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Angular momentum transport associated with finite plasma temperature in accreting disks

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Abstract

A "hot" plasma mode that can be excited in a rotating Keplerian disk imbedded in a magnetic field with a significant toroidal component, is found analytically. This mode is shown to transport angular momentum outward, that is the primary requirement for the occurrence of mass accretion on the central object. The relevant distinctive feature is that the mode depends on the effects of the finite plasma compressibility and finite anisotropic ion longitudinal (relative to the field) viscosity and sound velocity; the driving factor of the relevant instability being the combination of the finite plasma temperature and the (negative) radial gradient of the rotation frequency. Electrostatic velocity gradient driven modes, in a plasma column with an axial magnetic field, that depend on the effects of a longitudinal viscosity or a viscous-like term, finite electron temperature and compressibility have been similarly predicted earlier and one of them has been experimentally confirmed recently by other authors. © 1997 Published by Elsevier Science B.V.

1. Introduction

We consider an axisymmetric rotating plasma, where the centrifugal force is balanced by a centripetal gravity [1], immersed in a magnetic field that has a significant toroidal component. The plasma parameters are assumed to be such that $\Omega_{ci}^2 \gg \nu_i^2$, where Ω_{ci} is the ion cyclotron frequency and ν_i is the ion (classical or effective) collision frequency. Specifically, the regimes of interest are those for which the longitudinal, to the field, ion viscosity is much larger than the transverse one, that is $D_\mu^\parallel \gg D_\mu^\perp$, and, likewise, the parallel ion thermal conductivity prevails over the perpendicular.

Previously it was shown [2,3] that electrostatic modes, associated with the longitudinal viscosity and involving finite electron temperature and compressibility, can be excited with relatively low thresholds in magnetically confined plasmas where an inhomogeneous flow velocity directed along the field [2,3] exists. Recently, clear experimental evidence [4] for modes of this kind has been provided for partially ionized plasmas, to which the original theory had been extended by including the effects of ion–neutral collisions, given the possibility it offers to explain observed fluctuations in the ionosphere [5]. On the other hand, it can be argued easily that electrostatic modes of this kind, whose growth rate depends on the finiteness of the ratio of the particle gyro-radius to a macroscopic scale distance, are not significant when considering astrophysical objects.

Here we model a Keplerian plasma disk by a nearly

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cylindrical configuration. The modes which are shown to be capable of being excited are electromagnetic and depend on the effects of finite compressibility and finite plasma temperature as represented by the sound velocity c_s or the longitudinal ion viscosity. The instability's driving factor is the combination of the finite plasma temperature and the (negative) gradient of the rotation frequency. Therefore, these modes cannot be found in a cold plasma where c_s and $D_\mu^\|$ are negligible.

2. Model configuration

We consider an axisymmetric configuration whose radial equilibrium condition is given by

$$-g_R + \frac{v_\phi^2}{R} = v_R \frac{\partial}{\partial R} v_R + \frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{1}{c}(\boldsymbol{J}\times\boldsymbol{B})_R, \quad (1)$$

where v_ϕ and v_R are the toroidal and the radial components of the velocity, p is the pressure and g_R is the radial component of the gravity acceleration. In particular $|v_R|^2 \ll v_\phi^2$ and

$$c_s \sim \frac{p}{\rho} \ll v_\phi^2 . \quad (2)$$

The magnetic field in which this is imbedded has a significant toroidal component. Thus

$$\boldsymbol{B} = B_\phi \boldsymbol{e}_\phi + B_z \boldsymbol{e}_z \quad (3)$$

and we assume that

$$v_A^2 \ll v_\phi^2, \quad (4)$$

where $\hat{v}_A^2 = B^2/4\pi\rho$.

The perturbations that we consider, from this equilibrium configuration, are of the form

$$\hat{v}_R = \tilde{v}_R(R,z)\exp(-\mathrm{i}\omega t + \mathrm{i}n^0\phi) . \quad (5)$$

Since we intend to model a thin disk [1], we analyze in particular modes that vary over scale distances in the z-direction that are much smaller than those in the radial and the toroidal directions, that is

$$\left|\frac{n^0}{R}\hat{v}_R\right| < \left|\frac{\partial \hat{v}_R}{\partial R}\right| < \left|\frac{\partial}{\partial z}\hat{v}_R\right| , \quad (6)$$

$$\left|\frac{\hat{v}_R}{H}\right| < \left|\frac{\partial}{\partial z}\hat{v}_R\right| , \quad (7)$$

where $H(R)$ is the height of the disk, that is being simulated, and $n^0 > 1$. Then we define

$$\gamma \equiv -\mathrm{i}[\omega - n^0\Omega(R_0)] , \quad (8)$$

where $\Omega(R) \equiv v_\phi/R$ is the rotation frequency, and R_0 is the radius around which the mode is assumed to be localized. We consider the radial localization distance to be sufficiently short that

$$\left|n^0\frac{\mathrm{d}\Omega}{\mathrm{d}R}(R-R_0)\right| \ll |\gamma| \ll |n^0\Omega| \quad (9)$$

and that the initial equilibrium velocity is sufficiently small that

$$|\gamma| > \left|v_R \frac{\partial \hat{v}_R}{\partial R}\frac{1}{\hat{v}_R}\right| . \quad (10)$$

We consider the most elementary type of modes which satisfy the frozen-in-constraint $\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}/c = 0$. The gravity acceleration is given by $\boldsymbol{g} = -\nabla\Phi_G$, where $\Phi_G = -C_G/(R^2+z^2)^{1/2}$ and is mostly radial. To the extent that the equilibrium in the z-direction [1] is held by gravity, $H/R \sim c_s/v_\phi$.

3. Basic mode equations

In order to illustrate our point simply, we consider at first the limit where $B \simeq B_\phi(R)$ and $c_s^2/v_{A\phi}^2 \lesssim 1$, where $v_{A\phi}^2 = B_\phi^2/4\pi\rho$. Thus the ϕ-component of the momentum conservation equation is

$$\rho\left(\gamma\hat{v}_\phi + \hat{v}_R\frac{\kappa^2}{2\Omega}\right) = -\mathrm{i}\frac{n^0}{R}\hat{p} - (\nabla\cdot\Pi)_\| + \frac{1}{c}J_z\hat{B}_R, \quad (11)$$

where Π is the anisotropic viscous pressure tensor [6], and $\kappa^2 \equiv (2\Omega/R)\,\mathrm{d}(R^2\Omega/\mathrm{d}R)$, $\kappa = \Omega$ being the epicyclic frequency. In addition, we can adopt the classical expression for $(\nabla\cdot\Pi)_\|$ rewriting it as

$$-\frac{1}{\rho}(\nabla\cdot\hat{\Pi})_\| = 2D_\mu^\|\frac{\boldsymbol{B}}{B}\cdot\nabla\hat{G}, \quad (12)$$

where $D_\mu^\|$ is the longitudinal viscous diffusion coefficient, and

$$G = \frac{\boldsymbol{B}}{B}\cdot\nabla\left(\frac{\boldsymbol{B}}{B}\cdot\boldsymbol{v}\right) - \left(\frac{\boldsymbol{B}}{B}\cdot\nabla\frac{\boldsymbol{B}}{B}\right)\cdot\boldsymbol{v} - \tfrac{1}{3}(\nabla\cdot\boldsymbol{v}) . \quad (13)$$

Then,

$$\hat{G} \simeq \frac{\hat{B}_R}{B}\frac{\mathrm{d}}{\mathrm{d}R}v_\phi + \mathrm{i}\frac{n^0}{R}\hat{v}_\phi + \frac{1}{R}\hat{v}_R - \frac{v_\phi}{R}\frac{\hat{B}_R}{B} - \tfrac{1}{3}\nabla\cdot\hat{v}$$
$$= \mathrm{i}\frac{n^0}{R}\hat{v}_\phi + \frac{\hat{B}_R}{B}\frac{\mathrm{d}\Omega}{\mathrm{d}R}R + \frac{1}{R}\hat{v}_R - \tfrac{1}{3}\nabla\cdot\hat{v}. \quad (14)$$

Now we note that the equation $\partial\hat{B}/\partial t = \nabla\times(\hat{v}\times B + v\times\hat{B})$ yields

$$\frac{\hat{B}_R}{B} = \mathrm{i}\frac{n^0}{R}\frac{\hat{v}_R}{\gamma}, \quad \frac{\hat{B}_z}{B} = \mathrm{i}\frac{n^0}{R}\frac{\hat{v}_z}{\gamma}, \quad (15)$$

$$\frac{\hat{B}_\phi}{B} = \mathrm{i}\frac{n^0}{R}\frac{\hat{v}_\phi}{\gamma} - (\nabla\cdot\hat{v})\frac{1}{\gamma} + \frac{\hat{v}_R}{\gamma R}B\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{R}{B}\right)$$
$$+ \mathrm{i}n^0\frac{\mathrm{d}\Omega}{\mathrm{d}R}\frac{1}{\gamma^2}\hat{v}_R. \quad (16)$$

Moreover, $\hat{p} = (T_e + T_i)\hat{n} + n(\hat{T}_e + \hat{T}_i)$, and

$$\hat{n} \simeq -\frac{n}{\gamma}\nabla\cdot\hat{v}. \quad (17)$$

We consider $|n\hat{T}_e| < |\hat{n}T_e|$ due to the large electron thermal conductivity along the magnetic field. The expression for $\hat{T}_i$ depends on the coupling to the electron temperature and on the longitudinal ion thermal conductivity. Therefore, we may write

$$\frac{1}{\rho}\hat{p} \simeq -\frac{1}{\gamma}(\nabla\cdot\hat{v})c_s^2\mathcal{I}_i(\gamma), \quad (18)$$

where $c_s^2 \equiv p/\rho$,

$$\mathcal{I}_i = 1 + \frac{T_i}{T_e + T_i}\frac{\gamma}{3\gamma/2 + \nu_\parallel^T},$$

$\nu_\parallel^T \equiv k_\parallel^2 D_T^\parallel + \nu_i^e$, $k_\parallel = n^0/R$, $D_T^\parallel$ is the diffusion coefficient for the ion thermal energy along the field and ν_i^e represents the rate of ion thermal energy loss to the electrons. In the collisional limit $D_T^\parallel \sim D_\mu^\parallel$.

The z-component of the momentum conservation equation gives, on the basis of Eqs. (16) and (18),

$$(\gamma + \omega_A^2)\hat{v}_z = -\mathrm{i}k_z\left[-(c_s^2\mathcal{I}_i + v_A^2)\nabla\cdot\hat{v}\right.$$
$$\left.+ \mathrm{i}k_\parallel v_A^2\left(\hat{v}_\phi + \frac{\Omega' R}{\gamma}\hat{v}_R\right)\right], \quad (19)$$

where $\omega_A^2 \equiv k_\parallel^2 v_{A\phi}^2$. Consequently,

$$\nabla\cdot\hat{v} \simeq \mathrm{i}k_\parallel\left(\hat{v}_\phi + \frac{\Omega' R}{\gamma}\hat{v}_R\right)\frac{v_A^2}{v_A^2 + c_s^2\mathcal{I}_i}, \quad (20)$$

$$\gamma\frac{\hat{B}_\varphi}{B_\phi} \simeq \mathrm{i}k_\parallel\left(\hat{v}_\phi + \frac{\Omega' R}{\gamma}\hat{v}_R\right)\frac{c_s^2\mathcal{I}_i}{v_A^2 + c_s^2\mathcal{I}_i}, \quad (21)$$

$$\hat{G} \simeq \mathrm{i}k_\parallel\left(\hat{v}_\phi + \frac{\Omega' R}{\gamma}\hat{v}_R\right)\frac{2}{3}\left(1 + \frac{1}{2}\frac{c_s^2\mathcal{I}_i}{v_A^2 + c_s^2\mathcal{I}_i}\right). \quad (22)$$

Then, if we define

$$\bar{\nu}_\parallel \equiv \tfrac{4}{3}k_\parallel^2 D_\mu^\parallel[1 + \tfrac{1}{2}c_s^2\mathcal{I}_i/(v_A^2 + c_s^2\mathcal{I}_i)]$$

and

$$\bar{\omega}_{As}^2 \equiv k_\parallel^2 c_s^2\mathcal{I}_i v_A^2/(v_A^2 + c_s^2\mathcal{I}_i),$$

Eq. (11) reduces

$$\gamma\hat{v}_\phi + \hat{v}_R\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}(R^2\Omega)$$
$$\simeq -\left(\bar{\nu}_\parallel + \frac{\bar{\omega}_{As}^2}{\gamma}\right)\left(\hat{v}_\phi + \frac{R}{\gamma}\frac{\mathrm{d}\Omega}{\mathrm{d}R}\hat{v}_R\right). \quad (23)$$

Finally, we consider the following component of the total momentum conservation equation

$$\boldsymbol{e}_\phi\cdot\nabla\times\hat{\boldsymbol{A}} = 0, \quad (24)$$

where

$$\hat{\boldsymbol{A}} \equiv \rho\left(\frac{\partial}{\partial t}\hat{v} + \hat{v}\cdot\nabla v + v\cdot\nabla\hat{v}\right)$$
$$+ \hat{\rho}v\cdot\nabla v - (g\hat{\rho} - \hat{g}\rho)$$
$$- \frac{1}{4\pi}(\hat{B}\cdot\nabla B + B\cdot\nabla\hat{B}).$$

We note that Eq. (24) implies $\partial\hat{A}_R/\partial z - \partial\hat{A}_z/\partial R = 0$ and given the condition (6) considered at the start, this reduces to

$$\hat{A}_R \simeq \gamma\hat{v}_R - 2\hat{v}_\phi\Omega - \mathrm{i}\frac{n^0}{R}\frac{B_\phi}{4\pi\rho}\hat{B}_R \simeq 0 \quad (25)$$

when $g_R = v_\phi^2/R$.

4. Mode growth rate and induced transport

The dispersion equation resulting from combining Eq. (25) with (23) is

$$\gamma^2[\gamma^2 + \bar{\nu}_\parallel\gamma + (\omega_A^2 + \bar{\omega}_{As}^2 + \kappa^2)]$$
$$- (\bar{\omega}_{As}^2 + \bar{\nu}_\parallel\gamma)(\Gamma_0^2 - \omega_A^2) = 0, \quad (26)$$

where $\Gamma_0^2 \equiv -R\, d\Omega^2/dR$. Then, we can make the following observations.

(1) The unstable mode that is found from Eq. (26) is driven by the combination of $\bar{\omega}_{As}^2 \Gamma_0^2$ or $\bar{\nu}_{\parallel} \Gamma_0^2$, both of which involve the negative gradient of Ω and the finite plasma temperature. In this context, we mention that the dissipative electrostatic velocity gradient driven mode [2,3] is associated with finite electron and ion temperature effects.

(2) Dissipation as represented by the $\bar{\nu}_{\parallel}$ and $\nu_{\parallel}^T$ terms, does not suppress the instability. In fact, in the limit where $\bar{\nu}_{\parallel} \gtrsim \gamma$ and $\bar{\nu}_{\parallel}\gamma > \bar{\omega}_{As}^2$ the mode becomes dissipative in nature. We may argue, however, that the implied condition that $\bar{\nu}_{\parallel} > \bar{\omega}_{As}$ is not always satisfied if the collisional expression for $\bar{\nu}_{\parallel}$ is adopted. In this case $\bar{\nu}_{\parallel} \sim (k_{\parallel}\lambda_i) k_{\parallel} v_{thi}$, where λ_i is the collisional mean free path and v_{thi} is the ion thermal velocity. Since $|k_{\parallel}\lambda_i| < 1$, for the validity of the adopted fluid description, we need to consider regimes where $v_A^2 < c_s^2$. In particular, for $\bar{\nu}_{\parallel} > \gamma$ and $\bar{\nu}\gamma > \bar{\omega}_{As}^2$, Eq. (26) reduces to a simple cubic equation.

(3) In very high temperature regimes for which $|k_{\parallel}\lambda_i| > 1$, the finite longitudinal ion viscosity and thermal conductivity terms will have to be replaced by others resulting from relevant collisionless processes. These terms will have to be taken into account, given the fact that $\bar{\omega}_{As} \lesssim k_{\parallel} v_{thi}$.

(4) The non-dissipative form of the instability is obtained in the limit where $\bar{\nu}_{\parallel} \sim \nu_{\parallel}^T < \bar{\omega}_{As}^2/\gamma$ and $\gamma > \bar{\nu}_{\parallel} \sim \nu_{\parallel}^T$. Then $\mathcal{I}_i \simeq 1 + \frac{2}{3} T_i/(T_i + T_e)$ and Eq. (26) reduces to a quartic that can be easily solved. We note that the instability condition $\Gamma_0^2 > \bar{\omega}_A^2$, which implies

$$3\Omega^2 > k_{\parallel}^2 c_s^2 \mathcal{I}_i \frac{v_A^2}{v_A^2 + c_s^2 \mathcal{I}_i} \sim (k_{\parallel} H)^2 \Omega^2 \frac{v_A^2}{v_A^2 + c_s^2}$$

does not require necessarily that $c_s^2 > v_A^2$ as $k_{\parallel}^2 H^2$ is considerably smaller than unity. Regimes with both low and finite values of c_s^2/v_A^2 are included in the theory leading to Eq. (26) and it is evident that $\bar{\omega}_{As}^2$ approaches ω_A^2 and becomes nearly independent of temperature as c_s^2/v_A^2 increases above unity.

(5) If we consider, for simplicity, the case where

$$-R\frac{d\Omega}{dR}\left(\frac{\bar{\nu}_{\parallel}}{\gamma} + \frac{\bar{\omega}_{As}^2}{\gamma^2}\right) > \frac{1}{R}\frac{d}{dR}(R^2\Omega)$$

then $\hat{v}_\phi \simeq -(R\Omega')\hat{v}_R/\gamma$ and we can estimate the velocity of angular momentum transport as

$$v_J = \frac{1}{\rho R v_\phi}\left(v_\phi R\langle\hat{\rho}\hat{v}_R\rangle + R\rho\langle\hat{v}_\phi\hat{v}_R\rangle - \frac{1}{4\pi}R\langle\hat{B}_\phi\hat{B}_R\rangle\right)$$
$$\simeq \frac{\langle\hat{v}_\phi\hat{v}_R\rangle}{v_J} \simeq -\frac{1}{\gamma}\langle|\hat{v}_R|^2\rangle\frac{1}{\Omega}\frac{d\Omega}{dR} > 0.$$

Thus for an order of magnitude evaluation of v_J we need to give one for the expected saturation level of the instability. Since the mode is associated with the finite value of the ion temperature, we may take $\hat{v}_R \sim v_{thi}$ and obtain $v_J \sim v_{thi}^2/\Omega R$ which is consistent with the Shakura–Sunyaev [8] estimate $D_J \sim v_J R \sim c_s H$, for $H/R \sim c_s/v_\phi$.

The analysis that we have described has been extended to include a vertical component, $\hat{B}_z$, of the field with $B_z/B_\phi \sim n_0/k_z R < 1$. The relevant instability has been shown to persist. When $B_z \sim B_\phi$, another interesting ordering to consider, besides that given by Refs. [6,7], involves $|k_z B_z + n^0 B_\phi/R| < |k_z B_z|$ at $R = R_0$, with $H^{-1} < k_z \sim n^0/R \sim |\partial\hat{v}_R/\partial R|/|\hat{v}_R|$ and $\gamma \sim n^0(d\Omega/dR)(R - R_0)$. The differential equation relevant to this case, for the radial profile of $\hat{v}_R$, has been derived.

Finally, we note that the theory of the mode driven by the combined effects of the finite plasma temperature and the gradient of $\Omega(R)$ presented here was not included in the classic analysis of Velikhov [9], of a perfectly conducting differentially rotating fluid in a magnetic field, as it assumed incompressibility ($\nabla \cdot \hat{v} = 0$) and did not consider the effects of finite sound velocity and anisotropic ion viscosity. These limitations exist also for the more recent analytical treatments of the same (Velikhov) problem [10,11]. The temperature-independent axisymmetric ($n^0 = 0$, $k_z \neq 0$, $k_R \neq 0$) mode [9] that can be found in the presence of a mostly vertical magnetic field ($B_z \gg B_\phi$) by taking $\nabla\cdot\hat{v} = 0$ was, in fact, proposed to be relevant to accretion disks first in Ref. [12] and analyzed in Ref. [13]. Compressibility terms were retained only as perturbations in the analysis of a rotating disk, with no toroidal field, that is $B = B_z$, and $c_s^2 < v_{Az}^2$, reported in Ref. [14] and involving non-axisymmetric flute-like perturbations. On the other hand, for the purely numerical analyses presented in Refs. [15,16] the set of ideal MHD equations that is given includes an adiabatic equation of state for the plasma but does not retain the effects of finite longitudinal ion viscosity. Compressibility has been considered in the extension of the analysis of the axisymmetric mode [9,12], with

$n^0 = 0$ and $k_z \neq 0$, given in Ref. [17].

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