

Angular Momentum Transport in Thin Accretion Disks and Intermittent Accretion

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The plasma modes, transporting angular momentum in accretion disks, under minimally restrictive conditions when the magnetic energy density is significant relative to the thermal energy density, are shown to be singular if the ideal MHD approximation is adopted. A similarity with the modes producing magnetic reconnection in current carrying plasmas is established. The combined effects of finite plasma temperature, of plasma compressibility, of the gradient of the rotation frequency, and of appropriate transport processes (outside ideal MHD) are involved in the onset of these nonaxisymmetric and locally corotating modes.

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One of the basic problems in astrophysics is to explain how the angular momentum is transported outward in a disk rotating around a central object, whose gravity is dominant, so that accretion of matter onto the object can take place. This problem is relevant to a large variety of astrophysical sources where gravitational energy is released through accretion and where the rate of angular momentum transfer allowed by the familiar collisional viscosity is inadequate.

In the present Letter, we suggest that in thin accretion disks a significant rate of angular momentum transport can be provided by novel plasma collective modes that can be excited, under the most accessible conditions, in regimes where the magnetic energy density is significant relative to the thermal energy density. These modes are of the “singular” type, depend on the existence of a small level of “dissipation,” and are driven by the combined effects of the rotation frequency gradient and of the plasma temperature. In particular, these modes, that are nonaxisymmetric and corotate locally with the disk, have four characteristic double singularities in the radial direction, when described by the ideal MHD approximation, and depend on the plasma compressibility. In fact, in magnetically confined laboratory plasmas the macroscopic modes that are observed to be most easily excited are also of the singular type and depend on the finiteness of the plasma resistivity. The results of the analysis that are described lead to a specific form of the effective transport coefficient for angular momentum transfer that can be inferred from it and, in addition, suggest strongly that accretion involving disks is an inherently intermittent process.

We consider an axisymmetric thin disk [1], whose height is $2H(R) \ll R$, where R is the distance from the symmetry axis. The magnetic field in the disk has both a vertical component (B_z) and a toroidal one (B_ϕ) that can be of the same magnitude ($B_z \sim B_\phi$). We look for plasma modes which are localized around the surface $R = R_0$ over a distance $\Delta R \ll R_0$. Since the particle density n at the edge of the disk is much lower than at the corner of it, we argue

that the Alfvén velocity $v_A^2 \equiv B^2/(4\pi m_p n)$, which is associated with the stabilizing factor of the modes of interest, is much larger at the edge than at the center. Therefore, these modes will also have to be localized in the z direction, as shown by the relevant analytical formulation of the axisymmetric mode [2] which we give. In this region around the equatorial plane, we can consider the radial component of the B field to be negligible and the vertical equilibrium condition to reduce to $-\partial p/\partial z \approx \rho v_\phi^2 z/R_0^2$, given that the radial equilibrium reduces to the Keplerian condition $v_\phi^2 = GM_*/R_0$, where M_* is the mass of the accreting object and $\rho \equiv m_p n$. Thus $H \approx R_0 c_S/v_\phi$, where $c_S \ll v_\phi$ is the velocity of sound.

As can be verified *a posteriori*, the axisymmetric mode is most easily described by the perturbed velocity $\hat{v}_\phi \approx \tilde{v}_\phi(z) \exp[ik_R(R - R_0) + \gamma_0 t]$ and the associated finite radial displacement $\hat{\xi}_R$ when $\gamma_0^2/(k_R^2 v_A^2) \rightarrow 0$ and the marginal stability condition is approached. The R and ϕ components of the frozen-in-law $\partial \hat{\mathbf{B}}/\partial t = \nabla \times (\hat{\mathbf{v}} \times \mathbf{B} + \mathbf{v} \times \hat{\mathbf{B}})$, where $\hat{v}_R = d\hat{\xi}_R/dt$, give the perturbed radial field $\hat{B}_R = B_z \partial \hat{\xi}_R/\partial z$ and $0 \approx B_z \partial \hat{v}_\phi/\partial z + (R_0 d\Omega/dR) \hat{B}_R$. Therefore $\tilde{v}_\phi \approx -(d\Omega/dR) \hat{\xi}_R$, i.e., if a plasma element is displaced outward by $\hat{\xi}_R > 0$, $v_\phi + \tilde{v}_\phi$ will be increased for $d\Omega/dR < 0$.

Then, if we take the $\mathbf{e}_\phi \cdot \nabla \times$ component of the total momentum conservation equation $\rho(\partial \hat{\mathbf{v}}/\partial t + \hat{\mathbf{v}} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \hat{\mathbf{v}}) = -\nabla \hat{p} + 1/c(\hat{\mathbf{J}} \times \mathbf{B}) + \hat{\rho}(\mathbf{g} - \mathbf{v} \cdot \nabla \mathbf{v})$, we obtain

$$-2\Omega \rho(z) \hat{v}_\phi \approx \frac{1}{4\pi} \left(B_z \frac{\partial}{\partial z} \hat{B}_R - ik_R B_z \hat{B}_z \right), \quad (1)$$

where $\rho(z) \approx \rho_0 (1 - \alpha_n z^2/H^2)$ near $z = 0$. Equation (1) leads to a second order differential equation in $\hat{\xi}_R(z)$ whose ballooning [3] solution is $\hat{\xi}_R \approx \tilde{\xi}_0 \exp(-\sigma_z^2 z^2/2)$, where $\sigma_z^2 = \alpha_n k_R^2/H^2$ provided $k_R^2 > |d^2/dz^2| > 1/H^2$. In this case $\hat{B}_z \approx -ik_R B_z \hat{\xi}_R$, $k_R^2 = k_{R_0}^2 - \delta k_{R_0}^2$, where $\delta k_{R_0}^2 \ll k_{R_0}^2$, and

$$-2\Omega \frac{d\Omega}{dR} R_0 = k_{R_0}^2 v_{A_z}^2(z=0) \quad (2)$$

is the marginal stability condition. This means that $3v_\phi^2 = k_R^2 R_0^2 v_{A_z}^2 \gg (R_0^2/H^2)v_{A_z}^2$, and, if we take $R_0/H \sim v_\phi/c_S$, that $c_S^2 \gg v_A^2$.

Therefore, this kind of mode involves a rather severe excitation condition in terms of the requirement that the magnetic energy density is well below the plasma thermal energy density. We note that it is a feature of the ballooning mode to have a radial profile characterized by a periodic oscillation whose wavelength is related to the height of the disk. This feature is similar to that of the ballooning modes [4] that are driven by the plasma pressure gradient in axisymmetric toroidal plasmas where the ballooning profile along with the fields is coupled to the radial oscillation. On the other hand, in the long cylinder approximation considered in Ref. [2] where $\tilde{v}_\phi(z) \approx \tilde{v}_\phi^0 \exp(ik_z z)$, the radial wavelength is decoupled from the vertical one.

Now we may envision a heating process to take place, in a formed magnetized disk before accretion develops. In this case and in other realistic circumstances, corresponding, for instance, to the observation of magnetized jets originating from disks, the hard conditions for the excitation of the axisymmetric ballooning instability (ABI) may not be met. Then, the modes that should be excited first can be found outside the ideal MHD approximation, extend over macroscopic scale distances, and have similar to the modes that produce magnetic reconnection in current carrying plasmas, characteristic radial singularities when treated by the ideal MHD approximation. We call them singular bending modes [5,6]. These are of the nonaxisymmetric type [7]; their wave numbers along the magnetic field can be adjusted to provide the minimal instability threshold, and their spectrum can be relatively broad. In this case $\hat{v}_\phi \approx \tilde{v}_\phi(R - R_0) \exp(-i\omega t + in^0\phi + ik_z z)$, where the complex function $\tilde{v}_\phi(R - R_0)$ is determined by the relevant dispersion equation. Thus, the characteristic topology of this class of modes is the reciprocal of that of the ABI mode as it is ballooning in the R direction and oscillatory in the z direction. In particular, $k_{\parallel} B = \mathbf{k} \cdot \mathbf{B} = k_z B_z + n^0 B_\phi / R_0 < n^0 B_\phi / R_0$, with $(n^0)^2 \gg 1$. Thus $k_z \approx -(n^0/R_0)B_\phi/B_z$ can be

taken to be $>\pi/H$ without constraining the value of k_{\parallel} to be related to that of the small height of the disk. The term $k_{\parallel} v_A$ has a stabilizing effect and, unlike the case of the axisymmetric mode, the value of k_{\parallel} is chosen to correspond to the lowest relevant instability threshold. In particular, we may write $k_z = m^0 \pi/H$ so that each mode is characterized by the toroidal and poloidal mode numbers n^0 and m^0 . We note that the same set of mode numbers is usually adopted to describe the modes of a large aspect ratio (thin) toroidal plasma column with a circular cross section that can be viewed as a slightly bent cylinder.

The modes of interest are corotating with the disk at $R = R_0$ and have $\omega = n^0 \Omega(R_0) + i\gamma_0$ so that the Doppler shifted frequency $\varpi(R) \approx -n^0(d\Omega/dR)(R - R_0)$ is varying significantly with $R = R_0$. As will be shown, each mode is localized around a given radius $R = R_0$ over a radial distance $\Delta_0 \sim R_0/n^0 \ll R_0$. Therefore we envision a sequence of modes that can be excited at different radii so as to involve a large fraction of the disk in a similar fashion to that of the toroidal modes that can be excited at different (minor) radii of the plasma column. These modes are characterized by “microscopic” transition regions, that are contained within the localization distance Δ_0 and whose width $\Delta_1 \ll \Delta_0$. In the transition regions, the ideal MHD approximation is not valid and a class of small but appropriate dissipative effects has to be considered.

To gain further insight into the nature of these modes, we use, at first, the ideal MHD approximation, which includes the perturbed momentum conservation equation, the adiabatic equation of state $(\partial/\partial t + \mathbf{v} \cdot \nabla)\hat{p} = -c_S^2 \rho \nabla \cdot \hat{\mathbf{v}}$, the frozen-in-law indicated earlier, and Maxwell’s equations. We consider the limit near marginal stability, where $\gamma_0/|k_{\parallel} v_A| \ll 1$ and $|k_{\parallel} v_A| \sim \Omega$. The effects of finite compressibility, $\nabla \cdot \hat{\mathbf{v}}$, have to be taken into account [8], in spite of the complexity that they add to the derivation, as $\nabla \cdot \hat{\mathbf{v}}$ tends to become infinite near one of the pair of singularities that we shall discuss and because of the importance of the effects of finite plasma temperature associated with it.

Then we obtain the following equation [5,6] for the complex function $\tilde{\xi}(\bar{R}) \equiv \tilde{\xi}_R$:

$$\left\{ \left[\omega_A^2 - \overline{\omega}(\bar{R})^2 - i2\gamma_0 \overline{\omega}(\bar{R}) + 2\Omega b_\phi \left(R_0 \frac{d\Omega}{dR} + 2\Omega G_{AS} \right) \right] \right\} \tilde{\xi}(\bar{R}) = \left(\frac{b_\phi}{k_z} \right)^2 \frac{d}{d\bar{R}} \left\{ \left[\omega_A^2 - \overline{\omega}(\bar{R}) - i2\gamma_0 \overline{\omega}(\bar{R}) \right] \frac{d}{d\bar{R}} \tilde{\xi}(\bar{R}) \right\}, \quad (3)$$

where $\bar{R} \equiv R - R_0$, $b_\phi \equiv B_\phi/B$, $\overline{\omega}(\bar{R}) \propto \bar{R}$, $G_{AS} \equiv \frac{\overline{\omega}^2(\bar{R}) + i2\gamma_0 \overline{\omega}(\bar{R})}{\omega_{AS}^2 - \overline{\omega}(\bar{R})^2 - i2\gamma_0 \overline{\omega}(\bar{R})}$, $\omega_A^2 \equiv k_{\parallel}^2 v_A^2$, and $\omega_{AS}^2 = \omega_A^2 c_S^2 / (c_S^2 + v_A^2) < \omega_A^2$. The frequency ω_{AS} is that of the so-called “slow mode” in the limit $k_{\parallel}^2 \ll k_{\perp}^2$ in a homogeneous plasma, and ω_A that of the shear Alfvén mode. We note that the real coefficients of Eq. (3) are even in \bar{R} while the imaginary ones are odd, and we look for eigenfunctions $\tilde{\xi} = \text{Re}\tilde{\xi}(\bar{R}) + i \text{Im}\tilde{\xi}(\bar{R})$ with $\text{Re}\tilde{\xi}(\bar{R})$ even and $\text{Im}\tilde{\xi}(\bar{R})$ odd.

Both $\text{Re}\tilde{\xi}(\bar{R})$ and $\text{Im}\tilde{\xi}(\bar{R})$ undergo sharp variations at the surfaces where $\bar{R} = \bar{R}_1$ and $\bar{R} = \bar{R}_2$, corresponding to $\overline{\omega} = \omega_{AS}$ and $\overline{\omega} = \omega_A$, respectively. The two singularities, which appear as $\gamma_0/\omega_A \rightarrow 0$, are different in nature. At

$\bar{R} = \bar{R}_1$ where $|G_{AS}| \rightarrow \infty$, both $\text{Re}\tilde{\xi}(\bar{R})$ and $\text{Im}\tilde{\xi}(\bar{R})$ remain finite but their curvature tends to become infinite. At the second resonance, $\text{Im}\tilde{\xi}(\bar{R})$ has a logarithmic singularity while $\text{Re}\tilde{\xi}(\bar{R})$ has a jump downward in amplitude, which is reminiscent of that found [9] for modes producing magnetic reconnection and having $m^0 = 1$ and $n^0 = 1$ in a toroidal configuration with circular cross section and large aspect ratio. Instead, the first singularity is similar to that characterizing toroidal and cylindrical modes [10,11] with $m^0 > 1$, whose existence depends on magnetic reconnection. In the case of the toroidal and cylindrical plasma modes, $\text{Im}\tilde{\xi}$ has the same radial profile as $\text{Re}\tilde{\xi}$, and only one singularity per mode has to be considered. Instead, Eq. (3) makes it necessary to deal with four transitions at the same time: two for $\text{Re}\tilde{\xi}$ and two for $\text{Im}\tilde{\xi}$.

We note that, to resolve the second singularity, it is sufficient to consider the finiteness of γ_0/ω_A or to replace γ_0 by $\gamma_0 + \nu_D$, ν_D representing a small rate of dissipation, when letting $\gamma_0 \rightarrow 0$. For the first resonance, the introduction of a singular perturbation in the transition region Δ_1 around $\bar{R} = \bar{R}_1$ is hard to avoid. In particular, the derivation of Eq. (3) indicates that at $\bar{R} = \bar{R}_1$ the compressibility $\nabla \cdot \hat{\mathbf{v}}$, the perturbed longitudinal velocity $\hat{v}_{\parallel} = \hat{\mathbf{v}} \cdot \mathbf{B}/B$, the plasma density \hat{n} , and the temperature \hat{T} become infinite. Then the effects of small but finite transport processes affecting the thermal energy, the longitudinal momentum, and the electrical resistivity have to be taken into account. One of the simplest models [6], that can remove the singularity at $\bar{R} = \bar{R}_1$ and produce a finite growth rate, is a diffusion operator acting on the perturbed plasma pressure with a locally varying diffusion coefficient. In this case, we are led to solve an inhomogeneous second order differential equation within the Δ_1 region around $\bar{R} = \bar{R}_1$, as is the case of resistive modes. A simpler, algebraic model consists in introducing a locally enhanced rate of dissipation [6] in the thermal energy balance equation. We note that the lowest modes, with the largest width of localization that are found for $B_z \gtrsim B_{\phi}$, typically for $\alpha_z^2 \gtrsim 1$, have the highest amplitudes in the interval $-(1 + A_S)^{1/2} \lesssim r \lesssim (1 + A_S)^{1/2}$, which corresponds to a width $2(1 + A_S)^{1/2}\Delta_0$, for $A_S \equiv v_A^2/c_S^2$.

Considering a diffusion coefficient $D_p \ll \Omega R_0^2/(n^0)^2$ associated with the smoothing out of the solution in the transition region around $r = 1$, the order of magnitude of the growth rate that is obtained is $\gamma_0 \sim [D_p(n^0)^2/R_0^2]^{1/3}(R_0|d\Omega/dR|)^{2/3} f(c_S/v_A)$, where f is a finite function of c_S/v_A . The corresponding width Δ_1 of the transition region is of the magnitude $\Delta_1 \sim (D_p\Delta_0/\Omega)^{1/3}$. Clearly, the existence of the singular modes depends on having to deal with the singularity where $\varpi = \omega_{AS}$ in the theory, by considering the ratio $\sqrt{(\omega_A^2 - \omega_{AS}^2)/\omega_{AS}} = v_A/c_S$ as significant even when it is small, and on the essential role of the finite compressibility $\nabla \cdot \hat{\mathbf{v}}$. In particular, the mode growth rate depends on the nature of the transition of the eigenfunction at the

resonance $\bar{R} = \bar{R}_1$ and of the ‘‘dissipative’’ process that makes the transition possible. In fact, these dissipative modes resulting from singular perturbation theory were not included in preceding analyses [2,7,12].

The resulting angular momentum flux derived by the quasilinear approximation, considering the symmetries of the radial eigenfunctions of the modes, is found to be significant and directed outward. Clearly, the relevant transport process is not that of a diffusion equation. Nonetheless, we may give a rudimentary estimate of an effective diffusion coefficient, that a radial sequence of localized modes of the considered type can produce, as $D_J \sim \gamma_0\Delta_0^2 \sim D_p^{1/3}(\Omega R^2/n^0)^{2/3} \sim D_p^{1/3}(\Omega H^2/m^0)^{2/3}$. Given that, we have $D_J \sim D_p^{1/3}(c_S H/m^0)^{2/3}$. If we compare this to the Shakura-Sunyaev [13] coefficient $D_{ss} \approx \alpha_{ss}c_S H$, where α_{ss} is small, adjustable numerical parameter. Thus, our estimate gives the physical expression $D_p^{1/3}/[(c_S H)^{1/3}m^{0^{2/3}}]$ for α_{ss} .

Defining $\alpha_z \equiv 3B_z/(2B_{\phi})$, we note that, in the limit where $\alpha_z^2 \ll 1$ and the field is nearly toroidal, the relevant modes are strongly oscillatory [6] as a function of \bar{R} and are contained within the surface $\bar{R} = \bar{R}_1$. In particular, if we refer, for simplicity, to the radial profile of $\text{Re}\tilde{\xi}(\bar{R})$, the numerical analysis [14] has shown that this profile has no nodes for $\alpha_z \geq \alpha_0 \sim 1$, and $A_S \sim 1$, one node within a discrete interval $\alpha_{11} \leq \alpha_z \leq \alpha_{21}$, where $\alpha_{11} < 1$, two nodes within $\alpha_{21} \leq \alpha_z \leq \alpha_{22}$, where $\alpha_{12} < \alpha_{21}$, etc. In fact, the presence of a dissipative process can be shown to be necessary for the existence of the considered nonaxisymmetric modes for all values of α_z .

Finally, we may envision that, as a starting point, a cold disk is formed in a preexisting magnetic field. Then the disk is gradually heated until the plasma can be nearly frozen in with the magnetic field, as its resistivity is decreased. Following this, the macroscopic modes we have described can be excited. At this stage, a fast accretion process can take place at the rate allowed by the outward momentum transport produced by the excited modes. Thus, accretion can develop as an intrinsically intermittent process that is reminiscent of the sawtooth oscillations of radiation emission [15], which are commonly observed in toroidal laboratory plasmas. In the latter case, the plasma thermal energy at the center of the plasma column is increased until the plasma pressure gradient reaches a threshold for the onset of a $m^0 = 1$, $n^0 = 1$ mode [9] which redistributes the plasma thermal energy, cooling the central part. Then, reheating starts and the cycle repeats itself. We note that in the process of sawtooth oscillations the modes whose excitation causes the ‘‘crash’’ of the plasma temperature [15] are in fact of the singular type [9] when treated by the ideal MHD approximation.

A very simple analytical model, consisting of a set of two first order nonlinear differential equations simulating an intermittent accretion process, has been formulated and a typical solution of it is shown in Fig. 1.

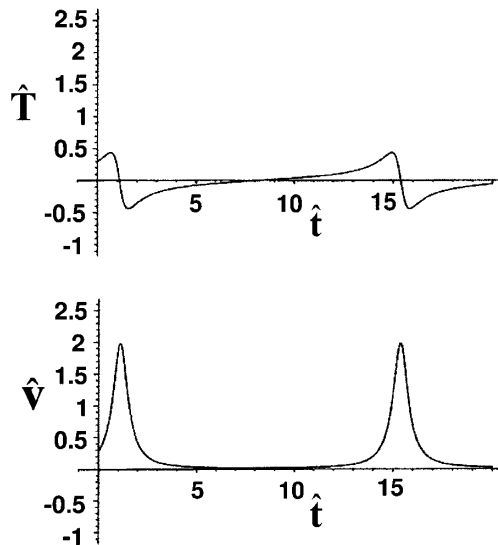


FIG. 1. Bursty time dependence of the inflow velocity, and corresponding variation of the plasma temperature, from an analytic nonlinear model of accretion allowed by finite temperature effects.

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