systems, including a globular cluster, a galactic cluster, and a dense galactic nucleus. As in Paper I, the present calculations refer only to the idealized case in which all stars have the same mass.

For convenience, we repeat the definitions of some of the important quantities used in the method of Paper I. We denote by $\rho(r)$ the mass density of stars at radius r, and we denote by u, v, w the velocity components in the radial and two transverse directions, respectively. In addition to the density ρ and the mean radial velocity $\langle u \rangle$, the following higher moments of the velocity distribution are to be calculated as functions of the radius r and the time t:

$$\left.\begin{aligned} \alpha &\equiv \langle (u-\langle u\rangle)^2 \rangle \\ \beta &\equiv \langle v^2 \rangle = \langle w^2 \rangle \\ \epsilon &\equiv \langle (u-\langle u\rangle)^3 \rangle \\ \xi &\equiv \langle (u-\langle u\rangle)^4 \rangle - 3\alpha^2. \end{aligned}\right\} \qquad (1)$$

As was explained in Paper I, α and β are the mean squared random velocities in the radial and transverse directions, ϵ represents an energy flux in the radial direction, and ξ represents an excess (if $\xi > 0$) or a deficiency (if $\xi < 0$) of high velocity stars relative to a Maxwellian distribution. Again the basic units of mass, length, and time are taken as $1 M_{\odot}$, 1 pc, and 10^6 yr; in this system the unit of velocity is 1 pc/10^6 yr $= 0.978$ km s^{-1}, and the value of the gravitational constant G is 4.50×10^{-3}.

2. THE BOUNDARY CONDITIONS

Since the method of Paper I utilizes an Eulerian computational scheme, the boundary conditions are most conveniently specified in Eulerian form, i.e. at a point fixed in space. In the example of Paper I this was easily done since the system was by assumption confined inside a rigid container, and the condition of perfectly reflecting walls was readily formulated by requiring that the odd-order moments of the velocity distribution vanish at the boundary. In the case of a real star cluster, however, it is necessary to allow for the escape of stars from the system, so some sort of porous or absorbing wall would clearly be a more appropriate boundary condition. Since the size of a star cluster is limited by galactic tidal forces, it seems most appropriate to set the boundary for the fluid-dynamical calculations at the 'tidal limiting radius' of the cluster, as defined, for example, by King (1962). One usually imagines that once an escaping star gets outside the tidal limit it will be effectively removed from the cluster by the galactic tidal field, and so need no longer be considered as a member of the cluster. This tidal removal of the escaping stars may be simulated in the fluid-dynamical model by assuming that the system is bounded by a perfectly absorbing wall, which effectively absorbs or removes all stars which reach it. The boundary condition on the velocity distribution is then that there are no stars moving inward at the boundary, i.e. that the velocity distribution is truncated for radial velocities $u < 0$.

The method of Paper I requires that the odd-order moments $\langle u \rangle$ and ϵ of the velocity distribution be specified at the boundary, or that they be related to the values of the other moments at the boundary. In the present case $\langle u \rangle$ and ϵ may be related to the second moment α if a specific form for the velocity distribution at the boundary is assumed. For this purpose we have assumed that the distribution of radial velocities u is gaussian for $u > 0$, i.e. $f(u) \propto \exp(-u^2/2b)$, and is truncated for $u < 0$. For a velocity distribution of this form we then have

$$\left.\begin{aligned} \langle u \rangle &= \left(\frac{2}{\pi - 2}\right)^{1/2} \alpha^{1/2} = 1.324\, \alpha^{1/2} \\ \epsilon &= \left(2 - \frac{\pi}{2}\right)\left(\frac{\pi}{2} - 1\right)^{-3/2} \alpha^{3/2} = 0.995\, \alpha^{3/2}. \end{aligned}\right\} \quad (2)$$

If the velocity distribution has a different form, or if the bounding wall is not perfectly absorbing as assumed, then the numerical coefficients in equations (2) would be different, but we would still have $\langle u \rangle \propto \alpha^{1/2}$ and $\epsilon \propto \alpha^{3/2}$. Fortunately, trial calculations made with different choices for the numerical coefficients in equations (2) show that the structure and evolution of a cluster are not very sensitive to the exact values of these coefficients, except in the region just inside the boundary.

3. EVOLUTION OF A GLOBULAR CLUSTER

As a typical mass for a globular cluster, we have adopted $M = 2 \times 10^5 M_{\odot}$. Also, for a typical tidal limiting radius we have taken $R = 100$ pc. The factor $\ln (D_{max}\langle V^2 \rangle / 2Gm)$ occurring in equation (25) of Paper I and in expressions for the relaxation time has been set equal to 10 throughout the calculations. Since the results are in some respects rather sensitive to the initial conditions chosen, we have made calculations for several different choices of the initial model for a globular cluster, taking the initial model in each case to be a hydrostatic equilibrium configuration. We describe first the results obtained when the initial model is assumed to be a polytrope of index $n = 5$ (' Plummer's model '; Plummer 1911). The polytrope of index 5 has often been used as a model for globular clusters because of its convenient analytical properties and because it has been claimed to represent with fair accuracy the observed structure of at least some globular clusters (of course, this does not make Plummer's model unique, since agreement with the observations has also been claimed for a wide variety of other models).

The density distribution for a polytrope of index 5 is of the form

$$\rho(r) = \rho_c[1 + (r/r_0)^2]^{-5/2}. \tag{3}$$

In the example to be discussed, we have set $r_0 = 6$ pc; with a total mass of $2 \times 10^5 M_{\odot}$ and a radius of 100 pc, this leads to a central density

$$\rho_c = 2 \cdot 2 \times 10^2 M_{\odot}/\text{pc}^3$$

and a central velocity dispersion in one coordinate of $\alpha_c^{1/2} = 5 \cdot 0$ pc/10^6 yr

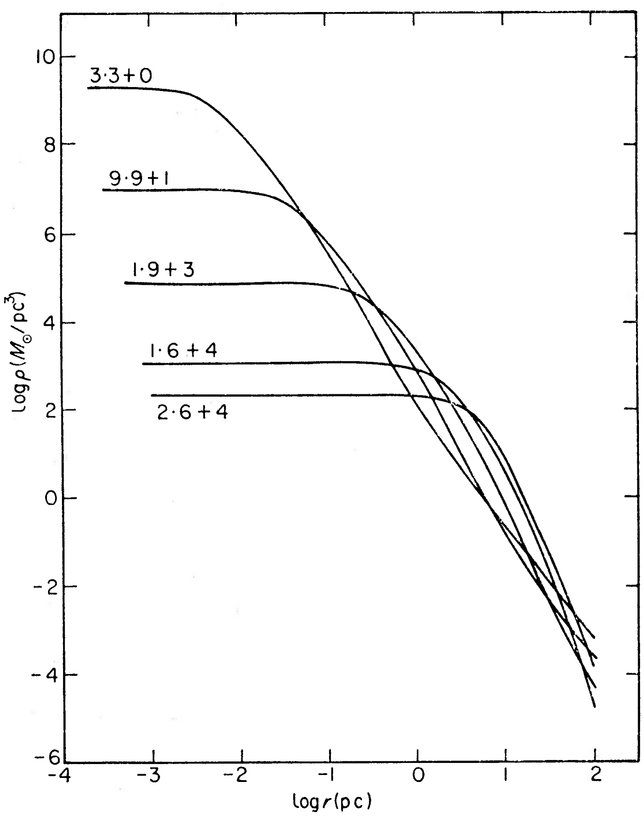

FIG. 1. *The time development of the density distribution for a globular cluster with initial density distribution given by equation* (3). *The curves are labelled with the corresponding values of* τ *in units of* 10^6 *yr.*

($4 \cdot 9$ km s^{-1}). The relaxation time at the centre (see Paper I or Chandrasekhar 1942, equation 2.379) is then $T_c = 9 \cdot 0 \times 10^8$ yr. In all of the calculations to be described, we have set $\langle u \rangle = (\alpha - \beta) = \epsilon = 0$ in the initial model; the initial distributions of $\alpha(r)$ and $\xi(r)$ have been calculated from the assumed initial density distribution $\rho(r)$ by solving equations (16) and (19) of Paper I with all time derivatives set equal to zero.

The time development of the density distribution for the initial conditions described above is illustrated in Fig. 1. As was found in the example studied in Paper I, the system becomes more and more centrally condensed, and the central density 'runs away' and tends to approach infinity after a finite time which in this case is about $2 \cdot 6 \times 10^{10}$ yr, or about 29 times the initial relaxation time at the centre. As in Paper I, it will be convenient to measure time from the instant t_0 when the central density becomes infinite; thus we define $\tau \equiv t_0 - t$, and henceforth we use τ in place of t as the time variable. The curves in Fig. 1 are labelled with the corresponding values of τ in units of 10^6 yr. We note that after 10^{10} yr have elapsed ($\tau = 1 \cdot 6 \times 10^4$), the central density has risen by a factor of 5 to about $1 \cdot 1 \times 10^3 \, M_\odot/\text{pc}^3$, a value which would be fairly representative for a dense globular cluster. The density distribution at this time has not changed much in form from the initial density distribution, except that it drops off slightly less steeply near the centre and more steeply near the outer boundary.

The variation of the central density and velocity dispersion with time is illustrated in Fig. 2, which shows log ρ_c and log α_c plotted vs. log τ. Although these

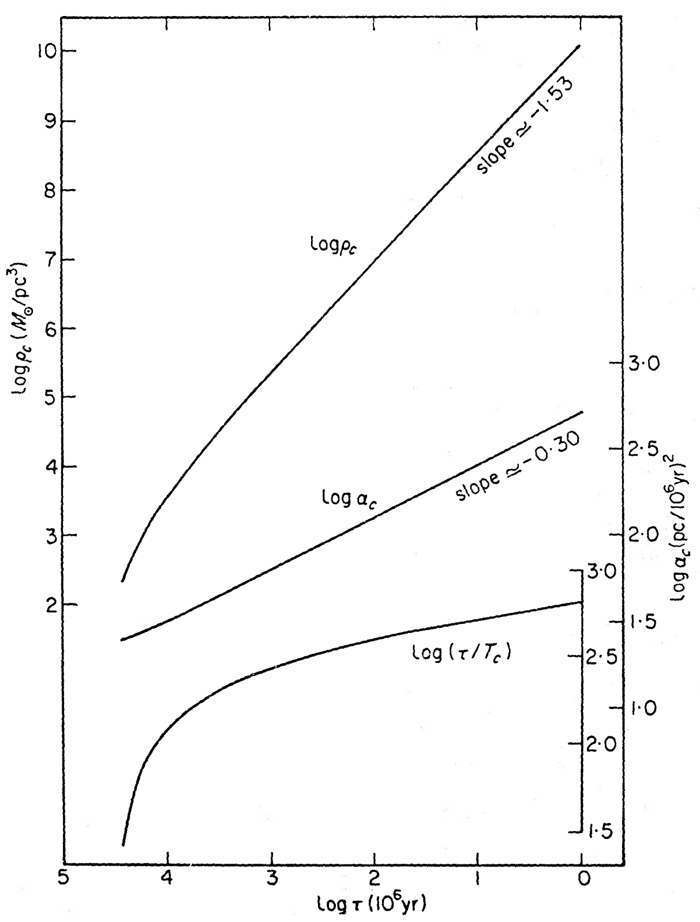

FIG. 2. *The variation with time of ρ_c, α_c, and τ/T_c for a globular cluster (Case 1).*

curves become nearly linear during the later stages of the evolution, it appears that, at least over the time interval covered by the calculations, they never become strictly linear as they did in the example studied in Paper I; thus one cannot determine a precise limiting slope, but only a mean slope over some specified range in τ. In the range $1 \gtrsim \log \tau \gtrsim 0$, the slopes of the two curves are about $-1{\cdot}53$ and $-0{\cdot}30$ respectively, so that we have approximately

$$\left.\begin{array}{l}\rho_c \propto \tau^{-1.53} \\ \alpha_c \propto \tau^{-0.30}\end{array}\right\} \qquad (4)$$

over this time interval.

Also illustrated in Fig. 2 is the variation with time of τ/T_c, the ratio of the time scale of evolution to the central relaxation time. This ratio varies by a large factor during the course of the evolution, increasing from an initial value of about 29 to a value of about $6{\cdot}7 \times 10^2$ at $\tau = 1$ (unit of τ is 10^6 yr). As far as the calculations have been carried ($\tau \sim 0{\cdot}3$), the ratio τ/T_c shows no sign of approaching an asymptotic limiting value; however, it still has not become as large as the limiting value of $8{\cdot}9 \times 10^2$ found for the example studied in Paper I. Examination of the results shows that the reason for the steady increase in τ/T_c can be traced to the continuing relaxation of the central part of the cluster toward an isothermal structure with a nearly Maxwellian velocity distribution. Initially the cluster deviates considerably from an isothermal structure, and correspondingly the velocity distribution is significantly non-Maxwellian ($\xi/\alpha^2 = -0{\cdot}42$ at the centre, indicating a deficiency of high velocity stars relative to a Maxwellian distribution). As the system evolves, the velocity distribution at the centre relaxes toward a more nearly Maxwellian form, and a nearly isothermal region begins to develop at the centre and grow outwards. At the latest time shown in Fig. 1 ($\tau = 3{\cdot}3$) the deviation from a Maxwellian velocity distribution has become quite small at the centre of the cluster ($\xi/\alpha^2 = -3{\cdot}5 \times 10^{-3}$), and the central part of the cluster is very nearly isothermal out to a radius where the density is approximately a factor of 10 smaller than the central density. As was discussed in Paper I, the rate of evolution of a stellar system is quite sensitive to how much the central part of the system deviates from an equilibrium isothermal structure; the closer it is to an isothermal structure and the closer the velocity distribution is to a Maxwellian form, the slower is the evolution. This accounts for the continual slowing down of the evolution relative to the relaxation time, as manifested by the steady increase in the ratio τ/T_c.

Fig. 3 shows the variation with radius of α and β, the mean squared random velocities in the radial and transverse directions respectively, for two times: $\tau = 1{\cdot}6 \times 10^4$ (i.e., $t = 10^{10}$ yr), and $\tau = 3{\cdot}3$. These curves demonstrate clearly the increasing anisotropy of the velocity distribution in the outer region of the cluster. After 10^{10} yr ($\tau = 1{\cdot}6 \times 10^4$), the ratio α/β has increased from $1{\cdot}0$ to a value of about $6{\cdot}8$ at the outer boundary, corresponding to a ratio of velocity dispersions of $(\alpha/\beta)^{1/2} = 2{\cdot}6$. We note, however, that if this state of evolution is taken as representative of the present structure of globular clusters, then it appears that this degree of anisotropy, while definitely a significant effect, is not as large as that postulated in some previous models of globular clusters (e.g., Michie 1961). As the system evolves, the anisotropy of the velocity distribution increases steadily; at $\tau = 3{\cdot}3$, the anisotropy at the outer boundary has increased to $\alpha/\beta \simeq 4 \times 10^3$, corresponding to $(\alpha/\beta)^{1/2} \simeq 63$.

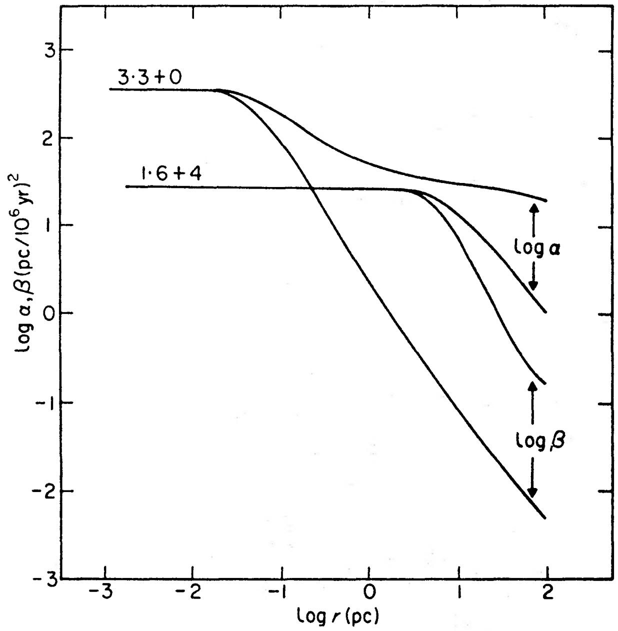

FIG. 3. *The variation with radius of α and β for a globular cluster at two stages of evolution (Case 1). The corresponding values of τ are marked on the curves in units of* 10^6 *yr*

Because of the presence of the absorbing wall at the outer boundary, the mass of the cluster decreases steadily as it evolves. The variation with time of the total mass M remaining inside the outer boundary is indicated by the solid curve in Fig. 4, which shows log M plotted vs. log τ. At $\tau = 1 \cdot 6 \times 10^4$, the total mass has decreased from $2 \cdot 0 \times 10^5 \, M_\odot$ to $1 \cdot 64 \times 10^5 \, M_\odot$, and at $\tau = 3 \cdot 3$ has decreased by a factor of about 6 to $3 \cdot 5 \times 10^4 \, M_\odot$. It is noteworthy that during the later stages of the evolution, not all of the mass inside the outer boundary at $r = 100$ pc remains bound to the cluster, since many of the stars in the outer regions have by this time acquired velocities in excess of the escape velocity (here taken as the velocity required to escape to infinity); in fact, even the mean velocity $\langle u \rangle$ begins to exceed the escape velocity in the outermost part of the system. In order to obtain an estimate of the mass which remains bound to the cluster, we have calculated a quantity M_b, defined as the mass inside the radius at which $\langle u \rangle$ becomes equal to the escape velocity. The time variation of M_b is indicated by the dashed curve in Fig. 4. At $\tau = 3 \cdot 3$, for example, the radius at which $\langle u \rangle$ becomes equal to the escape velocity is about 15 pc, and the mass inside this radius is about $2 \cdot 5 \times 10^4 \, M_\odot$.

It is of interest to compare the time scale for the escape of stars, defined here by $t_e \equiv |d \ln M/dt|^{-1}$, with the central relaxation time T_c. Over the first 10^9 yr of the evolution, during which there is little change in the structure of the cluster, we obtain $t_e \simeq |\Delta t/\Delta \ln M| = 3 \cdot 5 \times 10^{10}$ yr, which is about 39 times the initial relaxation time at the centre. Over the first 10^{10} yr we have

$$|\Delta t/\Delta \ln M| = 5 \cdot 0 \times 10^1 \; 5 \cdot 0 \times 10^{10} \text{ yr},$$

or about 56 times the initial value of T_c. At $t = 10^{10}$ yr the instantaneous value of t_e/T_c is about $2 \cdot 2 \times 10^2$, and thereafter this ratio increases continually, reaching a value of approximately 3×10^3 at $\tau = 3 \cdot 3$ (here we have used M_b in place of M in calculating t_e). Thus it is clear that the ratio t_e/T_c is quite sensitive to the structure of the cluster and increases by a large factor during the evolution.

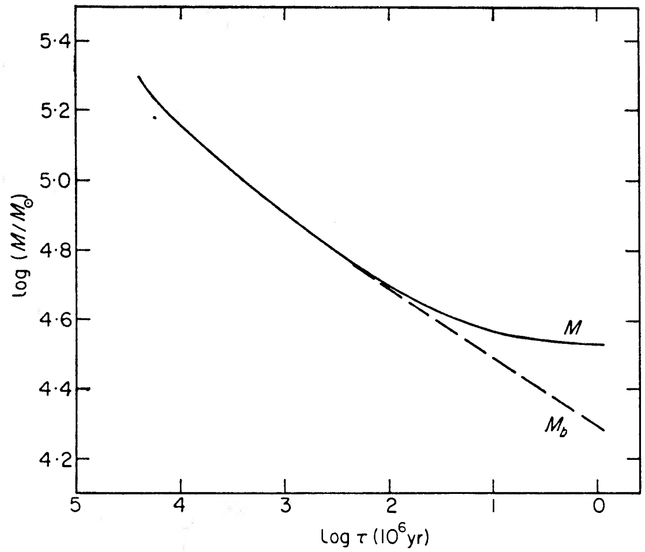

FIG. 4. *The variation with time of the mass of a globular cluster (Case 1). M denotes the total mass inside the outer boundary at r = 100 pc, and M_b is an approximate measure of the mass which remains gravitationally bound to the cluster (see text).*

Similar conclusions and a similar range of values for t_e/T_c were obtained by King (1966), who considered a sequence of simple tidally bounded cluster models with increasing central concentration. It is difficult to make a more quantitative comparison with King's results, however, since the present models differ somewhat in structure from those studied by King; also, we note that while King predicted a nearly constant mass-loss rate, the present results give a steadily increasing mass-loss rate. In any case, the present results and those of King (1966) make it clear that the more classical theories of the escape of stars from clusters, e.g. King (1958), do not adequately represent the escape rate since they do not take into account the effect of the varying structure of the system or the effect of a tidal boundary.

According to the formula given by King (1962), the tidal limiting radius of a cluster should vary proportionally to $M^{1/3}$ as the mass of the cluster decreases. To test the importance of this variation in radius, the calculations were repeated with a boundary radius varying as $M^{1/3}$; the radius then decreases by about a factor of 2 over the time interval of the calculations. The results are in all respects quite similar to those obtained with a fixed boundary; in fact, they would be closely reproduced by simply truncating the previously calculated models at a radius which decreases proportionally to $M^{1/3}$. Calculations were also made with a fixed radius of 200 pc instead of 100 pc, and again the results for the region inside $r = 100$ pc are very similar to those previously calculated. Thus it appears that the exact location of the boundary is not a matter of critical importance, at least as far as the evolution of the central part of the system is concerned.

In order to test the importance of the assumed initial conditions, calculations were made for a number of different initial models for a globular cluster. The various initial density distributions tried are listed in Table I, along with the initial values of T_c and τ/T_c. In all cases the total mass is $2 \times 10^5 M_\odot$ and the boundary radius R is 100 pc. In Cases 2, 3, and 4 the initial density distribution has been chosen such that the density vanishes at the outer boundary, as advocated by King (1962, 1966). It is evident from the values of τ/T_c listed in Table 1 that

Table I

Case No.	$\rho_0(r)$ $(M_\odot/\mathrm{pc}^3)$	T_c (10^8 yr)	τ/T_c
1	$218\left[\frac{1}{1+(r/6)^2}\right]^{5/2}$	9·0	29
2	$188\left[\frac{1}{1+(r/5)^2}-\frac{1}{1+(R/5)^2}\right]^2$	7·5	46
3	$239\left[\left(\frac{1}{1+(r/3\cdot5)^2}\right)^{1/2}-\left(\frac{1}{1+(R/3\cdot5)^2}\right)^{1/2}\right]^3$	4·9	82
4	$151\left[\frac{1}{1+(r/3\cdot5)^2}-\frac{1}{1+(R/3\cdot5)^2}\right]^{3/2}$	4·3	152

the rate of evolution of the cluster, as judged by the time required to reach infinite central density, is strongly dependent on the initial structure of the cluster. Examination of the initial models shows clearly that the rate of evolution is closely related to how nearly the central part of the cluster approaches an isothermal structure with a Maxwellian velocity distribution; the more closely it approaches an isothermal structure, the slower is the evolution.

As an example of the results obtained with a different initial density distribution, we present some results for Case 4 of Table I, which is the one differing most from Case 1 which we have already described. The time development of the density distribution for Case 4 is illustrated in Fig. 5. In this case the evolution proceeds more slowly than in Case 1, and after 10^{10} yr the central density has

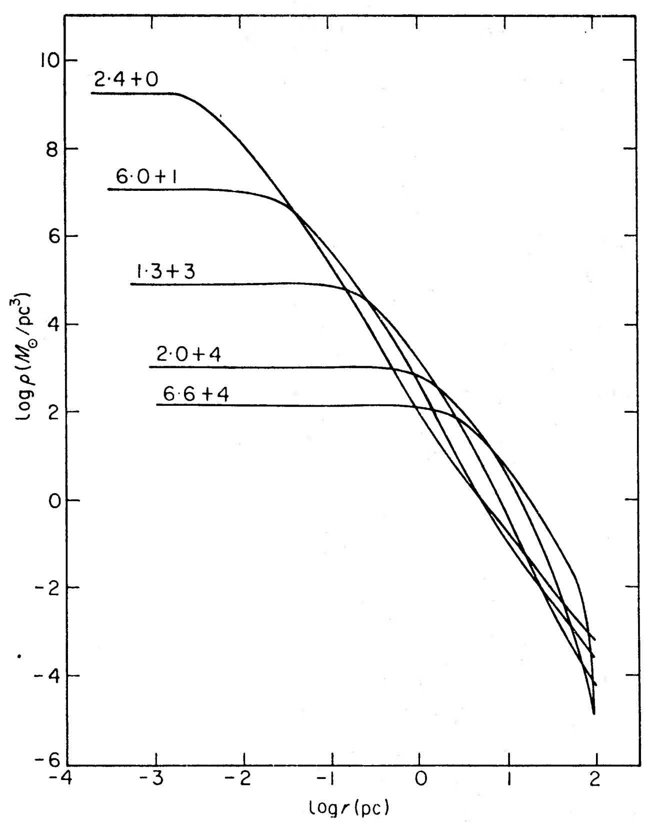

Fig. 5. *The time development of the density distribution for a globular cluster (Case 4). The curves are labelled with the corresponding values of τ in units of 10^6 yr.*

increased by only about 45 per cent to $2 \cdot 2 \times 10^2 \, M_\odot/\mathrm{pc}^3$. Although the density distribution for Case 4 is initially rather different from Case 1, the two density distributions become more nearly similar as the evolution proceeds, and for central densities above about $10^5 \, M_\odot/\mathrm{pc}^3$ they become almost indistinguishable in form. However, they still differ significantly for central densities characteristic of real globular clusters ($\lesssim 10^3 \, M_\odot/\mathrm{pc}^3$); thus it is clear that the presently observed structure of globular clusters cannot be explained solely on the basis of relaxation or evolutionary effects, but must still reflect the initial conditions. Thus if the globular clusters are all as similar as was claimed by King (1962), this must be ascribed to some similarity in the conditions of formation, unless additional relaxation is produced by some mechanism not considered here. Alternatively, the apparent similarity in structure of the globular clusters may just reflect the well-known ease of fitting the observations with a wide variety of theoretical models.

We note also that, although the initial model has a density distribution which vanishes at the boundary, the evolved models all have a finite boundary density which increases with time. This occurs because of the finite and steadily increasing flux of stars escaping from the cluster, which requires a finite density at the boundary. In fact, there will be a finite density even outside the boundary, due to stars which have recently escaped from the cluster, and the density distribution will vary continuously across the boundary. Thus the concept of a discrete boundary

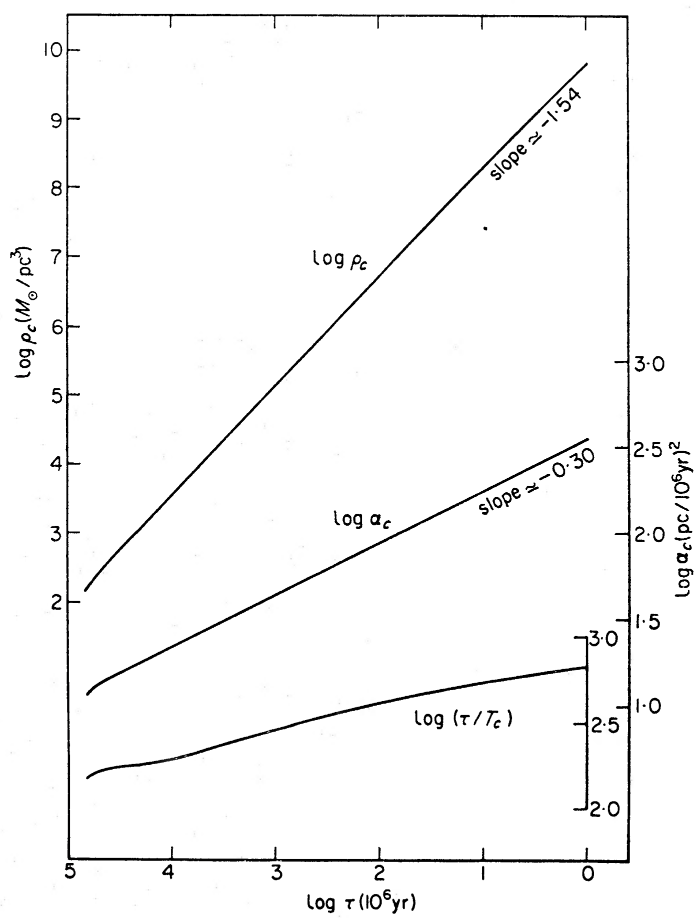

FIG. 6. *The variation with time of ρ_c, α_c, and τ/T_c for a globular cluster (Case 4).*

characterized by a sharp cutoff in the density distribution is seen to be only an idealization; in general, a cluster will not show a sharp cutoff in the density distribution, and the boundary might be difficult to define with any precision.

The variation of log ρ_c, log α_c, and log (τ/T_c) with log τ in Case 4 is shown in Fig. 6. During the later stages of the evolution the curves have very nearly the same slopes as for Case 1; however, for a given τ, ρ_c is smaller by a factor of 0·57 and α_c is smaller by a factor of 0·69 than in Case 1. Thus although the density and temperature distributions eventually become almost identical in form in Cases 1 and 4, there remain scale factors which differ between the two cases and which depend on the initial conditions. Similarly, the way in which the mass varies with time is much the same in the two cases, except that for a given value of τ the mass is smaller in Case 4 than in Case 1 by a factor of about 0·77.

4. EVOLUTION OF A GALACTIC CLUSTER

In the case of small clusters with $n \approx 100$ stars, it is no longer valid to assume, as is the case with large systems, that relaxation effects occur primarily as the result of a sequence of small perturbations to the stellar motions; large perturbations become quite important. Therefore the assumptions underlying the Fokker–Planck equation begin to break down, and theories such as the present one which are based on the Fokker–Planck equation cannot be expected to give very accurate results. Nevertheless, it seems interesting to try this case and see how the results compare with previous theories for small clusters.

As a typical mass for a galactic cluster we have adopted $M = 100\,M_\odot$, and for a tidal limiting radius we have taken $R = 10$ pc. The factor $\ln\,(D_{\max}\langle V^2\rangle/2Gm)$ occurring in the relaxation terms has been set equal to a constant value of 3 throughout the calculations. For the initial model we have adopted a polytrope of index 5 with a density distribution given by

$$\rho(r) = 3\cdot 1\left[\frac{1}{1+(r/2)^2}\right]^{5/2} M_\odot/\text{pc}^3. \tag{5}$$

The initial velocity dispersion at the centre is $\alpha_c^{1/2} = 0\cdot 20$ pc/10^6 yr, and the corresponding relaxation time is $T_c = 1\cdot 3\times 10^7$ yr. The evolution of the density distribution with time is illustrated in Fig. 7, where the curves are again labelled with the corresponding values of τ in units of 10^6 yr. The time required to reach infinite central density is $4\cdot 1\times 10^8$ yr, or about 31 times the initial value of T_c. In the case of a galactic cluster the central part of the system runs out of stars at a relatively early stage in the evolution; in fact, at the last time shown in Fig. 7 ($\tau = 7\cdot 0$), the mass inside the point where the density drops to half its central value is only about $0\cdot 8\,M_\odot$. Thus it would not make much sense to carry the calculations any farther, if indeed this far.

The variation of log ρ_c, log α_c, and log (τ/T_c) with log τ is shown in Fig. 8. The curves are seen to differ somewhat from those for a globular cluster (Fig. 2); in particular, we note that the curve for log (τ/T_c) in Fig. 8 continues to rise steeply with no sign of levelling off. The primary reason for the difference in evolution between a globular cluster and a galactic cluster appears to be that in the case of the globular cluster the relaxation time T_c is always much longer than the dynamical time, as measured for example by the free-fall time t_f at the centre, whereas in the galactic cluster T_c is of the same order as t_f. In the present example,

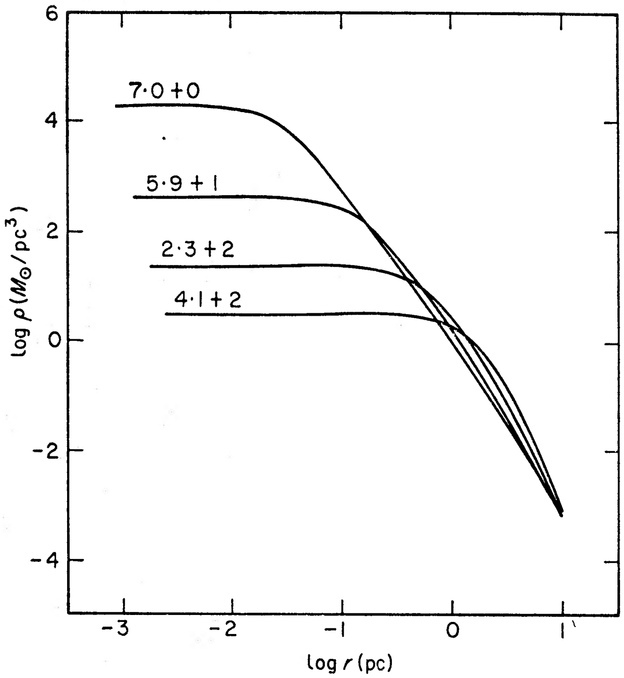

FIG. 7. *The time development of the density distribution for a galactic cluster with initial density distribution given by equation* (5). *The curves are labelled with the corresponding values of* τ *in units of* 10^6 *yr.*

the relaxation time T_c is initially about 3 times longer than the free-fall time t_f at the centre; at $\tau \simeq 1 \cdot 6 \times 10^2$, T_c becomes equal to t_f, and after this time T_c is smaller than t_f. In this case, since the orbital periods of the stars are for the most part considerably longer than t_f, the orbital periods are also longer than the relaxation time T_c; consequently the orbital motions are unable to respond immediately to relaxation processes, and the rate of evolution of the system becomes limited by the time scale of the orbital motions. In fact, we find in the present example that the ratio τ/t_f is more nearly constant than the ratio τ/T_c, and it is closer to unity during the later stages of the evolution; for example, at $\tau = 7 \cdot 0$ we have $\tau/t_f \simeq 120$ and $\tau/T_c \simeq 560$. Thus the time scale of evolution appears to be more closely related to the dynamical time than to the relaxation time.

The variation with time of the total mass of the cluster is shown in Fig. 9. In this case the mean outward velocity $\langle u \rangle$ never exceeds the escape velocity, so there is no distinction between M and M_b. It is evident in Fig. 9 that the mass-loss rate remains nearly constant as the system evolves, instead of increasing with time as was found for a globular cluster. The reason for this appears to be that the mass-loss rate is limited by the time scale of the orbital motions of the stars in the outer part of the cluster; this time scale, as we have seen, is considerably longer than the relaxation time at the centre, and it does not change much as the system evolves, so the mass-loss rate remains roughly constant. It is clear also that the existence of a tidal limit, represented here by an absorbing wall, is a dominant influence in facilitating the loss of stars from the system, since most of the stars which cross the boundary have velocities too small to allow them to escape to infinity; therefore the mass-loss rate is considerably higher than it would be for a completely isolated cluster.

In order to find how the evolution would proceed for a more nearly isolated

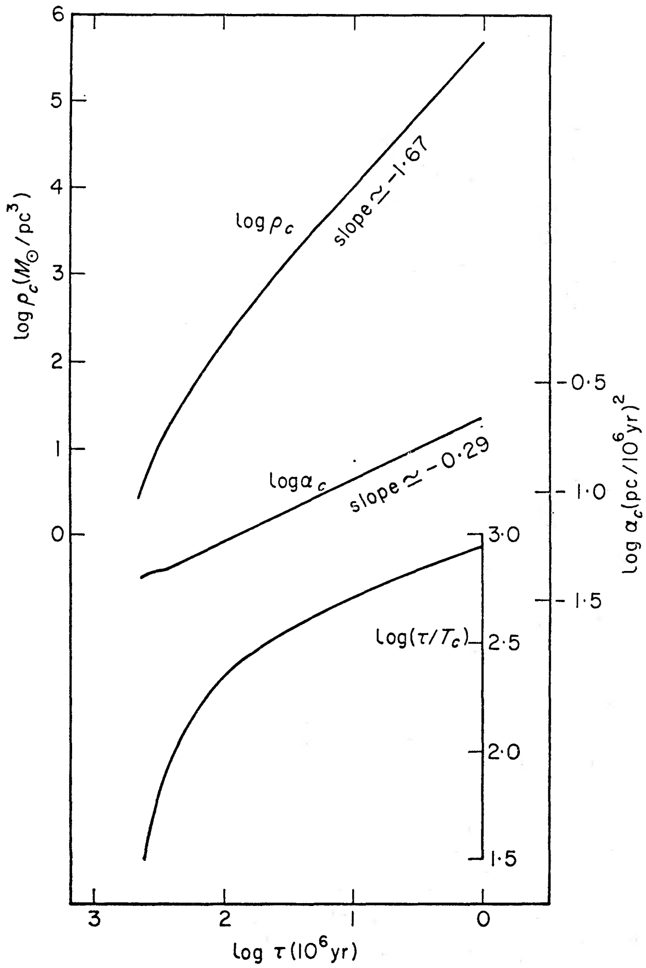

FIG. 8. *The variation with time of ρ_c, α_c, and τ/T_c for a galactic cluster.*

cluster, the calculations were repeated with a boundary radius of 100 pc instead of 10 pc. In this case the evolution of the part of the cluster inside $r = 10$ pc is much the same as previously calculated; however, as might be expected because of the larger boundary radius, the mass-loss rate for the system is considerably

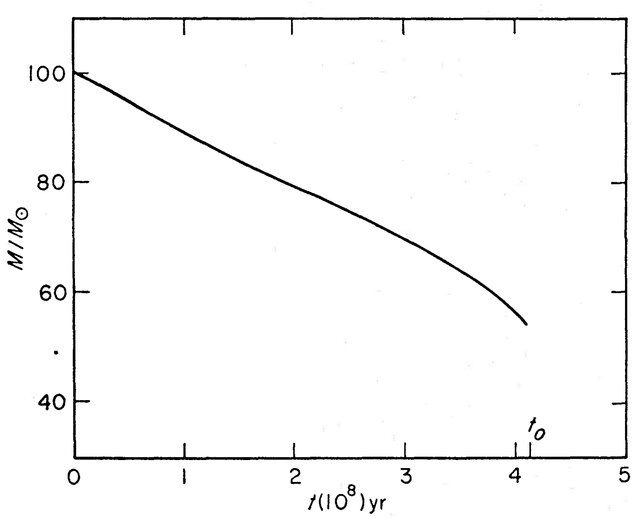

FIG. 9. *The variation with time of the mass of a galactic cluster with a tidal limit at $r = 10$ pc.*

reduced. The variation of mass with time is shown in Fig. 10, where the solid curve gives the total mass inside the boundary at $r = 100$ pc, and the dashed curve gives M_b as previously defined. A calculation made with a boundary radius of 1000 pc yielded results for M_b and for the mass inside $r = 100$ pc which are not significantly different from those shown in Fig. 10; thus the results shown in Fig. 10 may be taken as nearly the same as for a completely isolated cluster.

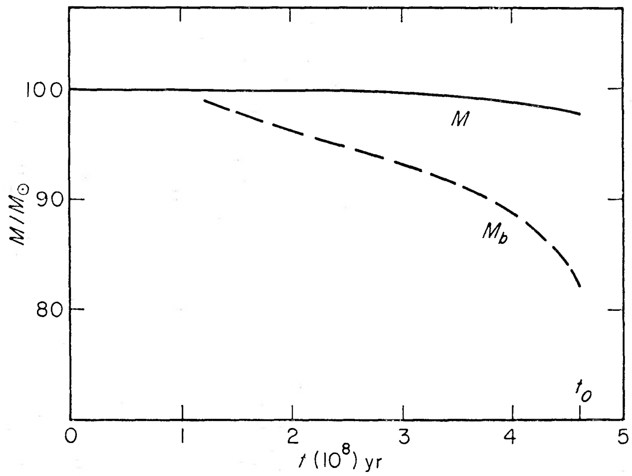

FIG. 10. *The variation with time of M and M_b for a galactic cluster with a tidal limit at $r = 100$ pc.*

It is of interest to compare the mass-loss rate found here with that predicted by previous authors for the same cluster model (i.e., a polytropic model of index $n = 5$ with no tidal cutoff). If we consider a time interval of $2 \cdot 5 \times 10^8$ yr so as to be able to compare with the predictions tabulated by Wielen (1968), we find that M_b decreases by $5 \cdot 0\, M_\odot$ during this time interval. Comparison with Wielen's Table 2 shows that this result is in good agreement, at least in order of magnitude, with the various classical estimates of the escape rate for an isolated cluster. For example, the theory of King (1958), when applied to the same initial model as considered here, predicts a mass loss of $3 \cdot 0\, M_\odot$ in $2 \cdot 5 \times 10^8$ yr. If we were to put $\ln (D_{\max}\langle V^2 \rangle / 2Gm) = 3 \cdot 0$ in King's formulas, as has been done in the present work, they would predict a mass loss of $3 \cdot 8\, M_\odot$ in $2 \cdot 5 \times 10^8$ yr.

Unfortunately, neither the present results nor the classical theories agree with the results of n-body calculations for isolated clusters, which in the case of equal masses show a much smaller escape rate than that predicted classically (Wielen 1968). Hénon (1960) argued that the escape rate for an isolated cluster should be much smaller than that predicted classically, since the usual assumption that a star changes its energy through a succession of small increments will not lead to the escape of the star, but only to a gradual increase in the size of its orbit; the only stars which escape, according to Hénon, are those which acquire the necessary energy in a single encounter. It is not clear to what extent this argument is applicable to a galactic cluster, where the time scale for large changes in the energy of a star is comparable with or less than the orbital period; however, the predicted mode of escape, i.e. through a single large change in energy, appears to be verified by the n-body calculations, at least for small clusters with $n \approx 100$ stars. In any case, it seems clear that the concept of a gradual diffusion of stars in velocity space, which is implicit in the Fokker–Planck equation and therefore also in the present calculations, does not adequately predict the rate of escape of stars from small isolated clusters.

If we return to the more realistic case of a cluster with a tidal boundary, however, the diffusion process can still lead to the escape of stars, since stars need only reach the tidal boundary in order to escape; thus the present method may in fact give better results in this case than in the more artificial case of a completely isolated cluster. Wielen (1968) found that the inclusion of tidal effects led to a great increase in the escape rate; over a period of $2 \cdot 5 \times 10^8$ yr he obtained a mass loss of 60–70 $M_\odot$, which is about $2 \cdot 5$ times as large as the mass loss of 25 $M_\odot$ found in the present calculations (Fig. 9). In this case the difference may be due to the fact that Wielen's calculations for the tidally bounded case were made with a distribution of masses, which is known (at least in the case of an isolated cluster) to lead to a substantially higher escape rate than is the case with equal masses.

5. EVOLUTION OF A DENSE GALACTIC NUCLEUS

The possibility that some exceptionally dense galactic nuclei may evolve to the stage where collisions between the stars become important has been a subject of considerable interest in recent years because of its possible connection with quasi-stellar objects (Spitzer & Saslaw 1966; Spitzer & Stone 1967). Von Hoerner (1968) has discussed the evolution of dense galactic nuclei using a simple theory for the dynamical evolution of a stellar system, and has shown how a dense galactic nucleus might within 10^{10} years evolve into a collision-dominated state resembling a typical quasi-steller object in its principal properties. According to Von Hoerner, this could be achieved if the central part of the nucleus originally has a mass of about $2 \times 10^7 M_\odot$ and a radius of about $0 \cdot 6$ pc, corresponding to a central density of about $2 \times 10^7 M_\odot/\text{pc}^3$. It must be assumed that this high initial central density was produced at the time of formation of the system, before all of the protogalactic material had been converted into stars. The formation of a centrally condensed system with a sufficiently high central density might occur in the way that was described by Larson (1969).

To illustrate how a dense galactic nucleus might evolve, we have considered a system having a mass of $10^8 M_\odot$, a boundary radius of 10^4 pc, and an initial density distribution given by

$$\rho(r) = 2 \cdot 8 \times 10^7 \left[\frac{1}{1+(r/0 \cdot 5)^2}\right]^{1 \cdot 7} M_\odot/\text{pc}^3. \qquad (6)$$

For this density distribution the central velocity dispersion is

$$\alpha_c^{1/2} = 2 \cdot 0 \times 10^2 \text{ pc}/10^6 \text{ yr}$$

and the central relaxation time is $T_c = 2 \cdot 1 \times 10^8$ yr. The quantity

$$\ln (D_{\max}\langle V^2\rangle/2Gm)$$

has in this case been set equal to 20 throughout the calculations.

The evolution of the density distribution in the region inside $r = 100$ pc (a region which contains ~ 90 per cent of the total mass) is shown in Fig. 11. In this case the time required to reach infinite central density is $2 \cdot 6 \times 10^{10}$ yr, which is about 124 times the initial value of T_c. As we have noted previously,

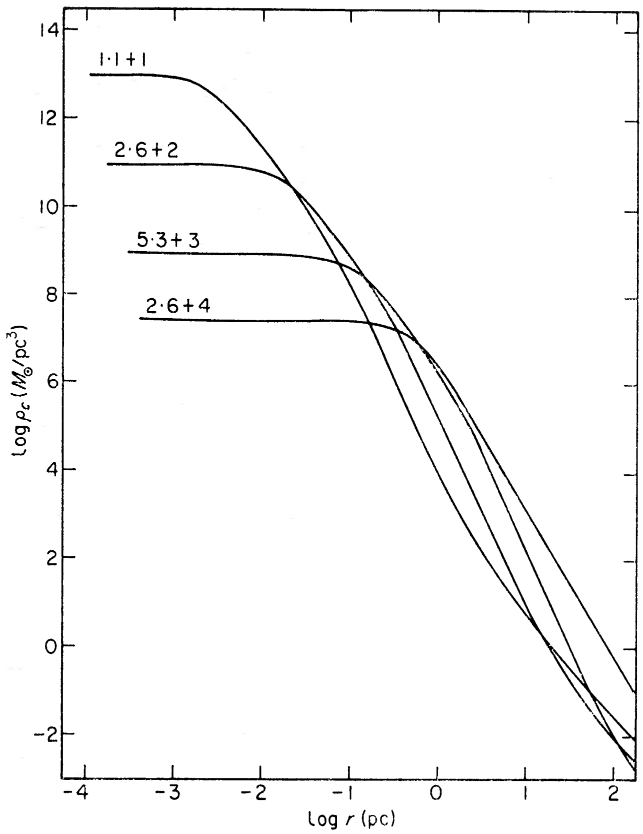

FIG. 11. *The time development of the density distribution for a dense galactic nucleus with initial density distribution given by equation* (6). *The curves are labelled with the corresponding values of* τ *in units of* 10^6 *yr.*

the time required to reach infinite central density is quite sensitive to the structure of the system, and can be as short as 30 times the initial value of T_c, or less; thus no special significance attaches to the evolution time of $2 \cdot 6 \times 10^{10}$ yr found here, and it is possible that with different initial conditions the evolution to infinite central density could have occurred in 10^{10} yr or less.

It is interesting to note the form of the density distribution found in the present results and to compare it with the form predicted by the theory of Von Hoerner (1968). At the latest time shown in Fig. 11 ($\tau = 11$), the curve is approximately linear between $r \sim 4 \times 10^{-3}$ pc and $r \sim 4 \times 10^{-2}$ pc, with a mean slope $d \ln \rho / d \ln r \simeq -2 \cdot 6$; this is not very different from the value of $-2 \cdot 4$ predicted by Von Hoerner for the inner part of an evolving cluster. (In the case of a globular cluster, the corresponding slopes obtained from Figs 1 and 5 are about $-2 \cdot 5$ and $-2 \cdot 6$ respectively.) Between $r \sim 10^{-1}$ pc and $r \sim 1$ pc there is another nearly linear section of the curve with a mean slope of about $-4 \cdot 5$, which is somewhat steeper than the slope of $-3 \cdot 75$ predicted by Von Hoerner for the outer part of an evolving cluster. Here the difference may be related to the fact that in the present calculations the velocity distribution becomes strongly anisotropic in the region in question ($4 \lesssim \alpha/\beta \lesssim 50$), whereas Von Hoerner's theory assumes that the velocity distribution is always isotropic. In the outermost part of the system ($r \gtrsim 30$ pc), the curve is again nearly linear, with a slope of about $-2 \cdot 2$. This outermost region is populated mainly by stars which are in the process of escaping from the system, and is characterized by nearly constant values of the mean outward velocity ($\langle u \rangle \sim 90$ pc/10^6 yr) and of the velocity dispersion ($\alpha^{1/2} \sim 70$ pc/10^6 yr).

In reality the escape of stars from a galactic nucleus would presumably be inhibited by the gravitational field of the whole galaxy, which is much more massive than the nucleus; however, in view of the insensitivity of the results to the exact boundary conditions, this would probably not make much difference to the evolution of the central part of the nucleus.

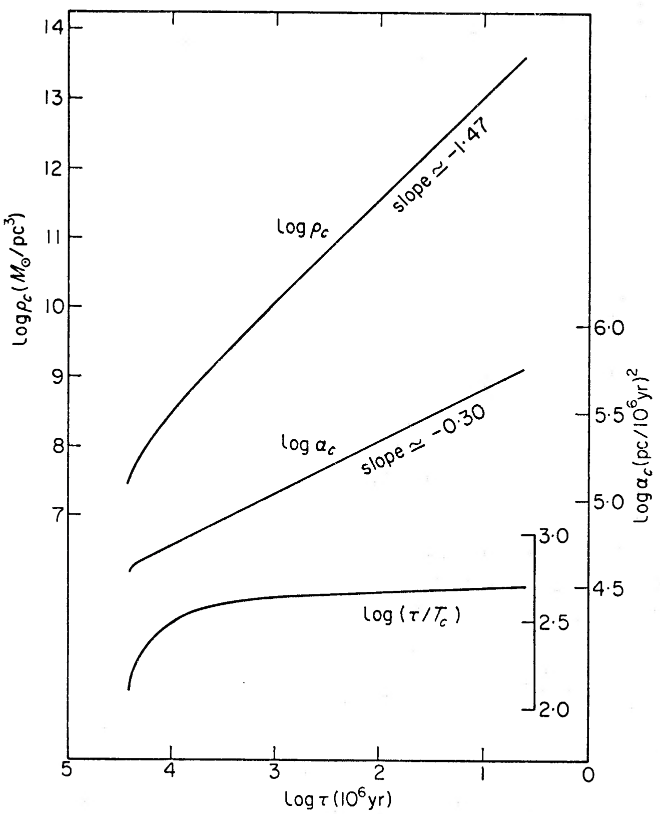

FIG. 12. *The variation with time of ρ_c, α_c, and τ/T_c for a dense galactic nucleus.*

The variation of log ρ_c, log α_c, and log (τ/T_c) with log τ is illustrated in Fig. 12. During the later stages of the evolution we have approximately

$$\left.\begin{array}{l}\rho_c \propto \tau^{-1\cdot 47} \\ \alpha_c \propto \tau^{-0\cdot 30},\end{array}\right\} \qquad (7)$$

and the ratio τ/T_c has a value of about 5×10^2 and is increasing only very slowly with time. The exponents in equations (7) are somewhat different from the values of $-1\cdot 33$ and $-0\cdot 22$ predicted by the theory of Von Hoerner (1968); however, they deviate in the opposite direction from the values of $-1\cdot 56$ and $-0\cdot 37$ predicted by the earlier theories of Von Hoerner (1958) and King (1958). The closest agreement appears to be obtained with the simple derivation of Miller & Parker (1964) which assumes that the escaping stars carry away no energy; this gives $\rho_c \propto \tau^{-1\cdot 43}$ and $\alpha_c \propto \tau^{-0\cdot 29}$. This agreement is perhaps not surprising, since examination of the results shows that the escaping stars do indeed carry away only a very small amount of energy—the mean energy of an escaping star is only $\approx 10^{-2}$ times the mean energy of the stars in the system.

We note that the calculations which we have described for a galactic cluster, a globular cluster, and a dense galactic nucleus all fall into a sequence of increasing mass, increasing size, increasing central concentration, and increasing (i.e. less negative) values for the slope $d \ln \rho_c / d \ln \tau$. Examination of the results shows that, because of the increasing central concentration along this sequence, the stars in the outermost part of the system, including the escapers, carry a smaller and smaller fraction of the total energy of the cluster. Simple derivations such as those given by King (1958) and Miller & Parker (1964) predict that the smaller the energy carried away by the escaping stars, the less negative is the value of $d \ln \rho_c / d \ln \tau$, the limiting value for no energy loss being equal to $-1\cdot43$. These predictions fit well with the results of the present calculations.

At the latest time shown in Fig. 11 ($\tau = 11$), the central density ρ_c is $9\cdot1 \times 10^{12}$ $M_\odot/\text{pc}^3$ and the velocity dispersion $\alpha_c^{1/2}$ is $6\cdot4 \times 10^2$ pc/10^6 yr, which implies a relaxation time $T_c = 2\cdot3 \times 10^4$ yr. Under these conditions collisions between the stars become quite important. The collision time may be estimated as

$$t_{\text{coll}} = (N\sigma V)^{-1},$$

where N is the number density of stars, σ is the collision cross-section, and $V = (6\alpha)^{1/2}$ is the mean relative velocity between two stars. If we take $\sigma \approx \pi (R_\odot)^2$ as a representative cross section for a strongly disruptive collision and use the values quoted above for the central density and velocity dispersion, we obtain a collision time $t_{\text{coll}} \sim 2 \times 10^4$ yr at the centre. Since this collision time is comparable to the relaxation time T_c and much shorter than the evolutionary time scale τ, it is clear that the collisions must by this time have an important and probably dominant effect on the further development of the system. The radius and mass over which the density and the collision time are within an order of magnitude of their central values are roughly 5×10^{-3} pc and $10^6\, M_\odot$ respectively, and the total collision rate is of the order of 15 strongly disruptive collisions per year. These numbers are quite similar to the results obtained by Von Hoerner (1968), and they seem capable of accounting for some of the principal properties of quasi-stellar objects, particularly when one considers that the system is rapidly evolving into a state with even more extreme properties (Spitzer & Saslaw 1966; Spitzer & Stone 1967).

6. CONCLUSIONS

In so far as a comparison can be made, the results of the present investigation appear to be in general agreement with the predictions of classical relaxation theory. Thus the present technique may be regarded as essentially just a more elaborate and precise formulation of classical relaxation theory, which however is more powerful in that it allows the evolution of a stellar system to be computed in considerably more detail than was previously possible.

The general conclusions of this project are summarized below. For the most part these conclusions are not new, but they are demonstrated particularly clearly in the present calculations.

(1) The evolution of a stellar system is always such that the central part of the system tends to relax toward an isothermal structure with a nearly Maxwellian velocity distribution. (This contrasts with the results for the artificial problem studied in Paper I, where the evolution was *away* from an isothermal sphere; however, the final state obtained in Paper I was still more nearly isothermal than any of the models calculated in the present paper.)

(2) The rate of evolution and the rate of escape of stars from a cluster are strongly dependent on its structure, and are slower the more nearly the central region approaches an isothermal structure. Thus as a cluster evolves the rate of evolution slows down relative to the central relaxation time, although in absolute terms it continues to speed up.

(3) In no case was a very close approach to homologous evolution of the central part of the system obtained; even though the density distributions approach similar forms, there remain scale factors which differ from case to case and which are determined by the initial conditions. If the state of homologous evolution found in Paper I is ever reached, it must be at such an advanced stage of evolution as to be of little or no relevance for real stellar systems.

(4) The velocity distribution always tends to become strongly anisotropic in the outer part of the system, in the sense that the stellar motions tend to be preferentially in the radial direction. However, the degree of anisotropy appears not to be as extreme as that postulated in some previous cluster models.

(5) The ratio of the relaxation time to the dynamical time is an important parameter for the evolution of a star cluster, and this leads to some qualitative differences in the evolution of globular and galactic clusters.

(6) The existence of a tidal boundary has a strong influence on the escape rate, particularly in the case of galactic clusters, where the escape rate is increased by a large factor compared with an isolated cluster.

(7) In general the density distribution varies smoothly across the ' boundary ' of a cluster, and does not suffer a cutoff or a sharp drop at the boundary; this might make it difficult in some cases to distinguish observationally any well-defined boundary for a star cluster.

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