AN APPLICATION OF HENYEY'S APPROACH TO THE INTEGRATION OF THE EQUATIONS OF STELLAR STRUCTURE

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ABSTRACT

A method is described for the numerical solution of the equations of stellar structure by a technique especially designed for the solution of boundary-value problems. The numerical solution incorporates the boundary conditions from the start and eliminates the necessity for the trial-and-error calculations involved in many previous methods. Model-atmosphere calculations and mixing-length theory can readily be incorporated for accurate treatment of convective envelopes, and the method is capable of treating more or less exactly the energy generation and opacity laws and the equation of state for partial degeneracy. In addition, the method is well suited for the computation of evolutionary sequences of models. The method has been successfully used at the University of Toronto for a number of purposes, including the computation of evolutionary tracks from the main sequence for the purpose of estimating the ages of old star clusters.

I. INTRODUCTION

With the advent in recent years of large high-speed computers such as the IBM 7090 it has been possible to improve considerably on earlier methods for integration of the equations of stellar structure. In particular, it is now possible and highly desirable to devise computer programs which are fully automatic and can construct complete stellar models and even evolutionary sequences of models without requiring human intervention. In the following article, we shall describe a method for automatic computation of stellar models which we have successfully used on the IBM 7090 of the University of Toronto.

In the computation of stellar models we have to deal with a set of four first-order differential equations, the solutions of which must satisfy four corresponding boundary conditions, two at the center of the star and two at the surface. The methods most widely used in the past for solving this problem, such as those of Schwarzschild (1958), have required much cumbersome trial-and-error work and fitting of solutions before a solution satisfying all the boundary conditions could be obtained. For this reason they are not well suited to automatic computation of stellar models. More direct methods are, however, available for the solution of boundary-value problems such as the one we have to deal with; these methods, which have been described by Fox (1957), incorporate the boundary conditions from the start and eliminate the necessity for any fitting or trial-and-error work. In Section II of this paper, we shall outline the principles involved, and in the succeeding sections we shall describe how they may be applied to the problem of stellar structure.

A method similar to the one to be outlined was first used for computation of stellar models by Henyey, Wilets, Böhm, LeLevier, and Levée (1959), and several important features of the present method are patterned after Henyey's method.

II. GENERAL METHOD FOR SOLUTION OF BOUNDARY-VALUE PROBLEMS WITH ONE INDEPENDENT VARIABLE

In the method to be outlined the techniques described in the monograph of Fox (1957) are applied to the case of a set of simultaneous first-order differential equations.

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Let x be the independent variable, and suppose there are m dependent variables $y^1(x) \ldots y^m(x)$, each of which satisfies a first-order differential equation of the form

$$\frac{dy^k}{dx} = f^k(y^1 \ldots y^m, x).$$

Considering the y^k's and the f^k's as the components of two m-dimensional vectors $\boldsymbol{y}$ and $\boldsymbol{f}$, we can write the complete set of differential equations in the compact form

$$\frac{d\boldsymbol{y}}{dx} = \boldsymbol{f}(\boldsymbol{y}, x). \tag{1}$$

We suppose that equation (1) is to be solved over an interval $0 \leq x \leq X$ in the independent variable, and that the boundary conditions occur at the two points $x = 0$ and $x = X$.

The method to be used consists in dividing the interval $[0, X]$ in the independent variable into a number n of subintervals by means of $n + 1$ suitably chosen points $x_1 \ldots x_{n+1}$, with $x_1 = 0$ and $x_{n+1} = X$. In each of these subintervals the m differential equations are replaced by approximating difference equations relating the values of the dependent variables at adjacent points. Together with the m boundary conditions necessary to determine a solution, these difference equations, m for each of the n intervals, provide a set of m $(n + 1)$ equations which if they can be solved determine approximate values for the m $(n + 1)$ unknown quantities y^k (x_i), $i = 1 \ldots n + 1$, $k = 1 \ldots m$. The accuracy of these approximate values depends on the accuracy with which the difference approximations represent the differential equations.

The difference approximation which we shall employ is derived from the following equation, which holds rigorously for any function $y(x)$ with a bounded third derivative:

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \tfrac{1}{2}(f_{i+1} + f_i) - \tfrac{1}{12}(x_{i+1} - x_i)^2 y'''(\xi), \tag{2}$$

where

$$y_i = y(x_i), \qquad f_i = y'(x_i),$$

and

$$x_i \leq \xi \leq x_{i+1}.$$

Provided that the third derivative $y'''(x)$ does not become unduly large anywhere in the interval $[x_i, x_{i+1}]$ (y''' must at least be bounded in magnitude), and that the interval $[x_i, x_{i+1}]$ is made small enough, the last term in equation (2) will be small compared to the other terms and we obtain a sufficiently good approximation by neglecting it. This gives us our difference equations, which we can write, adding superscripts to distinguish the m dependent variables, in the form

$$\frac{y^k{}_{i+1} - y_i{}^k}{x_{i+1} - x_i} = \tfrac{1}{2}(f^k{}_{i+1} + f_i{}^k), \qquad i = 1 \ldots n, \qquad k = 1 \ldots m. \tag{3}$$

As shown by Fox, it is possible by using values of y^k and f^k at more than two adjacent points to construct difference equations which are more accurate than equations (3) in the sense that the error term contains a higher power of $(x_{i+1} - x_i)$ and therefore decreases more rapidly as the interval size is reduced. However, these higher-order difference equations are not convenient for our purposes for two reasons: (1) it is necessary to extrapolate values of y^k and f^k beyond the boundaries in order to apply these equations near the boundaries, and (2) their accuracy depends on the existence and boundedness of derivatives of higher order than the third, a condition which is not always satisfied in our work. Therefore, we prefer to use a simple formula such as equations (3), even though more points may be required to give good accuracy.

As may be seen from (2), the errors introduced in neglecting the last term depend on the square of the interval size, so one may expect that the error in the numerical solutions obtained by applying (3) will be approximately proportional to the square of the interval size. Thus if the number of points used were doubled, the errors would be reduced approximately by a factor 4. This provides a valuable means of estimating the error in the numerical solution of the differential equations.

a) Solution of the Difference Equations

If the f^k's were linear functions of the dependent variables $y^1 \ldots y^m$ the difference equations would be linear and could be solved by standard methods of linear algebra. However, in problems of stellar structure we have to deal with non-linear equations, and in this case the difference equations can be solved only by some iterative process, starting from an initial approximation to the solution.

Such an iterative process may be obtained as a straightforward generalization of Newton's method for solving one non-linear equation in one unknown. Given an initial approximation z_0 to the solution of the equation $g(z) = 0$, Newton's method consists in obtaining an improved approximation z_1 by adding to z_0 a correction δz_0 determined by setting

$$g(z_1) = g(z_0 + \delta z_0) \doteqdot g(z_0) + g'(z_0)\delta z_0 = 0 .$$

The problem at hand can be put into an analogous form by defining

$$g^k(\boldsymbol{y}_i, \boldsymbol{y}_{i+1}) \equiv \frac{y^k{}_{i+1} - y_i{}^k}{x_{i+1} - x_i} - \tfrac{1}{2}(f^k{}_{i+1} + f_i{}^k) . \tag{4}$$

The solution of the difference equations (3) is then the solution of the set of equations

$$g^k(\boldsymbol{y}_i, \boldsymbol{y}_{i+1}) = 0, \qquad i = 1 \ldots n, \qquad k = 1 \ldots m . \tag{5}$$

As in Newton's method for one unknown, given an initial approximation $y_i (i = 1 \ldots n + 1)$ to the solution of equations (5), we obtain an improved approximation by adding to the y_i's corrections δy_i whose components $\delta y_i{}^j (j = 1 \ldots m)$ are determined from the set of equations

$$g_k(\boldsymbol{y}_i + \delta \boldsymbol{y}_i, y_{i+1} + \delta \boldsymbol{y}_{i+1})$$

$$\doteqdot g^k(\boldsymbol{y}_i, \boldsymbol{y}_{i+1}) + \sum_{j=1}^{m} \left(\frac{\partial g^k}{\partial y^j}\right)_i \delta y_i{}^j + \sum_{j=1}^{m} \left(\frac{\partial g^k}{\partial y^j}\right)_{i+1} \delta y^j{}_{i+1} = 0$$

or

$$\sum_{j=1}^{m} \left(\frac{\partial g^k}{\partial y^j}\right)_i \delta y_i{}^j + \sum_{j=1}^{m} \left(\frac{\partial g^k}{\partial y^j}\right)_{i+1} \delta y^j{}_{i+1} = - g^k(\boldsymbol{y}_i, \boldsymbol{y}_{i+1}),$$

$$i = 1 \ldots n, \qquad k = 1 \ldots m . \tag{6}$$

Equations (6) give us mn linear equations in the $m(n + 1)$ unknowns

$$\delta y_i{}^k, \qquad i = 1 \ldots n + 1, \qquad k = 1 \ldots m .$$

When we supplement these with m additional equations derived from the boundary conditions, we have altogether $m(n + 1)$ linear equations in $m(n + 1)$ unknowns. This set of equations can then be solved by standard methods of linear algebra. We have found quite satisfactory the standard method of Gaussian elimination with pivotal interchanges, described in most books on numerical analysis (see, e.g., National Physical Laboratory 1961).

III. THE EQUATIONS OF STELLAR STRUCTURE

According to the outline in the previous section, our method requires that the range of the independent variable be specified in advance, as well as a set of points in this range at which the dependent variables are to be evaluated. Since we choose to construct models with a specified mass, the mass variable M_r is the only one whose range of variation is known in advance, and we must therefore take M_r as our independent variable. This choice is necessary also for purposes of calculating the change in hydrogen content X as the star evolves; the points at which X is evaluated must always correspond to the same fixed set of values of M_r.

Accordingly we give below the equations of stellar structure in terms of M_r as independent variable. These equations are derived directly from the equations given by Schwarzschild (pp. 96 and 36), and they take account of both nuclear- and gravitational-energy sources. The gravitational-energy term in equation (8) holds for non-relativistic partial degeneracy independently of the degree of degeneracy. In equation (8) τ is the time variable; the other symbols all have the same meanings as in Schwarzschild's book.

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho}, \tag{7}$$

$$\frac{dL_r}{dM_r} = \epsilon + \underbrace{\tfrac{3}{2}\frac{P}{\rho}\frac{d}{d\tau}\ln\left(\frac{\rho^{5/3}}{P}\right)}_{\text{gravitational-energy term}}, \tag{8}$$

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}, \tag{9}$$

$$\frac{dT}{dM_r} = -\frac{3}{64\pi^2 a c}\,\kappa\,\frac{L_r}{T^3 r^4}, \qquad \text{(radiative)} \tag{10R}$$

$$\frac{dT}{dM_r} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dM_r} = -\left(1-\frac{1}{\gamma}\right)\frac{G}{4\pi}\frac{TM_r}{Pr^4}. \qquad \text{(convective)} \tag{10C}$$

Of expressions (10R) and (10C) we are to use whichever gives the smallest value of $|dT/dM_r|$, i.e., the largest (in an algebraic sense) value of dT/dM_r.

The boundary conditions which we shall use are those given by Schwarzschild (p. 97). For the present choice of independent variable, these become

$$r = 0, \quad L_r = 0 \quad \text{at} \quad M_r = 0,$$

and either

$$T = 0, \quad P = 0 \quad \text{at} \quad M_r = M \text{ (radiative)},$$

or

$$T = 0, \quad \frac{P}{T^{2.5}} = K \quad \text{at} \quad M_r = M \text{ (convective)}.$$

In the second case, which occurs when the star has a convective envelope, K is a function $K(L, M, R, X, Z)$ of the luminosity, mass, radius, and composition which must be calculated from a model atmosphere. K decreases with decreasing depth of the convective envelope, and the limiting case $K = 0$ corresponds to a star with no convective envelope.

Therefore, if we define K to be zero in the absence of a convective envelope, we can take account of both radiative and convective cases with the boundary conditions

$$\begin{aligned} r = 0\,, \qquad & L_r = 0 \qquad \text{at} \qquad M_r = 0\,, \\ T = 0\,, \qquad & \frac{P}{T^{2.5}} = K \qquad \text{at} \qquad M_r = M\,. \end{aligned} \tag{11}$$

IV. TRANSFORMATIONS AND CHOICE OF VARIABLES

If one plots the four dependent variables of equations (7)–(10) as functions of M_r, one finds that the resulting curves have singularities which make them unsuitable for the application of numerical techniques. For example, the curve of r versus M_r (see Fig. 1) has singularities at both ends at which all of the derivatives, including the third, become infinite. The difference equation (3), therefore, gives very poor accuracy near the boundaries where the neglected term in expression (2) becomes large.

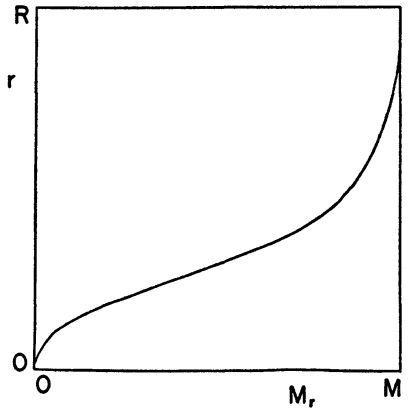

FIG. 1.—Run of r as a function of M_r in a solar-type star

This difficulty can be overcome while still maintaining a fixed relation between the independent variable and M_r by using as independent variable not M_r itself but a function of M_r in terms of which all the dependent variables are smoothly varying functions with no singularities. Such a transformed independent variable we shall denote by x, and the transformation relating x to M_r we shall write in the form

$$\frac{M_r}{M} = f(x)\,. \tag{12}$$

Here $f(x)$ is a function which varies from 0 to 1 as we go from the center to the surface of the star; the convenience, we can also choose the range of variation of x to be from 0 to 1.

The requirements on the transformation (12) will be satisfied if we choose $f(x)$ such that x varies nearly linearly with r; since the other dependent variables are smoothly varying functions of r, they will then also be smoothly varying functions of x. In constructing a function $f(x)$ with the required properties it is necessary first of all to insure that $f(x)$ has the correct asymptotic form at the boundaries to produce a linear dependence of r on x near the center and near the surface.

1. *Center.*—Close to the center where the density is almost constant we have $M_r/M \propto r^3$. Since we want $r \propto x$ in this region, we must have, using equation (12),

$$f(x) \propto x^3 \qquad \text{for} \qquad x \ll 1\,. \tag{13}$$

2. *Surface.*—The required asymptotic form of $f(x)$ at the surface depends on whether the star is convective or radiative at the surface.

i) *Convective at surface.*—Putting $\gamma = \frac{5}{3}$ in equations (7)–(10) and assuming the perfect gas law we can show that near the surface where $(1 - r/R) \ll 1$ we have

$$\frac{M_r}{M} \doteqdot 1 - \text{const.} \left(1 - \frac{r}{R}\right)^{2.5} .$$

Since r/R and x both approach 1 at the surface and we want a linear relation between r and x in this region, we must have

$$\left(1 - \frac{r}{R}\right) \propto (1 - x)$$

near the surface. Hence in the convective case we require

$$f(x) \doteqdot 1 - \text{const.}\ (1 - x)^{2.5} \text{ for } (1 - x) \ll 1 . \tag{14}$$

ii) *Radiative at surface.*—We assume that near the surface the opacity can be represented by a formula of the form

$$\kappa = \kappa_0 \rho^{\alpha} T^{-\beta} .$$

Then it can be shown from equations (7)–(10) (again assuming the perfect gas law) that near the surface we have

$$\frac{M_r}{M} \doteqdot 1 - \text{const.} \left(1 - \frac{r}{R}\right)^{\sigma} ,$$

where

$$\sigma = \frac{4 + \alpha + \beta}{1 + \alpha} .$$

Hence in the radiative case we require

$$f(x) \doteqdot 1 - \text{const.}\ (1 - x)^{\sigma} \qquad \text{for} \qquad (1 - x) \ll 1 . \tag{15}$$

A simple function $f(x)$ which satisfies the conditions (13)–(15) and serves for both convective and radiative cases is the following:

$$f(x) = [1 - (1 - x)^{2.5}]^{\lambda} [1 - (1 - x)^{\sigma}]^{(3-\lambda)} , \tag{16}$$

where $\lambda \propto K$, the limiting value of $P/T^{2.5}$ at the surface. For small x we obtain from equation (16)

$$f(x) \doteqdot 2.5^{\lambda} \sigma^{(3-\lambda)} x^3 , \tag{17}$$

as required. For small $(1 - x)$ we obtain in the convective case (K and λ both > 0)

$$f(x) \doteqdot 1 - \lambda(1 - x)^{2.5} \tag{18}$$

and in the radiative case ($K = \lambda = 0$)

$$f(x) \doteqdot 1 - 3(1 - x)^{\sigma} , \tag{19}$$

as required. In deriving expression (18) we have made use of the fact that in practical cases $\sigma > 2.5$ (e.g., with Kramers' opacity formula $\alpha = 1$, $\beta = 3.5$, $\sigma = 4.25$).

The constant of proportionality relating λ to K can be chosen to produce a smooth transition between models with convective and radiative surfaces; we have done this so as to make the value of dr/dx at the surface continuous in the transition from convective to radiative cases. It should be noted that in constructing main-sequence stars one is free to vary the parameters λ and σ in equation (16) to suit one's convenience, but in constructing evolutionary sequences of models λ and σ must be left fixed after the initial model has been constructed, so as to maintain a fixed relation between x and M_r.

a) Other Transformations

The pressure P varies by many orders of magnitude in a star, and it is preferable both for convenience and for better numerical accuracy to use in its place a variable which varies less strongly. A possible choice is P/T^n, where the variation of P is counteracted to some extent by a similar variation of T^n. For a star with a convective envelope the largest value of n we can use is 2.5, since for any larger value of n, P/T^n would become infinite at the surface. There are also two simplifications that arise from using $P/T^{2\,5}$ as a dependent variable in place of P: (1) The boundary condition $P/T^{2.5} = K$ becomes simply a specification of the boundary value of the variable $P/T^{2.5}$; (2) in cases where partial degeneracy is important, the functions $F_{1/2}(\psi)$ and $F_{3/2}(\psi)$ (Schwarzschild [1958], p. 61) depend on P and T only in the combination $P/T^{2\,5}$, and they therefore become functions of the single variable $P/T^{2.5}$.

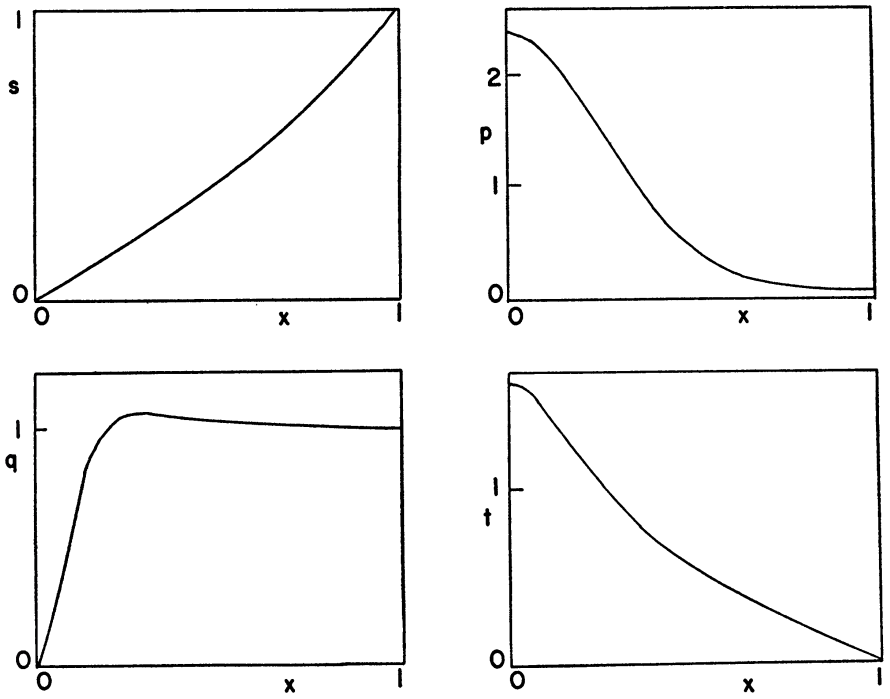

FIG. 2.—Runs of s, q, p, and t as functions of x

Since L_r increases sharply near the center ($L_r \propto r^3$), there will be some loss of accuracy in applying the difference equation (3) unless very small steps are taken in the independent variable x. This situation can be improved by using in place of L_r a variable which varies linearly with x near the center. Such a variable is

$$L_r\left(1+\frac{a}{x^2}\right)$$

where a is some small constant of order of magnitude 0.01.

For convenience we shall now define the following new set of dependent variables, which all are near unity in order of magnitude in a typical main-sequence star such as the Sun:

$$s \equiv \frac{r}{R_\odot}, \qquad q \equiv \frac{L_r}{L_\odot}\left(1+\frac{a}{x^2}\right), \qquad p \equiv 10\,\frac{P}{T^{2.5}}, \qquad t \equiv 10^{-7}T. \tag{20}$$

For illustration, we have plotted these four variables as functions of x in a model we have obtained for the present Sun (Fig. 2). It is apparent that s, q, p, and t are all smooth and well-behaved functions of x, as required for good numerical accuracy.

V. THE TRANSFORMED EQUATIONS

Substituting the definitions (20) into the four basic differential equations (7)–(10) and using equation (12) to transform from M_r to x as independent variable, we obtain

$$\frac{ds}{dx} = \frac{M}{4\pi R_\odot^3} \frac{f'(x)}{s^2\rho}, \tag{21}$$

$$\frac{dq}{dx} = \frac{M}{L_\odot}\left(1+\frac{a}{x^2}\right) f'(x)\epsilon - \frac{2aq}{x^3+ax} + \tfrac{3}{2}10^{16.5}\frac{M}{L_\odot}\left(1+\frac{a}{x^2}\right) f'(x)\frac{pt^{2.5}}{\rho}\frac{d}{d\tau}\ln\left(\frac{\rho^{5/3}}{pt^{2.5}}\right), \tag{22}$$

$$\frac{dp}{dx} = -10^{-16.5}\frac{GM^2}{4\pi R_\odot^4}\frac{f(x)f'(x)}{s^4t^{2.5}} - 2.5\frac{p}{t}\frac{dt}{dx}, \tag{23}$$

and

$$\frac{dt}{dx} = -10^{-28}\frac{3ML_\odot}{64\pi^2 a cR_\odot^4}\kappa\frac{f'(x)q}{s^4t^3(1+a/x^2)}, \quad \text{(radiative)} \tag{24R}$$

$$\frac{dt}{dx} = -\tfrac{2}{5}10^{-16.5}\frac{GM^2}{4\pi R_\odot^4}\frac{f(x)f'(x)}{s^4pt^{1.5}}, \quad \text{(convective)} \tag{24C}$$

where we have put $\gamma = \frac{5}{3}$.

VI. THE GAS-CHARACTERISTICS RELATIONS AND THE SURFACE BOUNDARY CONDITION

Before equations (21)–(24) can be integrated numerically, one must have some means of calculating the three quantities ρ, ϵ, and κ from the pressure, temperature, and composition. In addition it is necessary to have some means of calculating the constant K required for the surface boundary condition $P/T^{2.5} = K$.

a) The Equation of State

In the study of faint stars and in calculations of stellar evolution it is necessary to take into account partial degeneracy. For the case of interest in which the electrons may be partially degenerate but the nuclei still obey the perfect gas law, the density and pressure are related by the parametric equations (Schwarzschild [1958], pp. 58–61)

$$\rho = \mu_e H \frac{4\rho}{h^3}(2mkT)^{3/2}F_{1/2}(\psi), \tag{25}$$

$$P = \frac{8\pi}{3h^3}(2mkT)^{3/2}kTF_{3/2}(\psi) + \frac{k}{\mu_n H}\rho T, \tag{26}$$

where μ_e = mass of material per electron,

μ_n = mass of material per nucleus,

H = mass in grams of the unit in which μ_e and μ_n are measured (here taken as the mass of the H^1 atom),

and the other symbols have the same meaning as in Schwarzschild's book.

To calculate the density ρ from equations (25) and (26) our procedure is as follows: substituting equation (25) in equation (26) we obtain

$$\frac{4\pi}{h^3}(2mkT)^{3/2}kT\left[\tfrac{2}{3}F_{3/2}(\psi) + \frac{\mu_e}{\mu_n}F_{1/2}(\psi)\right] = P.$$

In terms of our variable $p = 10\ P/T^{2\ 5}$ this becomes

$$\tfrac{2}{3}F_{3/2}(\psi) + \frac{\mu_e}{\mu_n}F_{1/2}(\psi) = \frac{h^3}{40\pi(2mk)^{3/2}k}\,p = .13293p\,. \tag{27}$$

It is possible to represent $\frac{2}{3}\,F_{3/2}(\psi)$ quite accurately as a function of $F_{1/2}(\psi)$ by a simple analytical formula; we have found the following formula to be in error by less than ≈ 0.02 per cent for all ψ less than 30:

$$\tfrac{2}{3}F_{3/2}(\psi) = F_{1/2}(\psi)\frac{[\,1+0.1938F_{1/2}(\psi)\,]^{5/3}}{[\,1+0.12398F_{1/2}(\psi)\,]}\,. \tag{28}$$

When this is substituted in equation (27) we obtain an equation in the single unknown $F_{1/2}(\psi)$, which is readily solved by iteration. Once $F_{1/2}(\psi)$ is known ρ is calculated from equation (25) which becomes, in terms of $t \equiv 10^{-7}T$,

$$\rho = 288.36\ \mu_e t^{1\ 5}F_{1/2}(\psi)\,. \tag{29}$$

We shall later need also the derivative $\zeta \equiv dF_{1/2}(\psi)/dp$, which may be obtained from expression (27) by differentiating with respect to p:

$$\zeta \equiv \frac{dF_{1/2}(\psi)}{dp} = 0.13293\left[\tfrac{2}{3}\frac{dF_{3/2}(\psi)}{dF_{1/2}(\psi)} + \frac{\mu_e}{\mu_n}\right]^{-1}. \tag{30}$$

b) The Nuclear-Energy Generation Rate

The following formulae have been taken from Reeves (1964), with the slight modification that we use $t = T/10^7$ in place of Reeves's $T_6 = T/10^6$.

1. *Proton-proton cycle.*—The energy-generation rate ϵ_{pp} due to the proton-proton cycle is expressed as a sum of the contributions of the three branches of the cycle:

$$\epsilon_{pp} = \epsilon_{\rm I} + \epsilon_{\rm II} + \epsilon_{\rm III}\,,$$

where

$$\epsilon_{\rm I} = \epsilon_{\rm I}'(1-\gamma), \qquad \epsilon_{\rm II} = \epsilon_{\rm I}'\left(\frac{1.96\gamma}{1+w}\right), \qquad \text{and} \qquad \epsilon_{\rm III} = \epsilon_{\rm I}'\left(\frac{1.46\gamma w}{1+w}\right).$$

In these expressions

$$\epsilon_{\rm I}' = 4.44\times 10^5 f_{11}g_{11}X^2\rho t^{-2/3}\exp(-15.693/t^{1/3})\,,$$

$$\gamma = \alpha[\,(1+2/\alpha)^{1/2}-1\,],$$

where

$$\alpha = 1.94\times 10^{18}(Y/4X)^2\exp(-46.416/t^{1/3}),$$

and

$$w = 8.31\times 10^{15}f_{71}g_{71}\left(\frac{X}{1+X}\right)t^{-1/6}\exp(-47.623/t^{1/3}).$$

The quantities f_{11}, g_{11}, f_{71}, and g_{71} are correction factors given by Reeves as

$$f_{11} = 1+0.0079\rho^{1/2}t^{-3/2}\,, \qquad g_{11} = 1+0.026t^{1/3}+0.036t^{2/3}+0.006t\,,$$

$$f_{71} = 1+0.0316\rho^{1/2}t^{-3/2}\,, \qquad g_{71} = 1+0.0087t^{1/3}\,.$$

For the g's which vary little we have used the following approximations:

$$g_{11} \doteq 1.037 + 0.033t,\ g_{71} \doteq 1\,.$$

2. *Carbon-nitrogen cycle.*—For both branches of the carbon-nitrogen (-oxygen) cycle we have

$$\epsilon_{CN} = 1.711 \times 10^{27} f_{14,1}\, g_{14,1} X X_{14} \rho t^{-2/3} \exp\left(-70.697/t^{1/3}\right),$$

where

$$f_{14,1} = 1 + 0.0553\, \rho^{1/2} t^{-3/2},$$

$$g_{14,1} = 1 + 0.0058 t^{1/3} - 0.0172 t^{2/3} - 0.0007 t,$$

$$\doteq 0.995 - 0.008 t.$$

X_{14} is the abundance by weight of nitrogen 14. At temperatures less than about 16000000° K, only carbon and nitrogen participate in the energy-generation cycle, and since the reaction which consumes N_{14} is by far the slowest in the cycle, the nitrogen abundance becomes practically equal to the total initial abundance of carbon and nitrogen together. Using the general abundance table of Aller (1961) we estimate this to be about 0.19 times the total heavy-element abundance Z. At temperatures greater than about 16000000° K oxygen participates in the energy-generation cycle of X_{14} rises to about 0.96 times the total initial abundance of carbon, nitrogen, and oxygen together, which we estimate from Aller's table to be about 0.61 Z.

We shall later require the partial derivatives of $\epsilon = \epsilon_{pp} + \epsilon_{CN}$ with respect to p and t. In calculating these derivatives we can in good approximation neglect the variation with p and t of γ and w and of the f's and g's. The partial derivative with respect to p is obtained by making use of equations (29) and (30):

$$\frac{\partial \epsilon}{\partial p} = \frac{\epsilon}{F_{1/2}(\psi)}\, \zeta. \tag{31}$$

For the derivative with respect to t we again make use of equation (29) for ρ:

$$\begin{aligned}\frac{\partial \epsilon}{\partial t} &= \frac{\partial \epsilon_{pp}}{\partial t} + \frac{\partial \epsilon_{CN}}{\partial t} \\ &= \frac{\epsilon_{pp}}{t}\left(\tfrac{5}{6} + \frac{5.231}{t^{1/3}}\right) + \frac{\epsilon_{CN}}{t}\left(\tfrac{5}{6} + \frac{23.566}{t^{1/3}}\right).\end{aligned} \tag{32}$$

c) The Opacity

The most complete tabulation of radiative opacities presently available is that of Keller and Meyerott (1955). For purposes of formula-fitting or interpolation in these tables, it is convenient to separate out the electron-scattering opacity $\kappa_s = 0.19\,(1 + X)$; the logarithm of the remaining part of the opacity κ_t due to atomic transitions is then a nearly linear function of log ρ and log T. As a first rough approximation we may represent log κ_t as a linear function of log ρ and log T, which leads us to a Kramers-type formula of the form

$$\kappa_t \approx \kappa_0 \rho^{\alpha} T^{-\beta}. \tag{33}$$

For many purposes, however, formula (33) is not adequate and κ_t must be calculated more accurately either from a more elaborate formula or by interpolation in a table of log κ_t versus log ρ and log T. Whatever technique is used, it is assumed that logarithmic derivatives

$$\alpha = \left(\frac{\partial \log \kappa_t}{\partial \log \rho}\right)_T \quad \text{and} \quad -\beta = \left(\frac{\partial \log \kappa_t}{\partial \log T}\right)_\rho \tag{34}$$

can be calculated in addition to κ_t. It is important to insure that the calculated values of κ_t, α, and β are continuous functions of log ρ and log T; otherwise the convergence of the iterative method of Section II may be slowed down considerably.

The following partial derivatives which we shall require later are obtained from equations (34), using equations (29) and (30):

$$\left(\frac{\partial \log \kappa_t}{\partial \log p}\right)_t \equiv \eta = \alpha \frac{p}{F_{1/2}(\psi)} \zeta , \tag{35}$$

$$-\left(\frac{\partial \log \kappa_t}{\partial \log t}\right)_p \equiv \theta = \beta - 1.5\alpha . \tag{36}$$

The opacity calculations, which form a subroutine of the main computer program, may be easily modified to take into account any improvement in opacities on the Keller-Meyerott tables. The only requirements of the main program are for the opacity subroutine to provide the quantities α and β together with the opacity at each point.

d) The Calculation of K

Finally we require a means of calculating the surface value of our variable $p = 10\ P/T^{2.5}$. For this purpose we have adapted a program described elsewhere (Demarque and Geisler 1963) for integrating downward through the atmosphere and outer layers of a star of specified luminosity, mass, radius, and composition. The integration is stopped when a zone of adiabatic convection is reached, and K is calculated at this point from $K = P/T^{2.5}$; if no convection zone is encountered, K is set equal to zero. This calculation has been incorporated in the model construction as follows: K is first calculated from the radius and luminosity of the initial approximation, and this value of K is used in constructing the second approximation to the model; K is calculated again from the radius and luminosity of the second approximation and used to construct the third approximation, and so on. The process converges rapidly to a model with the correct K, provided that precautions are taken to insure that K does not change by too large a factor between models.

The above procedure is satisfactory for stars on or near the main sequence. Osterbrock (1953) has shown that the error introduced by assuming a constant ratio of specific heats in the calculation of the radius of red-dwarf models is small. This is due to the relatively small thickness of the hydrogen ionization zone. An accurate treatment of the superadiabatic transition layer is required mainly for the purpose of obtaining a realistic value of K in the adiabatic envelope, rather than for its direct effect on the computed radius of the star. However, the preceding remarks do not apply to high-luminosity red giants for which accurate surface conditions must be used and the exact effect of the hydrogen convection zone must be taken into account.

VII. THE DIFFERENCE EQUATIONS

In equations (21) and (22) we substitute ρ from equation (25), and in equation (23) we substitute dt/dx from equations (24R) or (24C) according to whether radiative or convective equilibrium prevails. Also in equation (22) we replace the time derivative $d/d\tau$ by the approximating difference operator $\Delta/\Delta\tau$, Δ denoting differences between the present model and the previous one in an evolutionary sequence. We can then write the basic differential equations (21)–(24) in the form

$$\frac{ds}{dx} \equiv DS = A \frac{1}{\mu_e} \frac{f'(x)}{s^2 t^{1.5} F_{1/2}(\psi)} , \tag{37}$$

$$\frac{dq}{dx} \equiv DQ \equiv DU - DV + DW\Delta G ,$$

where

$$DU = E\left(1+\frac{a}{x^2}\right) f'(x)\epsilon,$$

$$DV = 2aq/(x^3+ax),$$

$$DW = \frac{F}{\Delta\tau}\frac{1}{\mu_e}\left(1+\frac{a}{x^2}\right) f'(x)\frac{pt}{F_{1/2}(\psi)}, \tag{38}$$

$$G = \ln[\mu_e^{5/3} F_{1/2}(\psi)^{5/3} p^{-1}].$$

$$\frac{dp}{dx} \equiv DP = -B\frac{f(x)f'(x)}{s^4 t^{2.5}} + 2.5C\kappa\frac{f'(x)pq}{s^4 t^4(1+a/x^2)}, \quad \text{(radiative)} \tag{39R}$$

$$\frac{dp}{dx} \equiv DP = 0, \quad \text{(convective)} \tag{39C}$$

$$\frac{dt}{dx} \equiv DT = DR = -C\kappa\frac{f'(x)q}{s^4 t^3(1+a/x^2)}, \quad \text{(radiative)} \tag{40R}$$

$$\frac{dt}{dx} \equiv DT = DC = -\tfrac{2}{5}B\frac{f(x)f'(x)}{s^4 p t^{1.5}}, \quad \text{(convective)} \tag{40C}$$

where

$$A = 10^{-10.5}\frac{h^3 M_\odot}{16\pi^2 H(2mk)^{3/2} R_\odot^3}\left(\frac{M}{M_\odot}\right), \qquad B = 10^{-16.5}\frac{GM_\odot^2}{4\pi R_\odot^4}\left(\frac{M}{M_\odot}\right)^2,$$

$$C = 10^{-28}\frac{3M_\odot L_\odot}{64\pi^2 a c R_\odot^4}\left(\frac{M}{M_\odot}\right), \qquad E = \frac{M_\odot}{L_\odot}\left(\frac{M}{M_\odot}\right),$$

and

$$F = 10^6\frac{3h^3 M_\odot/L_\odot}{8\pi H(2mk)^{3/2}}\left(\frac{M}{M_\odot}\right).$$

Using $M_\odot = 1.991 \times 10^{33}$ gm, $R_\odot = 6.96 \times 10^{10}$ cm, and $L_\odot = 3.86 \times 10^{33}$ erg/sec, we obtain

$$A = 0.0016296\,(M/M_\odot), \quad B = 0.028355\,(M/M_\odot)^2, \quad C = 0.00068588\,(M/M_\odot),$$

$$E = 0.51580\,(M/M_\odot), \qquad F = 8.4846 \times 10^{13}\,(M/M_\odot).$$

In terms of the quantities DS, DQ, DP, and DT defined in equations (37)–(40) we can now write the four difference equations corresponding to the differential equations (37)–(40) in the following form:

$$g_i^1 \equiv \frac{s_{i+1}-s_i}{x_{i+1}-x_i} - \tfrac{1}{2}(DS_{i+1}+DS_i) = 0, \tag{41}$$

$$g_i^2 \equiv \frac{q_{i+1}-q_i}{x_{i+1}-x_i} - \tfrac{1}{2}(DQ_{i+1}+DQ_i) = 0, \tag{42}$$

$$g_i^3 \equiv \frac{p_{i+1}-p_i}{x_{i+1}-x_i} - \tfrac{1}{2}(DP_{i+1}+DP_i) = 0, \tag{43}$$

$$g_i^4 \equiv \frac{t_{i+1}-t_i}{x_{i+1}-x_i} - \tfrac{1}{2}(DT_{i+1}+DT_i) = 0, \tag{44}$$

where i takes values from 1 to n, n being the number of intervals into which the range of x is divided.

To solve equations (41)–(44) we follow the method outlined in Section II. Putting $y^1 = s$, $y^2 = q$, $y^3 = p$, $y^4 = t$, and using the quantities g_i^k defined in equations (41)–(44), we can apply equations (6) with $m = 4$:

$$\sum_{j=1}^{4}\left(\frac{\partial g^k}{\partial y^j}\right)_i \delta y_i{}^j + \sum_{j=1}^{4}\left(\frac{\partial g^k}{\partial y^j}\right)_{i+1} \delta y_{i+1} = -g_i{}^k, \qquad i = 1 \ldots n, \qquad k = 1 \ldots 4.$$

We shall write these equations in the following form:

$$\begin{aligned}
&a_{11}{}^i \delta s_i + a_{12}{}^i \delta q_i + a_{13}{}^i \delta p_i + a_{14}{}^i \delta t_i \\
&+ a_{15}{}^i \delta s_{i+1} + a_{16}{}^i \delta q_{i+1} + a_{17}{}^i \delta p_{i+1} + a_{18}{}^i \delta t_{i+1} = b_1{}^i, \\
&a_{21}{}^i \delta s_i + a_{22}{}^i \delta q_i + \ldots + a_{28}{}^i \delta t_{i+1} = b_2{}^i, \\
&a_{31}{}^i \delta s_i + a_{32}{}^i \delta q_i + \ldots + a_{38}{}^i \delta t_{i+1} = b_3{}^i, \\
&a_{41}{}^i \delta s_i + a_{42}{}^i \delta q_i + \ldots + a_{48}{}^i \delta t_{i+1} = b_4{}^i,
\end{aligned} \tag{45}$$

where $i = 1 \ldots n$. The coefficients in this set of equations are partial derivatives of the g_i^k's with respect to s_i, $q_i, \ldots, t_{i+1}$, and they are obtained by differentiating the expressions in equations (41)–(44), using equations (37)–(40) for DS, DQ, DP, and DT.

VIII. EXPRESSIONS FOR THE COEFFICIENTS

We give below the coefficients of δs_i, δq_i, δp_i, and δt_i in equations (45). The coefficients of $\delta s_{i+1} \ldots \delta t_{i+1}$ are quite similar in form and need not be written out separately.

Using equations (41), (37), and (30) we obtain

$$a_{11}{}^i = \frac{-1}{x_{i+1} - x_i} + \frac{DS_i}{s_i}, \qquad a_{12}{}^i = 0, \qquad a_{13}{}^i = \tfrac{1}{2}\frac{DS_i}{F_{1/2}(\psi_i)}\,\zeta_i,$$

$$a_{14}{}^i = \tfrac{3}{4}\frac{DS_i}{t_i}, \qquad b_1{}^i = -\frac{s_{i+1} - s_i}{x_{i+1} - x_i} + \frac{DS_{i+1} + DS_i}{2}.$$

In calculating the partial derivative of DU with respect to t, we set $DU = DUP + DUC$ where

$$DUP = E\,(1 + a/x^2) f'(x) \epsilon_{pp}, \qquad DUC = E(1 + a/x^2) f'(x) \epsilon_{\mathrm{CN}}.$$

From equations (42), (38), (31), and (32) we then obtain

$$a_{21}{}^i = 0$$

$$a_{22}{}^i = \frac{-1}{x_{i+1} - x_i} + \tfrac{1}{2}\frac{DV_i}{q_i},$$

$$a_{23}{}^i = -\tfrac{1}{2}\frac{(DUP + DUC)_i}{F_{1/2}(\psi_i)}\,\zeta_i + \tfrac{1}{2} DW_i\left[\left(\frac{\zeta_i}{F_{1/2}(\psi_i)} - \frac{1}{p_i}\right)\Delta G_i + \frac{1}{p_i} - \tfrac{5}{3}\frac{\zeta_i}{F_{1/2}(\psi_i)}\right],$$

$$a_{24}{}^i = -\tfrac{1}{2}\frac{DUP_i}{t_i}\left(\tfrac{5}{6} + \frac{5.231}{t_i^{1/3}}\right) - \tfrac{1}{2}\frac{DUC_i}{t_i}\left(\tfrac{5}{6} + \frac{23.566}{t_i^{1/3}}\right) - \tfrac{1}{2}\frac{DW_i}{t_i}\Delta G_i,$$

$$b_2{}^i = -\frac{q_{i+1} - q_i}{x_{i+1} - x_i} + \frac{DQ_{i+1} + DQ_i}{2}.$$

From equations (43), (39), (40), (35), and (36) we obtain:

a) Radiative case ($DR > DC$)

$$a_{31}{}^{i} = 2\,\frac{DP_i}{s_i}, \qquad a_{32}{}^{i} = \tfrac{5}{4}\frac{p_i}{t_i}\frac{DR_i}{q_i},$$

$$a_{33}{}^{i} = \frac{-1}{x_{i+1}-x_i} + \tfrac{5}{4}\frac{DR_i}{t_i}\left[\eta_i\left(1-\frac{\kappa_s}{\kappa}\right)_i + 1\right],$$

$$a_{34}{}^{i} = \tfrac{5}{4}\frac{p_i}{t_i{}^2}\left\{\tfrac{5}{2}DC_i - \left[\theta_i\left(1-\frac{\kappa_s}{\kappa}\right)_i + 4\right]DR_i\right\}$$

$$b_3{}^{i} = -\frac{p_{i+1}-p_i}{x_{i+1}-x_i} + \frac{DP_{i+1}+DP_i}{2},$$

where

$$DP_i = \tfrac{5}{2}\frac{p_i}{t_i}(DC_i - DR_i).$$

b) Convective case ($DR \leq DC$)

$$a_{31}{}^{i} = 0\,, \qquad a_{32}{}^{i} = 0\,, \qquad a_{33}{}^{i} = \frac{-1}{x_{i+1}-x_i}, \qquad a_{34}{}^{i} = 0\,, \qquad b_2{}^{i} = -\frac{p_{i+1}-p_i}{x_{i+1}-x_i}.$$

From equations (44), (40), (35), and (36) we obtain:

a) Radiative case ($DR > DC$)

$$a_{41}{}^{i} = 2\,\frac{DR_i}{s_i}, \qquad a_{42}{}^{i} = -\tfrac{1}{2}\frac{DR_i}{q_i},$$

$$a_{43}{}^{i} = -\frac{\eta_i}{2}\left(1-\frac{\kappa_s}{\kappa}\right)_i\frac{DR_i}{p_i},$$

$$a_{44}{}^{i} = \frac{-1}{x_{i+1}-x_i} + \tfrac{1}{2}\frac{DR_i}{t_i}\left[\theta_i\left(1-\frac{\kappa_s}{\kappa}\right)_i + 3\right],$$

$$b_4{}^{i} = -\frac{t_{i+1}-t_i}{x_{i+1}-x_i} + \frac{DT_{i+1}+DT_i}{2},$$

where $DT_i = DR_i$.

b) Convective case ($DR \leq DC$)

$$a_{41}{}^{i} = 2\,\frac{DC_i}{s_i}, \qquad a_{42}{}^{i} = 0\,, \qquad a_{43}{}^{i} = \tfrac{1}{2}\frac{DC_i}{p_i},$$

$$a_{44}{}^{i} = \frac{-1}{x_{i+1}-x_i} + \tfrac{3}{4}\frac{DC_i}{t_i}, \qquad b_4{}^{i} = -\frac{t_{i+1}-t_i}{x_{i+1}-x_i} + \frac{DT_{i+1}+DT_i}{2},$$

where $DT_i = DC_i$.

The coefficients $a_{15}{}^{i}$, $a_{16}{}^{i}$, etc., of $\delta s_{i+1} \ldots \delta t_{i+1}$ in equations (45) have the same form as the corresponding coefficients of $\delta s_i \ldots \delta t_i$, the only differences being that all quantities are evaluated at point $i+1$ instead of point i, and the term $1/(x_{i+1}-x_i)$ occurs with the opposite sign (+ instead of −).

As indicated above, the choice of coefficients corresponding to adiabatic or radiative equilibrium is made by application of the usual Schwarzschild criterion, i.e.,

$$DR \leq DC$$

for convective instability.

Since in the present method all variables are calculated at the two end points of each interval, no difficulty was experienced in the transition from a radiative to a convective region. This procedure is identical to that successfully used by Schwarzschild and Härm (1963) in their calculations of stellar evolution.

a) Coefficients at the Boundaries

1. *Center.*—Since the quantities DS, DU, etc., all contain either s or x in a denominator, they cannot be evaluated directly at the center point where $s = x = 0$; we must instead evaluate their limiting values as $x \rightarrow 0$. These limiting values are readily obtained from the asymptotic solutions of equations (37)–(40) at the center. We make use of equation (17) to obtain the asymptotic form of $f'(x)$ for small x; we have

$$f'(x) \doteqdot kx^2 \qquad \text{for} \qquad x \ll 1 , \tag{46}$$

where

$$k = 3 \times 2.5^{\lambda} \sigma^{(3-\lambda)} .$$

Substituting expressions (46) in equations (37)–(40), we derive

$$DS_1 = \left[\frac{kA}{(\mu_e)_1 t_1^{1.5} F_{1/2}(\psi_1)}\right]^{1/3}, \qquad DU_1 = kaE\epsilon_1 ,$$

$$DW_1 = ka \frac{F}{\Delta\tau}\left(\frac{1}{\mu_e}\right)_1 \frac{p_1 t_1}{F_{1/2}(\psi_1)}, \qquad DV_1 = \tfrac{2}{3}(DU_1 + DW_1 \Delta G_1), \tag{47}$$

$$DP_1 = DT_1 = 0 \text{ (radiative or convective)} .$$

Using equations (47) we can now calculate the coefficients of δp_1 and δt_1 from the formulae given previously. The coefficients of δs_1 and δq_1 are not required since $\delta s_1 = \delta q_1 = 0$ in virtue of the boundary conditions.

2. *Surface.*—Two cases arise according to whether the star is radiative or convective at the surface.

i) *Convective surface.*—Since the surface layers are for practical purposes completely non-degenerate, we can put $\frac{2}{3}F_{3/2}(\psi) = F_{1/2}(\psi)$ in expression (27) to obtain

$$F_{1/2}(\psi) = 0.13293 \frac{\mu}{\mu_e} p , \tag{48}$$

where $1/\mu = 1/\mu_e + 1/\mu_n$. From equation (18) we obtain

$$f'(x) \doteqdot 2.5\lambda(1 - x)^{1.5} \tag{49}$$

for small $(1 - x)$. Substituting equations (48) and (49) in equations (37)–(40), we derive the following limiting values as $x \rightarrow 1$:

$$DS_{n+1} \doteqdot -\frac{2.5A}{0.13293B}\left(\frac{1}{\mu}\right)_{n+1} s_{n+1}^2 DT_{n+1}, \qquad DP_{n+1} = 0 ,$$

$$DT_{n+1} = -\left(\frac{\lambda B}{s_{n+1}^4 p_{n+1}}\right)^{2/5}, \qquad DQ_{n+1} \doteqdot -DV_{n+1} = -\frac{2aq_{n+1}}{1+a}. \tag{50}$$

In calculating DQ_{n+1} we have made use of the fact that DU and DW are negligibly small near the surface.

ii) *Radiative surface.*—We assume that near the surface the opacity can be represented by a Kramers-type formula of the form

$$\kappa = \kappa_0 \rho^{\alpha} T^{-\beta} .$$

Since degeneracy is negligible near the surface we have $\rho \propto pt^{1.5}$ and we can write

$$\kappa = \kappa_0' p^a t^{-\theta}, \tag{51}$$

where $\theta = \beta - 1.5a$. We also make use of equations (19) to obtain

$$f'(x) \doteq 3\sigma(1 - x)^{\sigma-1} \tag{52}$$

for small $(1 - x)$, where $\sigma = (4 + a + \beta)/(1 + a)$.

Substituting expressions (48), (51), and (52) in equations (37)–(40) we derive

$$DS_{n+1} = -\frac{\sigma A}{0.13293B}\left(\frac{1}{\mu}\right)_{n+1} s_{n+1}^2 DT_{n+1}, \qquad DP_{n+1} = 0 \text{ (provided } \sigma > 3.5),$$

$$DT_{n+1} = -\left\{\frac{3B}{s_{n+1}^4}\left[\frac{\sigma C \kappa_0' q_{n+1}}{B(1+a)}\right]^{1/1+a}\right\}^{1/\sigma}, \qquad DQ_{n+1} \doteq -DV_{n+1} = -\frac{2a q_{n+1}}{1+a}. \tag{53}$$

Using equations (50) or (53) we can now calculate the coefficients of δs_{n+1} and δq_{n+1}. The coefficients of δp_{n+1} and δt_{n+1} are not required because $\delta p_{n+1} = \delta t_{n+1} = 0$ in virtue of the boundary conditions.

IX. CONSTRUCTION OF STELLAR MODELS

Taking the four linear equations (45) for each of the n intervals in x, we have altogether $4n$ linear equations in the $4(n + 1)$ quantities δs_i, δq_i, δp_i, δt_i, $i = 1, \ldots, n + 1$. Four of these quantities are however already determined by the boundary conditions, which give $\delta s_1 = \delta q_1 = \delta p_{n+1} = \delta t_{n+1} = 0$. We are therefore left with $4n$ equations in $4n$ unknowns which we can solve by elimination. Once the corrections $\delta s_i, \ldots, \delta t_i$, $i = 1, \ldots, n + 1$ have been determined, we obtain an improved approximation to our stellar model by adding these corrections to the corresponding variables of the initial approximation. The process is repeated until the corrections become negligibly small compared with the level of accuracy required.

Under favorable conditions, when a reasonably good first approximation is available, two or three iterations may suffice to produce convergence to an accuracy of better than 1 per cent in all the variables. Under other conditions, when the initial approximation differs appreciably in some respect from the final model, the iterations may converge slowly or even diverge. In such cases it may be necessary to bridge the gap between the initial approximation and the desired final model by means of intermediate models.

As for numerical accuracy, we have found that for a typical main-sequence star near the Sun an accuracy of better than 1 per cent in all the variables is obtained with a uniform distribution of forty points in the interval from $x = 0$ to $x = 1$. For evolved models in which energy generation occurs in a thin shell, it is necessary for equivalent accuracy to take more points in the region of the shell, but fair accuracy may still be obtained even with a coarse distribution of points.

X. APPLICATION TO STELLAR EVOLUTION

The method described in the preceding sections can readily be applied to the automatic computation of stellar evolution in the manner described by Henyey *et al.* (1959). The main feature of the method is that the change of hydrogen content X from one model to the next in an evolutionary sequence is calculated using the average rate of change of X for the two models. Thus, if we use subscripts 1 and 2 to distinguish the two models we have for the formula to be applied at each point

$$x_2 = x_1 + \tfrac{1}{2}\left[\left(\frac{dX}{d\tau}\right)_1 + \left(\frac{dX}{d\tau}\right)_2\right]\Delta\tau. \tag{54}$$

After each iteration the term $(dX/d\tau)_2$ is recalculated from the improved values of temperature and density in the second model, and equation (54) is applied again to obtain improved values for the hydrogen content X_2.

In calculating the rate of change of X it is necessary to take separate account of the four processes of energy generation: the three branches of the proton-proton cycle, and the carbon-nitrogen cycle. These processes release differing amounts of energy per He^4 nucleus formed because of the differing energy losses in the form of neutrinos. From Reeves (1964) we take the following figures for the non-neutrino energy released per He^4 nucleus formed: for Branches I, II, and III of the proton-proton cycle, 26.21 MeV, 25.65 MeV, and 19.1 MeV, respectively, and for the carbon-nitrogen cycle 25.04 MeV.

Finally, the possibility of the presence of a convective core at some stage of the evolution of the star is taken into account. Since mixing leads to chemical homogeneity within a convective region, the existence of a convective core may considerably influence the course of the subsequent evolution of the star. At each evolutionary step, the new run of the hydrogen content is calculated neglecting convective mixing. The computer then tests for stability against convection at each point. Within the convective core, if it exists, the hydrogen content is set equal to a weighted mean of the hydrogen contents at all points throughout the convective region. The weighting function is $f'(x)$ since the mass of an infinitesimal shell centered at x is $Mf'(x)dx$.

XI. CONCLUDING REMARKS

We have described a method of constructing stellar models which takes full advantage of the capabilities of large computers and is entirely automatic, requiring no work to be done outside the computer except for providing the initial approximation. Programming is more complicated than for the older methods of Schwarzschild, but the extra work required in programming is made up for by the ease with which models are obtained; with an IBM 7090 an accurate model incorporating all the refinements we have described can be obtained in less than a minute of computing time, and a whole sequence of models (such as an evolutionary sequence) can be obtained in one run on the computer.

The method has proven to be particularly well suited for the study of main-sequence and early evolutionary phases of stars near 1 $M_\odot$. We have used the method in a number of investigations involving such stars, the results of which will be published separately.

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REFERENCES

Aller, L. H. 1961, *The Abundance of the Elements* (New York: Interscience Publishers, Inc.)
Demarque, P. R., and Geisler, J. E. 1963, *Ap. J.*, **137**, 1102.
Fox, L. 1957, *Two-Point Boundary Value Problems* (Oxford: Clarendon Press).
Henyey, L. G., Wilets, L., Böhm, K., LeLevier, R., and Levée, R. D. 1959, *Ap. J.*, **129**, 628.
Keller, G., and Meyerott, R. E. 1955, *Ap. J.*, **122**, 32.
National Physical Laboratory. 1961, *Modern Computing Methods* (Notes on Applied Science, No. 16. [London: H.M. Stationery Office]).
Osterbrock, D. E. 1953, *Ap. J.*, **118**, 529.
Reeves, H. 1964, "Stellar Energy Sources," in *Stellar Structure*, Vol. VIII of *Stars and Stellar Systems*, ed. L. H. Aller and D. McLaughlin (Chicago: University of Chicago Press).
Schwarzschild, M. 1958, *Structure and Evolution of the Stars* (Princeton, N.J.: Princeton University Press).
Schwarzschild, M., and Härm, R. 1963, private communication.