

THE FLUID-DYNAMICAL METHOD

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The use of a fluid-dynamical technique for computing the evolution of stars clusters has been described briefly earlier in this volume, and in more detail by Larson (1970a, b). For convenience we repeat here the fundamental definitions and equations used in this method. We consider a spherical stellar system described by a distribution function of the form $f(r, u, v, w, t)$, where u is the velocity component in the r -direction and v and w are the two transverse velocity components. As the basic fluid-dynamical variables to be solved for, we define the following six moments of the velocity distribution at each point in space and time: the density of stars ϱ , the mean outward velocity $\langle u \rangle$, and the higher-order moments

$$\left. \begin{aligned} \alpha &\equiv \langle (u - \langle u \rangle)^2 \rangle \\ \beta &\equiv \langle v^2 \rangle = \langle w^2 \rangle \\ \varepsilon &\equiv \langle (u - \langle u \rangle)^3 \rangle \\ \xi &\equiv \langle (u - \langle u \rangle)^4 \rangle - 3\alpha^2. \end{aligned} \right\} \quad (1)$$

Here α and β are the squares of the radial and transverse velocity dispersions, ε represents an outward energy flux or 'heat flow', and ξ represents an excess or deficiency of high velocity stars relative to a Maxwellian distribution.

For the variables defined above we obtain six fluid-dynamical equations by taking the corresponding moments of the Boltzmann equation. The system of moment equations has been closed by approximating the velocity distribution by a low-order expansion in Legendre polynomials; this allows the various unknown moments to be related to quantities already defined, as described by Larson (1970a). The resulting set of moment equations is as follows:

$$\frac{\partial \varrho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \varrho \langle u \rangle = 0 \quad (2)$$

$$\frac{\partial \langle u \rangle}{\partial t} + \langle u \rangle \frac{\partial \langle u \rangle}{\partial r} + \frac{1}{\varrho} \frac{\partial}{\partial r} \varrho \alpha + \frac{2}{r} (\alpha - \beta) + \frac{\partial \Phi}{\partial r} = 0 \quad (3)$$

$$\frac{\partial \alpha}{\partial t} + \langle u \rangle \frac{\partial \alpha}{\partial r} + 2\alpha \frac{\partial \langle u \rangle}{\partial r} + \frac{1}{\varrho} \frac{\partial}{\partial r} \varrho \varepsilon + \frac{2\varepsilon}{r} \left(1 - \frac{2}{3} \frac{\beta}{\alpha} \right) = -\frac{4}{5} \frac{(\alpha - \beta)}{T} \quad (4)$$

$$\frac{\partial \beta}{\partial t} + \langle u \rangle \frac{\partial \beta}{\partial r} + 2\beta \frac{\langle u \rangle}{r} + \frac{1}{3\varrho} \frac{\partial}{\partial r} \varrho \frac{\beta}{\alpha} \varepsilon + \frac{4}{3} \frac{\beta}{\alpha} \frac{\varepsilon}{r} = +\frac{2}{5} \frac{(\alpha - \beta)}{T} \quad (5)$$

$$\frac{\partial \varepsilon}{\partial t} + \langle u \rangle \frac{\partial \varepsilon}{\partial r} + 3\varepsilon \frac{\partial \langle u \rangle}{\partial r} + 3\alpha \frac{\partial \alpha}{\partial r} + \frac{1}{\varrho} \frac{\partial}{\partial r} \varrho \xi + \frac{2\xi}{r} \left(1 - \frac{\beta}{\alpha} \right) = -\frac{87}{160} \frac{\varepsilon}{T} \quad (6)$$

$$\frac{\partial \xi}{\partial t} + \langle u \rangle \frac{\partial \xi}{\partial r} + 4\xi \frac{\partial \langle u \rangle}{\partial r} + 6\varepsilon \frac{\partial \alpha}{\partial r} + 4\alpha \frac{\partial \varepsilon}{\partial r} = - \frac{3}{35} \frac{[7\xi - 15\alpha(\alpha - \beta)]}{T}. \quad (7)$$

The terms on the right-hand sides of the above equations represent the relaxational effects of encounters between the stars, and they have been evaluated from the Fokker-Planck equation using the assumption that deviations from a Maxwellian velocity distribution are small. The quantity

$$T = \frac{1}{16} \left(\frac{3}{\pi} \right)^{1/2} \frac{\langle V^2 \rangle^{3/2}}{G^2 m \varrho \ln(D_{\max} \langle V^2 \rangle / 2Gm)} \quad (8)$$

is the classical relation time defined by Chandrasekhar (1942). Here $\langle V^2 \rangle = \alpha + 2\beta$ is the mean squared random velocity, m is the stellar mass (all stars being assumed to have the same mass), and D_{\max} is the dimension of the region over which relaxation effects are important.

The fluid-dynamical Equations (2)–(7) may be solved numerically by techniques similar to ones which have previously been used for gas-dynamical problems with spherical symmetry. For the present problem an Eulerian method is most suitable, since the boundary condition is specified in Eulerian form and since the use of an Eulerian grid ensures that we always have reasonable spatial resolution. The grid points r_i are most conveniently spaced at equal intervals in $\log r$; we have generally used a spacing of 0.1 in $\log r$, which is adequate to provide numerical accuracies of the order of 10%. Following a common procedure in numerical hydrodynamics, we suppose that the odd-order moments $\langle u \rangle$ and ε as well as the mass m inside radius r are assigned values at the grid points r_i , whereas the even-order moments ϱ , α , β , and ξ are assigned values at a second set of points $r_{i-1/2}$ half way between the regular grid points. To ensure stability for time steps which may be much greater than the dynamical time, the difference equations have been written in implicit form with backward time differences. Considerable arbitrariness is possible in the way the difference expressions are constructed, particularly in the method of averaging quantities in adjacent zones. After some experimentation, the equations given below appeared to represent a reasonable compromise between accuracy and stability requirements, although it is not claimed that this is the best possible set of difference equations.

To allow easy calculation of the mass variable m appearing in the gravitational acceleration term $\partial \Phi / \partial r = Gm/r^2$, the continuity Equation (2) has been replaced by the 2 equations

$$\frac{\partial m}{\partial t} = -4\pi r^2 \varrho u \quad (9)$$

$$\frac{\partial m}{\partial r} = 4\pi r^2 \varrho \quad (10)$$

(for convenience we henceforth write u in place of $\langle u \rangle$.) The differential Equations

(9), (10), and (3)–(7) are then approximated by the difference equations given below; here quantities with a superscript n refer to a time t^n , whereas quantities with no superscript all refer to the advanced time $t^{n+1} = t^n + \Delta t$.

$$m_i = m_i^n - 4\pi r_i^2 (q_{i-1/2} q_{i+1/2})^{1/2} u_i \Delta t \quad (11)$$

$$\frac{m_i - m_{i-1}}{r_i^3 - r_{i-1}^3} = \frac{4\pi}{3} q_{i-1/2} \quad (12)$$

$$\begin{aligned} \frac{u_i - u_i^n}{\Delta t} + u_i \frac{u_{i+1} - u_i}{r_{i+1} - r_i} + \left(\frac{\alpha_{i-1/2} + \alpha_{i+1/2}}{2} \right) \frac{\ln(q\alpha)_{i+1/2} - \ln(q\alpha)_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} \\ + \frac{(\alpha - \beta)_{i-1/2} + (\alpha - \beta)_{i+1/2}}{r_i} + \frac{Gm_i}{r_i^2} = 0 \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\alpha_{i-1/2} - \alpha_{i-1/2}^n}{\Delta t} + u_i \frac{\alpha_{i+1/2} - \alpha_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} + 2\alpha_{i-1/2} \frac{u_i - u_{i-1}}{r_i - r_{i-1}} \\ + \frac{1}{q_{i-1/2}} \frac{(q_{i-1/2} q_{i+1/2})^{1/2} \varepsilon_i - (q_{i-3/2} q_{i-1/2})^{1/2} \varepsilon_{i-1}}{r_i - r_{i-1}} \\ + 2 \frac{\varepsilon_{i-1} + \varepsilon_i}{r_{i-1} + r_i} \left(1 - \frac{2\beta_{i-1/2}}{3\alpha_{i-1/2}} \right) = - \frac{4(\alpha - \beta)_{i-1/2}}{5 T_{i-1/2}} \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{\beta_{i-1/2} - \beta_{i-1/2}^n}{\Delta t} + u_i \frac{\beta_{i+1/2} - \beta_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} + 2\beta_{i-1/2} \frac{u_{i-1} + u_i}{r_{i-1} + r_i} \\ + \frac{1}{3q_{i-1/2}} \frac{[(\beta q/\alpha)_{i-1/2} (\beta q/\alpha)_{i+1/2}]^{1/2} \varepsilon_i - [(\beta q/\alpha)_{i-3/2} (\beta q/\alpha)_{i-1/2}]^{1/2} \varepsilon_{i-1}}{r_i - r_{i-1}} \\ + \frac{4\beta_{i-1/2} \varepsilon_{i-1} + \varepsilon_i}{3\alpha_{i-1/2} r_{i-1} + r_i} = + \frac{2(\alpha - \beta)_{i-1/2}}{5 T_{i-1/2}} \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{\varepsilon_i - \varepsilon_i^n}{\Delta t} + u_i \frac{\varepsilon_{i+1} - \varepsilon_i}{r_{i+1} - r_i} + 3\varepsilon_i \frac{u_{i+1} - u_i}{r_{i+1} - r_i} \\ + \frac{3\alpha_{i+1/2}^2 - \alpha_{i-1/2}^2}{2r_{i+1/2} - r_{i-1/2}} + \frac{2}{q_{i-1/2} + q_{i+1/2}} \frac{q_{i+1/2} \xi_{i+1/2} - q_{i-1/2} \xi_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} \\ + \frac{\xi_{i-1/2} + \xi_{i+1/2}}{r_i} \left(1 - \frac{\beta_{i-1/2} + \beta_{i+1/2}}{\alpha_{i-1/2} + \alpha_{i+1/2}} \right) = - \frac{87}{160} \frac{\varepsilon_i}{(T_{i-1/2} T_{i+1/2})^{1/2}} \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\xi_{i-1/2} - \xi_{i-1/2}^n}{\Delta t} + u_i \frac{\xi_{i+1/2} - \xi_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} + 4\xi_{i-1/2} \frac{u_i - u_{i-1}}{r_i - r_{i-1}} \\ + 3(\varepsilon_{i-1} + \varepsilon_i) \frac{(\alpha_{i-1/2} \alpha_{i+1/2})^{1/2} - (\alpha_{i-3/2} \alpha_{i-1/2})^{1/2}}{r_i - r_{i-1}} \\ + 4\alpha_{i-1/2} \frac{\varepsilon_i - \varepsilon_{i-1}}{r_i - r_{i-1}} = - \frac{3}{35} \frac{[7\xi - 15\alpha(\alpha - \beta)]_{i-1/2}}{T_{i-1/2}}. \end{aligned} \quad (17)$$

The difference approximations for terms of the form $u\partial u/\partial r$, $u\partial\alpha/\partial r$, etc. have been written in a form which is appropriate if $u < 0$, i.e. if the flow is inward; for $u > 0$, these expressions should be replaced by

$$u_i \frac{u_i - u_{i-1}}{r_i - r_{i-1}}, \quad u_{i-1} \frac{\alpha_{i-1/2} - \alpha_{i-3/2}}{r_{i-1/2} - r_{i-3/2}}, \quad \text{etc.}$$

to ensure stability. Unfortunately, the difference equations still exhibit unstable behavior in some circumstances, particularly in computing a solution which has large 'starting transients'. No cure was found for the instability; however, it was always possible to avoid it by choosing more realistic initial conditions which did not produce large transient effects.

In solving the difference equations, m_i may be treated as an auxiliary variable, leaving the six quantities $q_{i-1/2}$, u_i , $\alpha_{i-1/2}$, $\beta_{i-1/2}$, ε_i , and $\xi_{i-1/2}$ to be solved for in each zone; thus if the numerical grid contains N zones we have a total of $6N$ unknowns to be solved for. Correspondingly we have the six difference Equations (12)–(17) for each of the N zones except the outermost one, where Equations (13) and (16) cannot be applied but must be replaced by boundary conditions for u_N and ε_N . The assumption of a perfectly absorbing boundary together with an assumed form of the velocity distribution leads to boundary conditions of the form

$$u_N \propto \alpha_N^{1/2}, \quad \varepsilon_N \propto \alpha_N^{3/2}$$

(Larson, 1970b). Incorporating these boundary conditions into the difference equations, we then have enough equations to allow a solution for all of the unknowns.

The standard method for solving a large number of simultaneous non-linear equations is the Newton-Raphson iterative technique, described for example by Larson and Demarque (1964). This method involves starting with an approximate solution for each of the unknowns; the exact solution is then expressed as the approximate value plus a correction, and the equations are linearized by expanding to first order in the correction terms, yielding a set of linear equations for the corrections. These linear equations are then solved by standard techniques of linear algebra, and the resulting corrections are used to obtain an improved solution to the difference equations. The correction procedure is repeated several times until adequate convergence is achieved. The details of application of the method are straightforward but lengthy, and will not be reproduced here. For the present problem, the first approximation to the value of any quantity at time t^{n+1} is conveniently taken as its value at time t^n . It was found that 3 iterations were generally sufficient to solve the difference equations with a fractional error less than 10^{-3} , which is considerably smaller than the truncation error of the difference equations for the fairly coarse grid used. With $N=65$, as used in calculating the evolution of globular clusters, the computing time on an IBM 7094 amounts to about 5 s per time step or about 5 min for a complete evolutionary calculation.

References

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