ON THE EXISTENCE AND STABILITY OF INTERMEDIATE SHOCKS

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ABSTRACT

The question of the existence of intermediate shock waves is re-examined. It is pointed out that the MHD Rankine-Hugoniot conditions allow up to four types of intermediate shock discontinuity and that one of these is evolutionary. Recent numerical simulations, analyses of spacecraft data and theoretical arguments suggest that intermediate shocks can form and be stable. Intermediate discontinuities are classified graphically and the pre-shock conditions under which they can exist are elucidated. No definitive conclusion is reached concerning the physical reality of intermediate shocks.

Keywords: Magnetohydrodynamics, Intermediate Shocks, MHD Rankine-Hugoniot Relations

1. INTRODUCTION

If recent indications prove correct, ideal MHD theory may yet hold some surprises. As is well-known, there are three characteristic propagation speeds for small amplitude waves: fast, intermediate and slow. Using a simple perturbation analysis, it can be shown that finite amplitude perturbations propagating as fast and slow waves steepen into fast and slow shocks. By contrast, intermediate waves don't steepen and are instead associated with a rotational discontinuity in which the perpendicular (to the normal to the discontinuity) component of magnetic field reverses. This set of discontinuities is complete in the sense that information from an arbitrary set of boundary conditions can be propagated upstream using a combination of them.

However, the MHD Rankine-Hugoniot conditions also admit up to four intermediate shock solutions in which the intermediate Mach number is reduced below unitv and the perpendicular component of the magnetic field changes sign as the fluid traverses the shock. The physical relevance of these additional shock solutions has been controversial. According to formal evolutionary conditions for a strictly hyperbolic system, the gas velocity relative to the shock is allowed to cross only *one* propagation speed. Taking MHD to be strictly hyperbolic, this immediately eliminates from consideration all but one of the intermediate shock solutions (since the other three cross two or more propagation speeds).

Unfortunately, this remaining shock may be difficult to establish for two reasons. Firstly, the intermediate wave does not steepen. Secondly, the intermediate shock is coplanar (the velocities and magnetic fields before and after the shock lie in the same plane), while the intermediate characteristics only carry information about velocity and magnetic field perturbations *perpendicular* to this plane. An even more serious objection is that it may be unstable. Consider, (following Kantrowitz & Petschek (Ref. 1)) that an intermediate shock is set up ahead of a piston. Now create an arbitrarily small velocity perturbation in the direction perpendicular to the plane of the downstream velocity and magnetic field. The intermediate wave launched by this perturbation will be convected into the shock but will be unable to propagate upstream. We thus appear to have formed a non-coplanar shock crossing the intermediate speed in violation of the Rankine-Hugoniot relations. Kantrowitz & Petschek conjectured that if this type of discontinuity were set up, it would immediately break up to form a sequence of fast and slow shocks and rotational discontinuities propagating at distinct speeds individually satisfying the Rankine-Hugoniot conditions and collectively satisfying the originally imposed up- and downstream boundary conditions.

Three developments have recently cast doubt on this line of argument. First, in analyzing the the results of a one-dimensional Navier-Stokes MHD code, C.C. Wu (Ref. 2,3) found a persistent shock structure that apparently satisfied the Rankine-Hugoniot jump conditions for an intermediate shock. Second, Winterhalter and Kivelson (Ref. 4) reported detecting steepened intermediate waves in the interplanetary medium. Finally, a re-examination of MHD shock Kennel, Blandford, Wu, and Coppi (Ref. 5) has raised some objections to the above arguments.

The objections fall into two classes. First, the equations of perfect MHD are not strictly hyperbolic because the fast and intermediate characteristics propagate with iden-

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tical speeds parallel to the magnetic field. This might allow shocks to cross more than one wave speed. Secondly, ideal MHD includes no dissipation and can say nothing about the details of shock structure. Until the shock structure is known, stability arguments like those of Kantrowitz & Petschek (Ref. 1) remain conjectures. When dissipation is added, it also appears that the intermediate rotational discontinuity used by Kantrowitz & Petschek to match boundary conditions may really be the unphysical solution. (From the Rankine-Hugoniot relations, entropy is constant across a rotational discontinuity. Rotating the field, however, generally leads to resistive dissipation, ie. an increase in entropy.) It is by no means obvious then that intermediate shocks should not exist. In fact, if one studies the model equation of Cohen & Kulsrud (Ref. 6) which describes the *non*-hyperbolic limit of small amplitude waves propagating in the quasi-parallel direction, one finds that intermediate waves do indeed steepen. If one then adds some model resistive dissipation, one finds that *non*-evolutionary intermediate shock solutions also appear to exist. The question of the physical reality of intermediate shocks is thus once again open.

2. RANKINE-HUGONIOT ANALYSIS

A detailed examination of this question is beyond the scope of this paper. We will content ourselves here with laying the foundations for future work by seeing what ideal MHD alone may tell us about intermediate shocks. Let us consider then the MHD Rankine-Hugoniot relations. We work in the shock frame and orient our coordinate system so that the magnetic field and velocity lie in the $x-y$ plane and the shock normal lies parallel to the x direction. Note that the shock frame is not unique for a one dimensional shock. We can simplify our equations considerably by choosing the frame where the upstream transerve velocity , v_{y1}, is zero. Henceforth, it will be assumed this has been done. The Rankine-Hugoniot relations are then derived by demanding that the mass, energy and momentum fluxes (including electromagnetic contributions) be continuous across the shock. To close the system of equations, electromagnetic jump conditions (the continuity of the normal component of the magnetic field and the tangential component of the electric field) are imposed at the discontinuity and the ideal MHD "Ohm's Law" ($\mathbf{E} = -\frac{1}{c}\mathbf{v}\times\mathbf{B}$) is assumed away from the discontinuity. The final jump conditions are then:

$$[\rho v_x] = 0, [v_x B_y - v_y B_x] = 0, [B_x] = 0,$$

$$[\rho v_x^2 + P + \frac{B_y^2}{4\pi}] = 0,$$

$$[\rho v_x v_y - \frac{B_x B_y}{4\pi}] = 0,$$

$$[\frac{1}{2}\rho v_x(v_x^2+v_y^2) + \frac{\gamma}{\gamma-1}v_x P + \frac{1}{4\pi}(v_x B_y^2 - v_y B_x B_y)] = 0. \tag{1}$$

Here the bracket notation $[G]$ represents $G_2 - G_1$ where the indices 1 and 2 refer to upstream and downstream quantities respectively. P is the pressure (assumed isotropic), ρ is the density, γ is the ratio of specific heats, and $v_{x,y}$ and $B_{x,y}$ are the components of the velocity and magnetic field. Restricting our attention to the case $v_{x1} \neq 0$ (the case $v_{x1} = 0$ represents an "entropy" discontinuity we are not interested in here), we can then introduce the convenient set of renormalized variables:

$$r = \frac{v_x}{v_{x1}}, b = \frac{B_y}{B_1}$$

and

$$M_{A1}^2 = \frac{4\pi\rho_1 v_{x1}^2}{B_1^2}, M_{I1}^2 = \frac{4\pi\rho_1 v_{x1}^2}{B_{x1}^2}, M_{S1}^2 = \frac{\rho_1 v_{x1}^2}{\gamma P_1} \quad (2).$$

Here M_{A1}, M_{I1}, and M_{S1} are the upstream Alfvén, intermediate, and sonic Mach numbers respectively.

The jump conditions then reduce to

$$F(r,b) = Ar^2 + B(b)r + C(b) = 0,$$

$$Z(r,b) = bX - b_1 Y = 0 \tag{3}$$

where

$$X = r - \frac{1}{M_{I1}^2}, Y = 1 - \frac{1}{M_{I1}^2},$$

$$A = -\frac{1}{2}(\frac{\gamma+1}{\gamma-1}),$$

$$B = \frac{1}{(\gamma-1)M_{S1}^2} + \frac{\gamma}{\gamma-1}(1 - \frac{b^2-b_1^2}{2M_{A1}^2}),$$

$$C = -\frac{1}{2} - \frac{1}{(\gamma-1)M_{S1}^2} + \frac{(b^2-b_1^2) - Y(b^2 - 2bb_1 + b_1^2)}{2M_{A1}^2} \quad (4).$$

The curves described by equations (3) in the (r,b) plane play a role analogous to that of the shock adiabat in hydrodynamic shocks — see Fig. 1 and Liberman & Velikovich (Ref. 7), § 3.3.2.

The two equations (3) combine to give a quartic in r from which the trivial root $r = 1$ can be factored to leave a cubic possessing either one or three real roots. In the case that there are three real roots, note that, although we have normalized to an upstream state 1, the quantities v_{x1} and B_1 can also be regarded as parameters that measure the mass, momentum and energy fluxes and any pair of of the four solutions may describe a shock.

When there are four solutions, labeled 1,2,3,4 in order of decreasing r (cf. Fig. 1), it can be shown that the following inequalities hold:

$$\begin{aligned} v_x(1) &\geq C_F(1) \geq C_I(1) \\ C_F(2) &\geq v_x(2) \geq C_I(2) \\ C_I(3) &\geq v_x(3) \geq C_{SL}(3) \\ C_I(4) &\geq C_{SL}(4) \geq v_x(4) \end{aligned} \tag{5}$$

We can now use these inequalites to label the solutions when only two exist. As is apparent from Fig. 1, the pair of solutions will be either types 1 and 2, or types 3 and 4. We can now classify MHD shocks. There are twelve possible pairs of solution types of which only six are compressive (ie. decrease r) and increase entropy across the shock. The 1 → 2 and the 3 → 4 shocks "cross" the fast and slow speeds respectively and are the standard fast and slow MHD shocks. The four remaining shocks are intermediate shocks that cross the intermediate speed. (As a a corollary, then, intermediate shocks occur only when the momentum and energy fluxes allow four solutions.) The 2 → 3 combination describes the lone evolutionary intermediate shock discussed above.

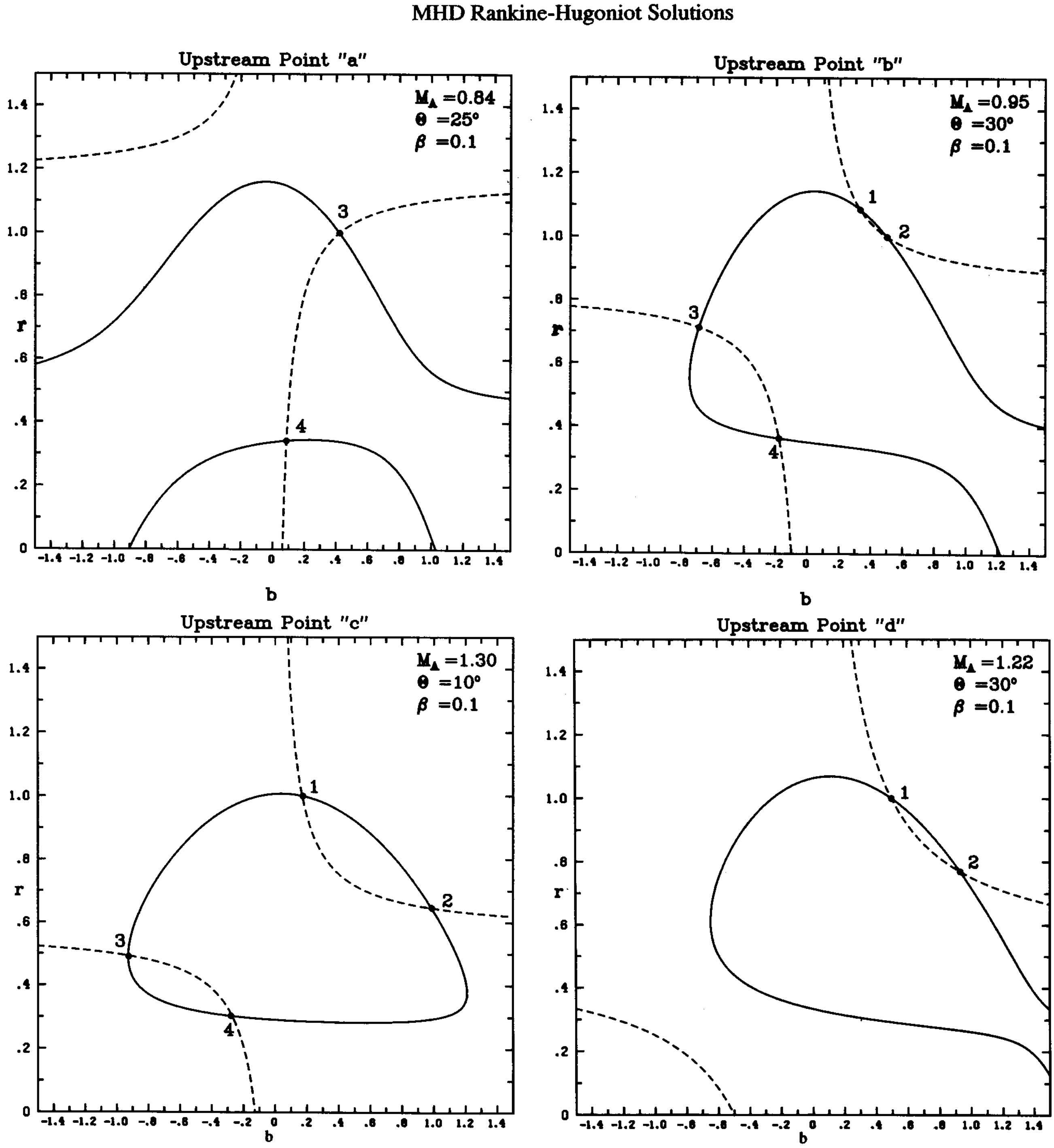

Fig. 1. Graphical method for locating MHD shocks. The bold solid curve graphs the relation $F(r,b) = 0$. All points on this curve correspond to flows that carry the same mass, momentum and energy. The dashed hyperbola represents the condition that the electric field component lying in the shock plane be continuous, ie $Z(r,b) = 0$. The four intersections satisfy both requirements. The point labeled 1 is the fastest such solution. The points labeled 2, 3, 4 correspond to alternative solutions. Any pair of solutions mutually satisfy the Rankine-Hugoniot jump conditions. The point labeled 1 need not correspond to the point chosen as the upstream state (which in these graphs is always the point sitting at the trivial solution $r = 1$.) Note that when the mass, momentum and energy fluxes are changed, the number of allowed solutions may decrease from four to two, and that alternative shapes are possible for the curve $F(r,b) = 0$. The labeling of the graphs shown corresponds to the choice of the corresponding points in Fig. 2 as the upstream states.

It is convenient to modify the Friedrichs diagram to exhibit the upstream conditions under which intermediate shocks exist (Fig.2). As can be seen, intermediate shocks are primarily *low* Alfvén Mach number, quasi-*parallel* shocks. As mentioned above, this is precisely the regime where MHD becomes non-hyperbolic and one might expect intermediate shock solutions. Note also that as a function of $\beta_1 = 8\pi P_1/B_1^2 (= 2M_{S1}^2/\gamma M_{A1}^2)$ the maximum Alfvén Mach number for which $1 \rightarrow 3,4$ shocks are allowed is the same as the maximum Mach number for which fast switch-on shocks may occur. (In the limit where the magnetic field points along the shock normal, the points of type 2 and 3 become degenerate. An "exactly" parallel $1 \rightarrow 3$ shock *is* a fast switch-on switch.) As β increases, this maximum Alfvén Mach number decreases, and for β greater than the threshold value $2/\gamma$, no $1 \rightarrow 3,4$ (and switch-on) shocks are allowed.

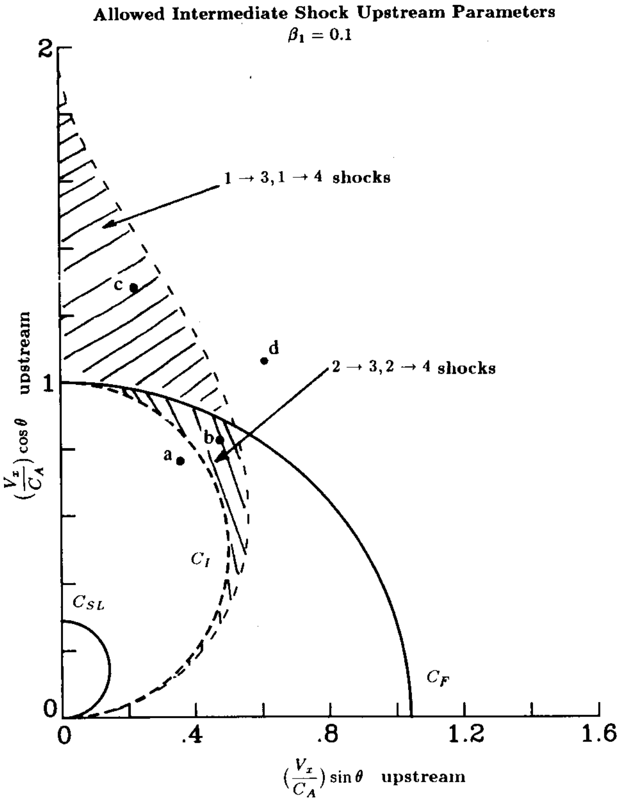
Allowed Intermediate Shock Upstream Parameters
$\beta_1 = 0.1$

Fig. 2. Modified Friedrichs diagram exhibiting the range of upstream Alfvén Mach numbers , M_{A1}, and angles between the magnetic field and the shock normal, θ_1, for which intermediate modes can be found. The figure is drawn for a value of $\beta_1 = 0.1$. The points labelled $a..d$ represent the upstream states for which F vs. Z plots are shown in Fig. 1.

3. REFERENCES

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