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1 Accretion as a source of energy

1.1 Introduction

For the nineteenth century physicists, gravity was the only conceivable source of energy in celestial bodies, but gravity was inadequate to power the Sun for its known lifetime. In contrast, at the beginning of the twenty-first century it is to gravity that we look to power the most luminous objects in the Universe, for which the nuclear sources of the stars are wholly inadequate. The extraction of gravitational potential energy from material which accretes on to a gravitating body is now known to be the principal source of power in several types of close binary systems, and is widely believed to provide the power supply in active galactic nuclei and quasars. This increasing recognition of the importance of accretion has accompanied the dramatic expansion of observational techniques in astronomy, in particular the exploitation of the full range of the electromagnetic spectrum from the radio to X-rays and γ-rays. At the same time, the existence of compact objects has been placed beyond doubt by the discovery of the pulsars, and black holes have been given a sound theoretical status. Thus, the new role for gravity arises because accretion on to compact objects is a natural and powerful mechanism for producing high-energy radiation.

Some simple order-of-magnitude estimates will show how this works. For a body of mass M and radius R_* the gravitational potential energy released by the accretion of a mass m on to its surface is

$$\Delta E_{\rm acc} = GMm/R_* \tag{1.1}$$

where G is the gravitation constant. If the accreting body is a neutron star with radius $R_* \sim 10\,\rm km$, mass $M \sim M_\odot$, the solar mass, then the yield $\Delta E_{\rm acc}$ is about 10^{20} erg per accreted gram. We would expect this energy to be released eventually mainly in the form of electromagnetic radiation. For comparison, consider the energy that could be extracted from the mass m by nuclear fusion reactions. The maximum is obtained if, as is usually the case in astrophysics, the material is initially hydrogen, and the major contribution comes from the conversion, (or 'burning'), of hydrogen to helium. This yields an energy release

$$\Delta E_{\rm nuc} = 0.007mc^2 \tag{1.2}$$

where c is the speed of light, so we obtain about 6×10^{18} erg g^{-1} or about one twentieth of the accretion yield in this case.

It is clear from the form of equation (1.1) that the efficiency of accretion as an energy release mechanism is strongly dependent on the compactness of the accreting object: the larger the ratio M/R_*, the greater the efficiency. Thus, in treating accretion on to objects of stellar mass we shall certainly want to consider neutron stars ($R_* \sim 10$ km) and black holes with radii $R_* \sim 2GM/c^2 \sim 3(M/M_\odot)$ km (see Section 7.7). For white dwarfs with $M \sim M_\odot$, $R_* \sim 10^9$ cm, nuclear burning is more efficient than accretion by factors 25–50. However, it would be wrong to conclude that accretion on to white dwarfs is of no great importance for observations, since the argument takes no account of the timescale over which the nuclear and accretion processes act. In fact, when nuclear burning does occur on the surface of a white dwarf, it is likely that the reaction tends to 'run away' to produce an event of great brightness but short duration, a nova outburst, in which the available nuclear fuel is very rapidly exhausted. For almost all of its lifetime no nuclear burning occurs, and the white dwarf (may) derive its entire luminosity from accretion. Binary systems in which a white dwarf accretes from a close companion star are known as *cataclysmic variables* and are quite common in the Galaxy. Their importance derives partly from the fact that they provide probably the best opportunity to study the accretion process in isolation, since other sources of luminosity, in particular the companion star, are relatively unimportant.

For accretion on to a 'normal', less compact, star, such as the Sun, the accretion yield is smaller than the potential nuclear yield by a factor of several thousand. Even so, accretion on to such stars may be of observational importance. For example, a binary system containing an accreting main-sequence star has been proposed as a model for the so-called symbiotic stars.

For a fixed value of the compactness, M/R_*, the luminosity of an accreting system depends on the rate $\dot{M}$ at which matter is accreted. At high luminosities, the accretion rate may itself be controlled by the outward momentum transferred from the radiation to the accreting material by scattering and absorption. Under certain circumstances, this can lead to the existence of a maximum luminosity for a given mass, usually referred to as the Eddington luminosity, which we discuss next.

1.2 The Eddington limit

Consider a steady spherically symmetrical accretion; the limit so derived will be generally applicable as an order-of-magnitude estimate. We assume the accreting material to be mainly hydrogen and to be fully ionized. Under these circumstances, the radiation exerts a force mainly on the free electrons through Thomson scattering, since the scattering cross-section for protons is a factor $(m_e/m_p)^2$ smaller, where $m_e/m_p \cong 5 \times 10^{-4}$ is the ratio of the electron and proton masses. If S is the radiant energy flux (erg $\mathrm{s^{-1}cm^{-2}}$) and $\sigma_T = 6.7 \times 10^{-25}$ cm^2 is the Thomson cross-section, then the outward radial force on each electron equals the rate at which it absorbs momentum, $\sigma_T S/c$. If there is a substantial population of elements other than hydrogen, which have retained some bound electrons, the effective cross-section, resulting from the absorption of photons in spectral lines, can exceed σ_{T} considerably. The attractive electrostatic Coulomb force between the electrons and protons means that as they move out the electrons drag the protons with them. In effect, the radiation pushes out electron–proton pairs against the total gravitational force $GM(m_{\mathrm{p}} + m_{\mathrm{e}})/r^2 \cong GMm_{\mathrm{p}}/r^2$ acting on each pair at a radial distance r from the centre. If the luminosity of the accreting source is $L(\mathrm{erg\ s^{-1}})$, we have $S = L/4\pi r^2$ by spherical symmetry, so the net inward force on an electron–proton pair is

$$\left(GMm_{\mathrm{p}} - \frac{L\sigma_{\mathrm{T}}}{4\pi c}\right)\frac{1}{r^2}.$$

There is a limiting luminosity for which this expression vanishes, the Eddington limit,

$$L_{\mathrm{Edd}} = 4\pi GMm_{\mathrm{p}}c/\sigma_{\mathrm{T}} \qquad (1.3)$$

$$\cong 1.3 \times 10^{38}(M/M_{\odot})\,\mathrm{erg\ s^{-1}}. \qquad (1.4)$$

At greater luminosities the outward pressure of radiation would exceed the inward gravitational attraction and accretion would be halted. If all the luminosity of the source were derived from accretion this would switch off the source; if some, or all, of it were produced by other means, for example nuclear burning, then the outer layers of material would begin to be blown off and the source would not be steady. For stars with a given mass–luminosity relation this argument yields a maximum stable mass.

Since L_{Edd} will figure prominently later, it is worth recalling the assumptions made in deriving expressions (1.3,1.4). We assumed that the accretion flow was *steady* and *spherically symmetric*. A slight extension can be made here without difficulty: if the accretion occurs only over a fraction f of the surface of a star, but is otherwise dependent only on radial distance r, the corresponding limit on the accretion luminosity is fL_{Edd}. For a more complicated geometry, however, we cannot expect (1.3,1.4) to provide more than a crude estimate. Even more crucial was the restriction to *steady* flow. A dramatic illustration of this is provided by supernovae, in which L_{Edd} is exceeded by many orders of magnitude. Our other main assumptions were that the accreting material was largely hydrogen and that it was fully ionized. The former is almost always a good approximation, but even a small admixture of heavy elements can invalidate the latter. Almost complete ionization is likely to be justified however in the very common case where the accreting object produces much of its luminosity in the form of X-rays, because the abundant ions can usually be kept fully stripped of electrons by a very small fraction of the X-ray luminosity. Despite these caveats, the Eddington limit is of great practical importance, in particular because certain types of system show a tendency to behave as 'standard candles' in the sense that their typical luminosities are close to their Eddington limits.

For accretion powered objects the Eddington limit implies a limit on the steady accretion rate, $\dot{M}(\mathrm{g\ s^{-1}})$. If all the kinetic energy of infalling matter is given up to radiation at the stellar surface, R_*, then from (1.1) the *accretion luminosity* is

$$L_{\mathrm{acc}} = GM\dot{M}/R_*. \qquad (1.5)$$

It is useful to re-express (1.5) in terms of typical orders of magnitude: writing the accretion rate as $\dot{M} = 10^{16}\dot{M}_{16}$ g s^{-1} we have

$$L_{\mathrm{acc}} = 1.3 \times 10^{33}\dot{M}_{16}(M/M_{\odot})(10^{9}\,\mathrm{cm}/R_{*})\,\mathrm{erg\ s}^{-1} \quad (1.6)$$

$$= 1.3 \times 10^{36}\dot{M}_{16}(M/M_{\odot})(10\,\mathrm{km}/R_{*})\,\mathrm{erg\ s}^{-1}. \quad (1.7)$$

The reason for rewriting (1.5) in this way is that the quantities $(M/M_{\odot})$, $(10^{9}\,\mathrm{cm}/R_{*})$ and $(M/M_{\odot})$, $(10\ \mathrm{km}/R_{*})$ are of order unity for white dwarfs and neutron stars respectively. Since 10^{16} g s^{-1}($\sim 1.5 \times 10^{-10} M_{\odot}$ yr^{-1}) is a typical order of magnitude for accretion rates in close binary systems involving these types of star, we have $\dot{M}_{16} \sim 1$ in (1.6,1.7), and the luminosities 10^{33} erg s^{-1}, 10^{36} erg s^{-1} represent values commonly found in such systems. Further, by comparison with (1.4) it is immediately seen that for steady accretion $\dot{M}_{16}$ is limited by the values $\sim 10^{5}$ and 10^{2} respectively. Thus, accretion rates must be less than about 10^{21} g s^{-1} and 10^{18} g s^{-1} in the two types of system if the assumptions involved in deriving the Eddington limit are valid.

For the case of accretion on to a black hole it is far from clear that (1.5) holds. Since the radius does not refer to a hard surface but only to a region into which matter can fall and from which it cannot escape, much of the accretion energy could disappear into the hole and simply add to its mass, rather than be radiated. The uncertainty in this case can be parametrized by the introduction of a dimensionless quantity η, the *efficiency*, on the right hand side of (1.5):

$$L_{\mathrm{acc}} = 2\eta G M\dot{M}/R_{*} \quad (1.8)$$

$$= \eta \dot{M} c^{2} \quad (1.9)$$

where we have used $R_{*} = 2GM/c^{2}$ for the black hole radius. Equation (1.9) shows that η measures how efficiently the rest mass energy, c^{2} per unit mass, of the accreted material is converted into radiation. Comparing (1.9) with (1.2) we see that $\eta = 0.007$ for the burning of hydrogen to helium. If the material accreting on to a black hole could be lowered into the hole infinitesimally slowly - scarcely a practical proposition - all of the rest mass energy could, in principle, be extracted and we should have $\eta = 1$. As we shall see in Chapter 7 the estimation of realistic values for η is an important problem. A reasonable guess would appear to be $\eta \sim 0.1$, comparable to the value $\eta \sim 0.15$ obtained from (1.8) for a solar mass neutron star. Thus, despite its extra compactness, a stellar mass black hole may be no more efficient in the conversion of gravitational potential energy to radiation than a neutron star of similar mass.

As a final illustration here of the use of the Eddington limit we consider the nuclei of active galaxies and the closely related quasars. These are probably the least understood class of object for which accretion is thought to be the ultimate source of energy. The main reason for this belief comes from the large luminosities involved: these systems may reach 10^{47} erg s^{-1}, or more, varying by factors of order 2 on timescales of weeks, or less. With the nuclear burning efficiency of only $\eta = 0.007$, the rate at which mass is processed in the source could exceed 250 $M_{\odot}$ yr^{-1}. This is a rather severe requirement and it is clearly greatly reduced if accretion with an efficiency $\eta \sim 0.1$ is postulated instead. The accretion rate required is of order 20 $M_\odot$ yr^{-1}, or less, and rates approaching this might plausibly be provided by a number of the mechanisms considered in Chapter 7. If these systems are assumed to radiate at less than the Eddington limit, then accreting masses exceeding about $10^9 M_\odot$ are required. White dwarfs are subject to upper limits on their masses of 1.4 $M_\odot$ and neutron stars cannot exceed about 3 $M_\odot$ thus, only massive black holes are plausible candidates for accreting objects in active galactic nuclei.

1.3 The emitted spectrum

We can now make some order-of-magnitude estimates of the spectral range of the emission from compact accreting objects, and, conversely, suggest what type of compact object may be responsible for various observed behaviour. We can characterize the continuum spectrum of the emitted radiation by a temperature $T_{\rm rad}$ defined such that the energy of a typical photon, $h\bar{\nu}$, is of order $kT_{\rm rad}$, $T_{\rm rad} = h\bar{\nu}/k$, where we do not need to make the choice of $\bar{\nu}$ precise. For an accretion luminosity $L_{\rm acc}$ from a source of radius R, we define a blackbody temperature $T_{\rm b}$ as the temperature the source would have if it were to radiate the given power as a blackbody spectrum:

$$T_{\rm b} = (L_{\rm acc}/4\pi R_*^2 \sigma)^{1/4}. \qquad (1.10)$$

Finally, we define a temperature $T_{\rm th}$ that the accreted material would reach if its gravitational potential energy were turned entirely into thermal energy. For each proton–electron pair accreted, the potential energy released is $GM(m_{\rm p} + m_{\rm e})/R_* \cong GMm_{\rm p}/R_*$, and the thermal energy is $2 \times \frac{3}{2}kT$; therefore

$$T_{\rm th} \; = \; GMm_{\rm p}/3kR_*. \qquad (1.11)$$

Note that some authors use the related concept of the virial temperature, $T_{\rm vir} = T_{\rm th}/2$, for a system in mechanical and thermal equilibrium. If the accretion flow is optically thick, the radiation reaches thermal equilibnum with the accreted material before leaking out to the observer and $T_{\rm rad} \sim T_{\rm b}$. On the other hand, if the accretion energy is converted directly into radiation which escapes without further interaction (i.e. the intervening material is optically thin), we have $T_{\rm rad} \sim T_{\rm th}$. This occurs in certain types of shock wave that may be produced in some accretion flows and we shall see in Chapter 3 that (1.11) provides an estimate of the shock temperature for such flows. In general, the radiation temperature may be expected to lie between the thermal and blackbody temperatures, and, since the system cannot radiate a given flux at less than the blackbody temperature, we have

$$T_{\rm b} \lesssim T_{\rm rad} \lesssim T_{\rm th}.$$

Of course, these estimates assume that the radiating material can be characterized by a single temperature. They need not apply, for example, to a non-Maxwellian distribution of electrons radiating in a fixed magnetic field, such as we shall meet in Chapter 9.

Let us apply the limits (1.10), (1.11) to the case of a solar mass neutron star. The upper limit (1.11) gives $T_{\rm th} \sim 5.5 \times 10^{11}$ K, or, in terms of energies, $kT_{\rm th} \sim$50 MeV. To evaluate the lower limit, $T_{\rm b}$, from (1.10), we need an idea of the accretion luminosity, $L_{\rm acc}$; but $T_{\rm b}$ is, in fact, very insensitive to the assumed value of $L_{\rm acc}$, since it is proportional to the fourth root. Thus we can take $L_{\rm acc} \sim L_{\rm Edd} \sim 10^{38}$ erg s^{-1} for a rough estimate; if, instead, we were to take a typical value $\sim 10^{36}$ erg s^{-1} (equation (1.10)) this would change $T_{\rm b}$ only by a factor of ~ 3. We obtain $T_{\rm b} \sim 10^7$K or $kT_{\rm b} \sim$1 keV, and so we expect photon energies in the range

$$1 \text{ keV} \lesssim h\bar{\nu} \lesssim 50 \text{ MeV}$$

as a result of accretion on to neutron stars. Similar results would hold for stellar mass black holes. Thus we can expect the most luminous accreting neutron star and black hole binary systems to appear as medium to hard X-ray emitters and possibly as γ-ray sources. There is no difficulty in identifying this class of object with the luminous galactic X-ray sources discovered by the first satellite X-ray experiments, and added to by subsequent investigations.

For accreting white dwarfs it is probably more realistic to take $L_{\rm acc} \sim 10^{33}$ erg s^{-1} in estimating $T_{\rm b}$ (cf. (1.6)). With $M = M_\odot$, $R_* = 5 \times 10^8$ cm, we obtain

$$6 \text{ eV} \lesssim h\bar{\nu} \lesssim 100 \text{ keV}.$$

Consequently, accreting white dwarfs should be optical, ultraviolet and possibly X-ray sources. This fits in neatly with our knowledge of cataclysmic variable stars, which have been found to have strong ultraviolet continua by the Copernicus and IUE satellite experiments. In addition, some of them are now known to emit a small fraction of their luminosity as thermal X-ray sources. We shall see that in many ways cataclysmic variables are particularly useful in providing observational tests of theories of accretion.

1.4 Accretion theory and observation

So far we have discussed the amount of energy that might be expected by the accretion process, but we have made no attempt to describe in detail the flow of accreting matter. A hint that the dynamics of this flow may not be straightforward is provided by the existence of the Eddington limit, which shows that, at least for high accretion rates, forces other than gravity can be important. In addition, it will emerge later that, certainly in many cases and probably in most, the accreting matter possesses considerable angular momentum per unit mass which, in realistic models, it has to lose in order to be accreted at all. Furthermore, we need a detailed description of the accretion flow if we are to explain the observed spectral distribution of the radiation produced: crudely speaking, in the language of Section 1.3, we want to know whether $T_{\rm rad}$ is closer to $T_{\rm b}$ or $T_{\rm th}$.

The two main tools we shall use in this study are the equations of gas dynamics and the physics of plasmas. We shall give a brief introduction to gas dynamics in Sections 2.1–2.4 of the next chapter, and treat some aspects of plasma physics in Chapter 3. In addition, the elements of the theory of radiative transfer are summarized in the Appendix. The reader who is already familiar with these subjects can omit these parts of the text. The rest of the book divides into three somewhat distinct parts. First, in Chapters 4 to 6 we consider accretion by stellar mass objects in binary systems. In these cases, we often find that observations provide fairly direct evidence for the nature of the systems. For example, there is sometimes direct evidence for the importance of angular momentum and the existence of accretion discs. This contrasts greatly with the subsequent discussion of active galactic nuclei in Chapters 7 to 10. Here, the accretion theory arises at the end of a sequence of plausible, but not unproblematic, inductions. Furthermore, there appears to be no absolutely compelling evidence for, or against, the existence of accretion *discs* in these systems. Thus, whereas we normally use the observations of stellar systems to test the theory, for active nuclei we use the theory, to some extent, to illustrate the observations. This is particularly apparent in the final part of the book, where, in Chapters 9 and 10, we discuss two quite different models for powering an active nucleus by an accretion disc around a supermassive black hole. Finally, in Chapter 11 we review all possible accretion flows, most of which have already been studied in earlier chapters, classifying them according to which physical effects dominate their properties and behaviour. We also describe in some detail recent advances in our understanding of accretion flows, with particular emphasis on the class of advection dominated accretion flows or ADAFs.

2 Gas dynamics

2.1 Introduction

All accreting matter, like most of the material in the Universe, is in a gaseous form. This means that the constituent particles, usually free electrons and various species of ions, interact directly only by *collisions*, rather than by more complicated short-range forces. In fact, these collisions involve the electrostatic interaction of the particles and will be considered in more detail in Chapter 3. On average, a gas particle will travel a certain distance, the *mean free path*, λ, before changing its state of motion by colliding with another particle. If the gas is approximately uniform over lengthscales exceeding a few mean free paths, the effect of all these collisions is to randomize the particle velocities about some mean velocity, the velocity of the gas, $\mathbf{v}$. Viewed in a reference frame moving with velocity $\mathbf{v}$, the particles have a Maxwell–Boltzmann distribution of velocities, and can be described by a temperature T. Provided we are interested only in lengthscales $L \gg \lambda$ we can regard the gas as a continuous *fluid*, having velocity $\mathbf{v}$, temperature T and density ρ defined at each point. We then study the behaviour of these and other fluid variables as functions of position and time by imposing the laws of conservation of mass, momentum and energy. This is the subject of gas dynamics. If we wish to look more closely at the gas, we have to consider the particle interactions in more detail; this is the domain of plasma physics, or, more strictly, plasma kinetic theory, about which we shall have something to say in Chapter 3. Note that the equations of gas dynamics may not always be applicable. For example, these equations may themselves predict large changes in gas properties over lengthscales comparable with λ; under these circumstances the fluid approximation is invalid and we must use the deeper but more complicated approach of the plasma kinetic theory.

2.2 The equations of gas dynamics

Here we shall write down the three conservation laws of gas dynamics, which, together with an equation of state and appropriate boundary conditions, describe any gas dynamical flow. We shall not give the derivations, which can be found in many books, for example Landau & Lifshitz, 1959, but merely point out the significance of the various terms.

Given a gas with, as before, a velocity field $\mathbf{v}$, density ρ and temperature T, all defined as functions of position $\mathbf{r}$ and time t, conservation of mass is ensured by the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{2.1}$$

Because of the thermal motion of its particles the gas has a pressure P at each point. An equation of state relates this pressure to the density and temperature. Astrophysical gases, other than the degenerate gases in white dwarfs and neutron stars and the cores of 'normal' stars, have as equation of state the *perfect gas law*:

$$P = \rho kT/\mu m_{\mathrm{H}}. \tag{2.2}$$

Here $m_{\mathrm{H}} \sim m_{\mathrm{p}}$ is the mass of the hydrogen atom and μ is the mean molecular weight, which is the mean mass per particle of gas measured in units of m_{H} or, equivalently, the inverse of the number of particles in a mass m_{H} of the gas. Hence, $\mu = 1$ for neutral hydrogen, $\frac{1}{2}$ for fully ionized hydrogen, and something in between for a mixture of gases with cosmic abundances, depending on the ionization state.

Gradients in the pressure in the gas imply forces since momentum is thereby transferred. Other, as yet unspecified, forces acting on the gases are represented by the force density, the force per unit volume, $\mathbf{f}$. Conservation of momentum for each gas element then gives the *Euler equation*:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \mathbf{f}. \tag{2.3}$$

This has the form (mass density) $\times$ (acceleration) $=$ (force density) and is, in fact, simply an expression of Newton's second law for a continuous fluid. The term $\rho \mathbf{v} \cdot \nabla \mathbf{v}$ on the left hand side of (2.3) represents the convection of momentum through the fluid by velocity gradients. The presence of this term means that *steady* motions are possible in which the time derivatives of the fluid variables vanish, but $\mathbf{v}$ is non-zero. An example of an external force is gravity: in this case $\mathbf{f} = -\rho \mathbf{g}$, where $\mathbf{g}$ is the local acceleration due to gravity. Another example would be the force due to an external magnetic field. Further important contributions to $\mathbf{f}$ can come from *viscosity*, which is the transfer of momentum along velocity gradients by random motions of the gas, especially turbulence and thermal motions. The inclusion of viscosity usually considerably complicates the momentum balance equation, so it is fortunate that in many cases it may be neglected. We anticipate some later results by stating that viscous effects are chiefly important in flows which show either large shearing motions or steep velocity gradients.

The third, and most complicated, conservation law is that of *energy*. An element of gas has two forms of energy: an amount $\frac{1}{2}\rho v^2$ of kinetic energy per unit volume, and internal or thermal energy $\rho\varepsilon$ per unit volume, where ε, the internal energy per unit mass, depends on the temperature T of the gas. According to the equipartition theorem of elementary kinetic theory, each degree of freedom of each gas particle is assigned a mean energy $\frac{1}{2}kT$. For a monatomic gas the only degrees of freedom are the three orthogonal directions of translational motion and

$$\varepsilon = \frac{3}{2}kT/\mu m_{\rm H}. \tag{2.4}$$

Molecular gases have additional internal degrees of freedom of vibration or rotation. In reality, cosmic gases are not quite monatomic and the effective number of degrees of freedom is not quite three; but in practice (2.4) is usually a good approximation.

The energy equation for the gas is

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}\rho v^2 + \rho\varepsilon\Big) + \nabla\cdot\Big[\Big(\frac{1}{2}\rho v^2 + \rho\varepsilon + P\Big)\mathbf{v}\Big] = \mathbf{f}\cdot\mathbf{v} - \nabla\cdot\mathbf{F}_{\rm rad} - \nabla\cdot\mathbf{q}. \tag{2.5}$$

The left hand side shows a family resemblance to the continuity equation (2.1), with the expected difference that the conserved quantity ρ is replaced by $(\frac{1}{2}\rho v^2 + \rho\varepsilon)$. The last term in the square brackets represents the so-called pressure work. Two new quantities appear on the right hand side: first, the radiative flux vector $\mathbf{F}_{\rm rad}=\int d\nu \int d\Omega \mathbf{n} I_\nu(\mathbf{n},\mathbf{r})$ where I_ν is the specific intensity of radiation at the point $\mathbf{r}$ in the direction $\mathbf{n}$ and the integrals are over frequency ν and solid angle $\boldsymbol{\Omega}$ (see the Appendix). The term $-\nabla\cdot\mathbf{F}_{\rm rad}$ gives the rate at which radiant energy is being lost by emission, or gained by absorption, by unit volume of the gas. In general, the specific intensity I_ν is itself governed by a further equation, the conservation of energy equation for the radiation field. Fortunately, we can often approximate the radiative losses quite simply. For example, let j_ν (erg s^{-1} cm^{-3} sr^{-1}) be the rate of emission of radiation per unit volume per unit solid angle; j_ν is the emissivity of the gas and is usually given as a function of ρ, T (and ν), but might also depend on external magnetic fields or the radiation field itself (examples are given in the Appendix). If the gas is optically thin, so that radiation escapes freely once produced and the gas itself reabsorbs very little, the volume loss is just $-\nabla\cdot\mathbf{F}_{\rm rad} = -4\pi\int j_\nu {\rm d}\nu$. For a hot gas radiating thermal bremsstrahlung (or 'free–free radiation'), this has the approximate form constant $\times\rho^2 T^{1/2}$. At the opposite extreme, if the gas is very optically thick, as in the interior of a star, then $\mathbf{F}_{\rm rad}$ approximates the blackbody flux and $-\nabla\cdot\mathbf{F}_{\rm rad}$ is given by the Rosseland approximation $\mathbf{F}_{\rm rad} = (16\sigma/3\kappa_{\rm R}\rho)T^3\nabla T$ where $\kappa_{\rm R}$ is a weighted average over frequency of the opacity. This Rosseland approximation is discussed in any book on stellar structure (see the Appendix).

The second new quantity in the energy equation (2.5) is the conductive flux of heat, $\mathbf{q}$. This measures the rate at which random motions, chiefly those of electrons, transport thermal energy in the gas and thus act to smooth out temperature differences. Standard kinetic theory (cf. equation (3.42)), shows that for an ionized gas obeying the requirement $\lambda \ll T/|\nabla T|$

$$\mathbf{q} \cong -10^{-6}T^{5/2}\nabla T \text{ erg s}^{-1}\text{ cm}^{-2}. \tag{2.6}$$

(See Section 3.6 for a discussion of transport processes.) Obviously the term $-\nabla\cdot\mathbf{q}$ raises the order of differentiation of T in the energy equation, so it is again fortunate that, in many cases, temperature gradients are small enough that this term can be omitted from (2.5).

The system of equations (2.1)–(2.6), supplemented, if necessary, by the radiative transfer equation and the specification of **f**, give, in principle, a complete description of the behaviour of a gas under appropriate boundary conditions. Of course, in practice one cannot hope to solve the equations in the fearsome generality in which they have been presented here, and all known solutions are either highly specialized or approximate in some sense. To show how useful information can be extracted from these equations, we shall discuss a number of simple solutions. Some of these will be of considerable importance later.

2.3 Steady adiabatic flows; isothermal flows

Let us consider first *steady* flows, for which time derivatives are put equal to zero, and let us specialize to the case in which there are no losses through radiation and no thermal conduction.

Our three conservation laws of mass, momentum and energy then become

$$\nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.7}$$

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \mathbf{f}, \tag{2.8}$$

$$\nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho\varepsilon + P\right)\mathbf{v}\right] = \mathbf{f} \cdot \mathbf{v}. \tag{2.9}$$

Substituting the first of these equations in the third implies

$$\rho\mathbf{v} \cdot \nabla\left(\frac{1}{2}v^2 + \varepsilon + P/\rho\right) = \mathbf{f} \cdot \mathbf{v}, \tag{2.10}$$

while (2.8), the Euler equation, shows that $\mathbf{f}\cdot\mathbf{v} = \rho\mathbf{v}(\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{v}\cdot\nabla P = \rho\mathbf{v}\cdot(\frac{1}{2}v^2) + \mathbf{v}\cdot\nabla P$; hence, eliminating $\mathbf{f}\cdot\mathbf{v}$ from (2.10) we get

$$\rho\mathbf{v} \cdot \nabla(\varepsilon + P/\rho) = \mathbf{v} \cdot \nabla P,$$

or, expanding $\nabla(P/\rho)$ and rearranging,

$$\mathbf{v} \cdot [\nabla\varepsilon + P\nabla(1/\rho)] = 0.$$

By the definition of the gradient operator, this means that, if we travel a small distance along a streamline of the gas, i.e. if we follow the velocity **v**, the increments $\mathrm{d}\varepsilon$ and $\mathrm{d}(1/\rho)$ in ε and $1/\rho$ must be related by

$$\mathrm{d}\varepsilon + P\mathrm{d}(1/\rho) = 0.$$

But from the expression for the internal energy (2.4) and the perfect gas law (2.2) this requires that

$$\frac{3}{2}\mathrm{d}T + \rho T\mathrm{d}(1/\rho) = 0,$$

which is equivalent to

$$\rho^{-1}T^{3/2} = \text{constant}$$

or

$$P\rho^{-5/3} = \text{constant} \tag{2.11}$$

using (2.2).

Equation (2.11) describes the so-called *adiabatic* flows. Although we have demonstrated only that the combination $P\rho^{-5/3}$ is constant along a given streamline, in many cases it is assumed that this constant is the same for each streamline, i.e. it is the same throughout the gas. This condition is equivalent to setting the entropy of the gas constant. The resulting flows are called *isentropic*. Note that adiabatic and isentropic are often used synonymously in the literature.

In a sense, our derivation of the adiabatic law (2.11) is 'back-to-front', since thermodynamic laws go into the construction of the energy equation (2.5). It is presented here to demonstrate the consistency of (2.5) with expectations from thermodynamics. If our gas were not monatomic, so that the numerical coefficient in (2.4) differed from $\frac{3}{2}$, we would obtain a result like (2.11), but with a different exponent for ρ:

$$P\rho^{-\gamma} = \text{constant}. \tag{2.12}$$

In this form γ is known as the *adiabatic index*, or the *ratio of specific heats*. A further important special type of flow results from the assumption that the gas temperature T is constant throughout the region of interest. This is called isothermal flow, and is obviously equivalent to postulating some unspecified physical process to keep T constant. This, in turn, means that the energy equation (2.5) is replaced in our system describing the gas by the relation $T =$ constant. Formally, this latter requirement can be written, using the perfect gas law (2.2), as

$$P\rho^{-1} = \text{constant},$$

which has the form of (2.12) with $\gamma = 1$.

2.4 Sound waves

An obvious class of solution to our gas equations is that corresponding to *hydrostatic equilibrium*. In this case, in addition to the restriction to steady flow, and the absence of losses assumed in Section 2.3 above, we take $\mathbf{v} = 0$. Then the only equation remaining to be satisfied is (2.8), which reduces to

$$\nabla P = \mathbf{f},$$

together with an explicit expression for $\mathbf{f}$, and the perfect gas law (2.2). Solutions of this type are, for example, appropriate to stellar, or planetary, atmospheres in radiative equilibrium.

Let us assume that we have such a solution, in which P and ρ are certain functions of position, P_0 and ρ_0, and consider small perturbations about it. We set

$$P = P_0 + P', \quad \rho = \rho_0 + \rho', \quad \mathbf{v} = \mathbf{v}'$$

where all the primed quantities are assumed small, so that we can neglect second and higher order products of them. In place of the energy equation (2.5), we assume that the perturbations are adiabatic, or isothermal: in reality, either of these cases can occur. Thus

$$P + P' = K(\rho + \rho')^{\gamma}, \qquad K = \text{constant} \tag{2.13}$$

with $\gamma = \frac{5}{3}$(adiabatic) or $\gamma = 1$ (isothermal). Linearizing the continuity equation (2.1) and the Euler equation (2.3), and using the fact that $\nabla P_0 = \mathbf{f}$, we get

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0, \tag{2.14}$$

$$\frac{\partial \mathbf{v}'}{\partial t} + \frac{1}{\rho_0} \nabla P' = 0. \tag{2.15}$$

From (2.13) P is purely a function of ρ, so $\nabla P' = (\mathrm{d}P/\mathrm{d}\rho)_0 \nabla \rho'$ to first order, where the subscript zero implies that the derivative is to be evaluated for the equilibrium solution, i.e. $(\mathrm{d}P/\mathrm{d}\rho)_0 = \mathrm{d}P_0/\mathrm{d}\rho_0$. Thus, (2.15) becomes

$$\frac{\partial \mathbf{v}'}{\partial t} + \frac{1}{\rho_0} \left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_0 \nabla \rho' = 0. \tag{2.16}$$

Eliminating $\mathbf{v}'$ from (2.16) and (2.14) by operating with $\nabla\cdot$ and $\partial/\partial t$ respectively and then subtracting, gives

$$\frac{\partial^2 \rho'}{\partial t^2} = c_{\mathrm{s}}^2 \nabla^2 \rho', \tag{2.17}$$

where we have defined

$$c_{\mathrm{s}} = \left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_0^{1/2}. \tag{2.18}$$

Equation (2.17) will be recognized as the *wave equation*, with the wave speed c_{s}. It is easy, now, to show that the other variables P', $\mathbf{v}'$ obey similar equations; this implies that small perturbations about hydrostatic equilibrium propagate through the gas as sound waves with speed c_{s}. From (2.13), (2.18) we see that the *sound speed* c_{s} can have two values:

$$\text{adiabatic}: \ c_{\mathrm{s}}^{\mathrm{ad}} = \left(\frac{5P}{3\rho}\right)^{1/2} = \left(\frac{5kT}{3\mu m_{\mathrm{H}}}\right)^{1/2} \propto \rho^{1/3}, \tag{2.19}$$

$$\text{isothermal}: \ c_{\mathrm{s}}^{\mathrm{iso}} = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{kT}{\mu m_{\mathrm{H}}}\right)^{1/2}. \tag{2.20}$$

The sound speeds $c_{\mathrm{s}}^{\mathrm{ad}}$, $c_{\mathrm{s}}^{\mathrm{iso}}$, are basic quantities which can be defined locally at any point of a gas. Note first that both $c_{\mathrm{s}}^{\mathrm{ad}}$ and $c_{\mathrm{s}}^{\mathrm{iso}}$ are of the order of the mean thermal speed of the ions of the gas, cf. equation (2.4). Numerically,

$$c_{\mathrm{s}} \cong 10(T/10^4 K)^{1/2} \ \mathrm{km\ s^{-1}} \tag{2.21}$$

where c_s stands for either sound speed.

Since c_s is the speed at which pressure disturbances travel through the gas, it limits the rapidity with which the gas can respond to pressure changes. For example, if the pressure in one part of a region of the gas of characteristic size L is suddenly changed, the other parts of the region cannot respond to this change until a time of order L/c_s, the sound crossing time, has elapsed. Conversely, if the pressure in one part of the region is changed on a timescale much longer than L/c_s the gas has ample time to respond by sending sound signals throughout the region, so the pressure gradient will remain small. Thus, if we consider *supersonic flow*, where the gas moves with $|\mathbf{v}| > c_s$, then the gas cannot respond on the flow time $L/|\mathbf{v}| < L/c_s$, so pressure gradients have little effect on the flow. At the other extreme, for *subsonic flow* with $|\mathbf{v}| < c_s$, the gas can adjust in less than the flow time, so to a first approximation the gas behaves as if in hydrostatic equilibrium.

These properties can be inferred directly from an order-of-magnitude analysis of the terms in the Euler equation (2.3). For example, for supersonic flow we have

$$\frac{|\rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v}|}{|\nabla P|} \sim \frac{v^2/L}{P/\rho L} \sim \frac{v^2}{c_s^2} > 1$$

and pressure gradients can be neglected in a first approximation. A very important property of the sound speed is its dependence on the gas density (2.19). This means that regions of higher than average density have higher than average sound speeds, a fact which gives rise to the possibility of *shock waves*. In a shock the fluid quantities change on lengthscales of the order of the mean free path λ and this is represented as a *discontinuity* in the fluid. Shock waves are important in physics and astrophysics and we shall return to them in Section 3.8.

2.5 Steady, spherically symmetric accretion

Let us now attack a real accretion problem and show how all of the apparatus we have developed in Sections 2.1–2.4 can be put to use. We consider a star of mass M accreting spherically symmetrically from a large gas cloud. This would be a reasonable approximation to the real situation of an isolated star accreting from the interstellar medium, provided that the angular momentum, magnetic field strength and bulk motion of the interstellar gas with respect to the star could be neglected. For other types of accretion flows, such as those in close binary systems and models of active galactic nuclei, spherical symmetry is rarely a good approximation, as we shall see. Nonetheless, the spherical accretion problem is of very great significance for the theory, as it introduces some important concepts which have much wider validity. Furthermore, it is possible to give a fairly exact treatment, allowing us to gain insight into more complicated problems. The problem of accretion of gas by a star in relative motion with respect to the gas was first considered by Hoyle and Lyttleton (1939) and later by Bondi and Hoyle (1944). The spherically symmetric case in a form similar to what is presented here arises when the accreting star is at rest with respect to the gas. This case was first studied by Bondi (1952), and is referred to as Bondi accretion.

Let us ask what we might hope to discover by analysing this problem. First, we should expect to be able to predict the steady accretion rate $\dot{M}$ (g s^{-1}) on to our star, given the ambient conditions (the density $\rho(\infty)$ and the temperature $T(\infty)$) in the parts of the gas cloud far from the star and some boundary conditions at its surface. Second, we might hope to learn how big a region of the gas cloud is influenced by the presence of the star. These questions can be answered in a natural and physically appealing way. In addition, we shall obtain an understanding of the relation between the gas velocity and the local sound speed which can be carried over quite generally to more complicated accretion flows.

To treat the problem mathematically we take spherical polar coordinates (r, θ, ϕ) with origin at the centre of the star. The fluid variables are independent of θ and ϕ by spherical symmetry, and the gas velocity has only a radial component $v_{\mathrm{r}} = v$. We take this to be negative, since we want to consider infall of material; $v > 0$ would correspond to a stellar wind. For steady flow, the continuity equation (2.1) reduces to

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho v\right) = 0 \tag{2.22}$$

using the standard expression for the divergence of a vector in spherical polar coordinates. This integrates to $r^2\rho v =$ constant. Since $\rho(-v)$ is the inward flux of material, the constant here must be related to the (constant) accretion rate $\dot{M}$; the relation is

$$4\pi r^2\rho(-v) = \dot{M}. \tag{2.23}$$

In the Euler equation the only contribution to the external force, **f**, is from gravity, and this has only a radial component

$$f_{\mathrm{r}} = -GM\rho/r^2$$

so that (2.3) becomes

$$v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0. \tag{2.24}$$

We replace the energy equation (2.5) by the polytropic relation (2.12):

$$P = K\rho^{\gamma},\ K = \text{constant}. \tag{2.25}$$

This allows us to treat both approximately adiabatic ($\gamma \cong \frac{5}{3}$) and isothermal ($\gamma \cong 1$) accretion simultaneously. After the solution has been found, the adiabatic or isothermal assumption should be justified by consideration of the particular radiative cooling and heating of the gas. For example, the adiabatic approximation will be valid if the timescales for significant heating and cooling of the gas are long compared with the time taken for an element of the gas to fall in. In reality, neither extreme is quite satisfied, so we expect $1 < \gamma < \frac{5}{3}$. In fact, the treatment we shall give is valid for $1 < \gamma < \frac{5}{3}$: the extreme values require special consideration (see e.g. (2.33), (2.34)). The interested reader may consult the article by Holzer & Axford (1970).

Finally, we can use the perfect gas law (2.2) to give the temperature

$$T = \mu m_{\rm H} P/\rho k \tag{2.26}$$

where $P(r)$, $\rho(r)$ have been found.

The problem therefore reduces to that of integrating (2.24) with the help of (2.25) and (2.23) and then identifying the unique solution corresponding to our accretion problem. We shall integrate (2.24) shortly, but it is instructive to see how much information can be extracted without explicit integration, since this technique is very useful in other cases when analytic integration is not possible, or not straightforward. We write first

$$\frac{{\rm d}P}{{\rm d}r} = \frac{{\rm d}P}{{\rm d}\rho}\frac{{\rm d}\rho}{{\rm d}r} = c_{\rm s}^2 \frac{{\rm d}\rho}{{\rm d}r}.$$

Hence, the term $(1/\rho)({\rm d}P/{\rm d}r)$ in the Euler equation (2.24) is $(c_{\rm s}^2/\rho)({\rm d}\rho/{\rm d}r)$. But, from the continuity equation (2.22),

$$\frac{1}{\rho}\frac{{\rm d}\rho}{{\rm d}r} = -\frac{1}{vr^2}\frac{\rm d}{{\rm d}r}\left(vr^2\right).$$

Therefore, (2.24) becomes

$$v\frac{{\rm d}v}{{\rm d}r} - \frac{c_{\rm s}^2}{vr^2}\frac{{\rm d}v}{{\rm d}r} + \frac{GM}{r^2} = 0$$

which, after a little rearrangement, gives

$$\frac{1}{2}\left(1 - \frac{c_{\rm s}^2}{v^2}\right)\frac{\rm d}{{\rm d}r}\left(v^2\right) = -\frac{GM}{r^2}\left[1 - \left(\frac{2c_{\rm s}^2 r}{GM}\right)\right]. \tag{2.27}$$

At first sight, we appear to have made things worse by these manipulations, since $c_{\rm s}^2$ is, in general, a function of r. However, the physical interpretation of $c_{\rm s}$ as the sound speed, plus the structure of equation (2.27), in which factors on either side can in principle vanish, allow us to sort the possible solutions of (2.27) into distinct classes and to pick out the unique one corresponding to our problem. First, we note that at large distances from the star the factor $[1 - (2c_{\rm s}^2 r/GM)]$ on the right hand side must be negative, since $c_{\rm s}^2$ approaches some finite asymptotic value $c_{\rm s}^2(\infty)$ related to the gas temperature far from the star, while r increases without limit. This means that for large r the right hand side of (2.27) is positive. On the left hand side, the factor ${\rm d}(v^2)/{\rm d}r$ must be negative, since we want the gas far from the star to be at rest, accelerating as it approaches the star with r decreasing. These two requirements (${\rm d}(v^2)/{\rm d}r < 0$, the r.h.s. of (2.27)$> 0$) are compatible only if at large r the gas flow is *subsonic*, i.e.

$$v^2 < c_{\rm s}^2 \quad \text{for large } r. \tag{2.28}$$

This is, of course, a very reasonable result, as the gas will have a non-zero temperature and hence a non-zero sound speed far from the star. As the gas approaches the star, r decreases and the factor $[1 - (2c_{\rm s}^2 r/GM)]$ must tend to increase. It must eventually reach zero, unless some way can be found of increasing c_s^2 sufficiently by heating the gas. This is very unlikely, since the factor reaches zero at a radius given by

$$r_s = \frac{GM}{2c_s^2(r_s)} \cong 7.5 \times 10^{13} \left(\frac{T(r_s)}{10^4\,\mathrm{K}}\right)^{-1} \left(\frac{M}{M_\odot}\right) \mathrm{cm} \tag{2.29}$$

where we have used (2.21) to introduce the temperature. The order of magnitude $r_s \cong 7.5 \times 10^{13}$ cm in (2.29) is so much larger than the radius, R_*, of any compact object ($R_* \lesssim 10^9$ cm) that very high temperatures would be required to make r_s smaller than R_*. In fact, it is clear from (2.16) and (1.7) that a gas temperature of order T_{th} is required: this can be achieved, for example, in a standing shock wave close to the stellar surface. We shall have much to say about this possibility later on, but it does not enter our analysis here as it requires discontinuous jumps in ρ, T, P, etc. A similar analysis of the signs in (2.27) for $r < r_s$ shows that the flow must be supersonic near the star:

$$v^2 > c_s^2 \quad \text{for small } r. \tag{2.30}$$

The discussion above shows that the problem we are considering is not mathematically well posed if we only give the ambient conditions at infinity. To specify the problem correctly we need a condition at or near the stellar surface also. Here we have imposed (2.30), which will have the effect of picking out just one solution (Type 1 below). Without (2.30) there would be another possible solution (Type 3).

The existence of a point r_s satisfying the implicit equation (2.29) is of great importance in characterizing the accretion flow. The direct mathematical consequence is that at $r = r_s$ the left hand side of (2.27) must also vanish: this requires

$$\textit{either} \quad v^2 = c_s^2 \text{ at } r = r_s, \tag{2.31}$$

$$\textit{or} \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = 0 \text{ at } r = r_s. \tag{2.32}$$

All solutions of (2.27) can now be classified by their behaviour at r_s, given by either (2.31) or (2.32), together with their behaviour at large r; for example, (2.28). This is very easy to see if we plot $v^2(r)/c_s^2(r)$ against r (Fig. 2.1).

From the figure it is clear that there are just six distinct families of solutions:

Type 1: $v^2(r_s) = c_s^2(r_s)$, $v^2 \to 0$ as $r \to \infty$
$(v^2 < c_s^2, r > r_s; v^2 > c_s^2, r < r_s)$;

Type 2: $v^2(r_s) = c_s^2(r_s)$, $v^2 \to 0$ as $r \to 0$
$(v^2 > c_s^2, r > r_s; v^2 < c_s^2, r > r_s)$;

Type 3: $v^2(r_s) < c_s^2(r_s)$ everywhere, $\frac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = 0$ at r_s;

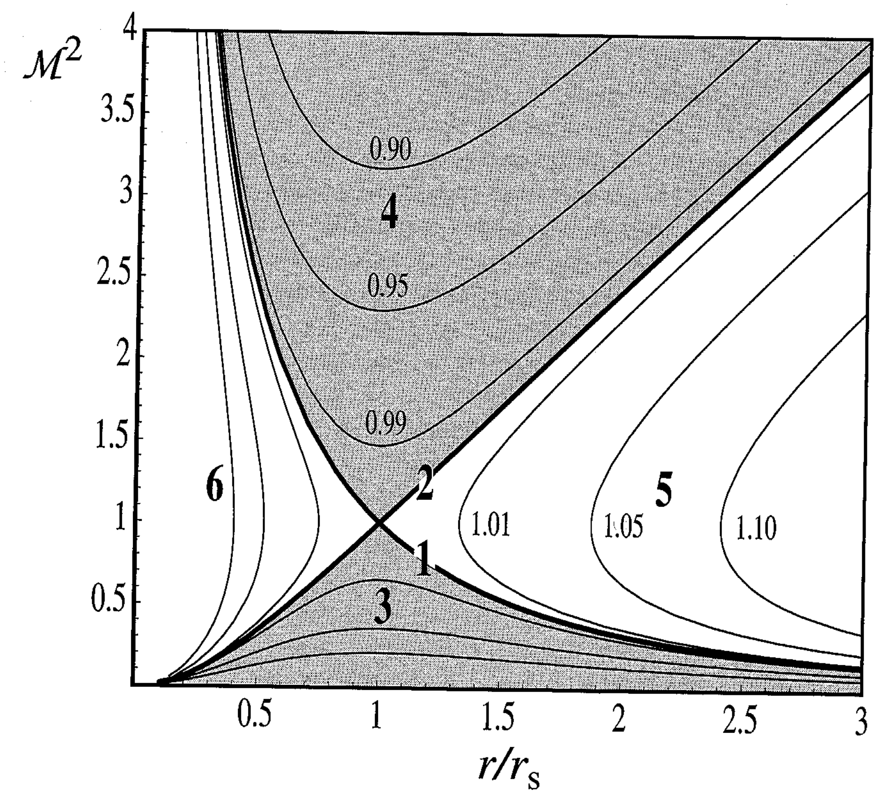

Fig. 2.1. Mach number squared $\mathcal{M}^2 = v^2(r)/c_s^2(r)$ as a function of radius r/r_s for spherically symmetrical adiabatic gas flows in the gravitational field of a star. For $v < 0$ these are accretion flows, while for $v > 0$ they are winds or 'breezes'. The two trans-sonic solutions **1**, **2** indicated by thick solid lines divide the remaining solutions into the families **3–6** described in the text (the case shown here is $\gamma = 4/3$, the integral curves are calculated and labelled as in Holzer & Axford (1970)).

Type 4: $\quad v^2(r_s) > c_s^2(r_s)$ everywhere, $\dfrac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = 0$ at r_s;

Type 5: $\quad \dfrac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = \infty$ at $v^2 = c_s^2(r_s)$; $\quad r > r_s$ always;

Type 6: $\quad \dfrac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = \infty$ at $v^2 = c_s^2(r_s)$; $\quad r < r_s$ always;

There is just one solution for each of Types 1 and 2: these are called *trans-sonic* as they make a transition between sub- and supersonic flow at r_s; r_s itself is known as the *sonic point* for these solutions. The occurrence of sonic points is a quite general feature of gas dynamical problems. Types 3 and 4 (shaded regions on Fig. 2.1) represent flow which is everywhere sub- or supersonic. Types 5 and 6 do not cover all of the range of r and are double-valued in the sense that there are two possible values of v^2 at a given r. We exclude these last two for these reasons, although they can represent *parts* of a correct solution if shocks are present. Types 2 and 4 must be excluded since they are supersonic at large r, violating (2.28), while Type 3 is subsonic at small r, violating (2.30). A solution of Type 2 with $v > 0$ describes a stellar wind: note that (2.27) is unchanged for $v \to -v$. Solutions of Type 3 with $v > 0$ give the so-called stellar 'breeze' solutions which are everywhere subsonic; if $v < 0$ this is a slowly sinking 'atmosphere'.

We are left finally with just the Type 1 solution: this has all the properties we want and is the unique solution to our problem. The sonic point condition (2.31) will lead us to the goal of relating the accretion rate $\dot{M}$ to the conditions at infinity.

With the question of uniqueness settled, we now integrate (2.24) directly, using the fact that (2.25) makes ρ a function of P:

$$\frac{v^2}{2} + \int \frac{\mathrm{d}P}{\rho} - \frac{GM}{r} = \text{constant}.$$

From (2.25) we have $\mathrm{d}P = K\gamma\rho^{\gamma-1}\mathrm{d}\rho$, and performing the integration, we obtain (for $\gamma \neq 1$)

$$\frac{v^2}{2} + \frac{K\gamma}{\gamma-1}\rho^{\gamma-1} - \frac{GM}{r} = \text{constant}.$$

But $K\gamma\rho^{\gamma-1} = \gamma P/\rho = c_s^2$, and we obtain the *Bernoulli integral*:

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma-1} - \frac{GM}{r} = \text{constant}. \tag{2.33}$$

(The strictly isothermal ($\gamma = 1$) case gives a logarithmic integral.) From the known property of our physical solution (Type 1) we have $v^2 \to 0$ as $r \to \infty$, so the constant in (2.33) must be $c_s^2(\infty)/(\gamma - 1)$, where $c_s(\infty)$ is the sound speed in the gas far away from the star. The sonic point condition now relates $c_s(\infty)$ to $c_s(r_s)$, since (2.31), (2.29) imply $v^2(r_s) = c_s^2(r_s)$, $GM/r_s = 2c_s^2(r_s)$, and the Bernoulli integral gives

$$c_s^2(r_s)\left[\frac{1}{2} + \frac{1}{\gamma-1} - 2\right] = \frac{c_s^2(\infty)}{\gamma-1}$$

or

$$c_s(r_s) = c_s(\infty)\left(\frac{2}{5-3\gamma}\right)^{1/2}. \tag{2.34}$$

We now obtain $\dot{M}$ from (2.23):

$$\dot{M} = 4\pi r^2\rho(-v) = 4\pi r_s^2\rho(r_s)c_s(r_s) \tag{2.35}$$

since $\dot{M}$ is independent of r. Using $c_{\rm s}^2 \propto \rho^{\gamma-1}$ we find

$$\rho(r_{\rm s}) = \rho(\infty)\left[\frac{c_{\rm s}(r_{\rm s})}{c_{\rm s}(\infty)}\right]^{2/(\gamma-1)}.$$

Putting this and (2.35) into (2.34) gives, after a little algebra, the relation we are looking for between $\dot{M}$ and conditions at infinity:

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_{\rm s}^3(\infty)}\left[\frac{2}{5-3\gamma}\right]^{(5-3\gamma)/2(\gamma-1)}. \tag{2.36}$$

Note that the dependence on γ here is rather weak: the factor $[2/(5-3\gamma)]^{(5-3\gamma)/2(\gamma-1)}$ varies from unity in the limit $\gamma = \frac{5}{3}$ to $e^{3/2} \cong 4.5$ in the limit $\gamma = 1$. For a value $\gamma = 1.4$, which would be typical for the adiabatic index of a part of the interstellar medium, the factor is 2.5.

Equation (2.36) shows that accretion from the interstellar medium is unlikely to be an observable phenomenon; reasonable values would be $c_{\rm s}(\infty) = 10$ km s^{-1}, $\rho(\infty) = 10^{24}$ g cm^{-3}, corresponding to a temperature of about 10 K and number density near 1 particle cm^{-3}. Then (2.36) gives (with $\gamma = 1.4$)

$$\dot{M} \cong 1.4 \times 10^{11}\left(\frac{M}{M_\odot}\right)^2\left(\frac{\rho(\infty)}{10^{-24}}\right)\left(\frac{c_{\rm s}(\infty)}{10 \text{ km s}^{-1}}\right)^{-3} \text{g s}^{-1}. \tag{2.37}$$

From (1.46) even accreting this on to a neutron star yields $L_{\rm acc}$ only of the order 2×10^{31} erg s^{-1}; at a typical distance of 1 kpc this gives far too low a flux to be detected.

To complete the solution of the problem to find the run of all quantities with r we could now get $v(r)$ in terms of $c_{\rm s}(r)$ from (2.35), using $c_{\rm s}^2 = \gamma P/\rho \propto \rho^{\gamma-1}$:

$$(-v) = \frac{\dot{M}}{4\pi r^2 \rho(r)} = \frac{\dot{M}}{4\pi r^2 \rho(\infty)}\left[\frac{c_{\rm s}(\infty)}{c_{\rm s}(r)}\right]^{2/(\gamma-1)}.$$

Substituting this into the Bernoulli integral (2.33) gives an algebraic relation for $c_{\rm s}(r)$; the solution of this then gives $\rho(r)$ and $v(r)$. In practice, the algebraic equation for $c_{\rm s}(r)$ has fractional exponents and must be solved numerically. However, the main features of the r-dependence can be inferred by looking at the Bernoulli integral (2.33). At large r the gravitational pull of the star is weak and all quantities have their 'ambient' values $(\rho(\infty), c_{\rm s}(\infty), v \cong 0)$. As one moves to smaller r, the inflow velocity increases until $(-v)$ reaches $c_{\rm s}(\infty)$, the sound speed at infinity. The only term in (2.33) capable of balancing this increase is the gravity term GM/r; since $c_{\rm s}(r)$ does not greatly exceed $c_{\rm s}(\infty)$ this must occur at a radius

$$r \cong r_{\rm acc} = \frac{2GM}{c_{\rm s}(\infty)^2} \cong 3 \times 10^{14}\left(\frac{M}{M_\odot}\right)\left(\frac{10^4\text{K}}{T(\infty)}\right) \text{cm}. \tag{2.38}$$

At this point $\rho(r)$ and $c_{\rm s}(r)$ begin to increase above their ambient values. At the sonic point $r = r_{\rm s}$ (see (2.31)) the inflow becomes supersonic and the gas is effectively in free fall: from (2.33) $v^2 \gg c_{\rm s}^2$ implies

$$v^2 \cong 2GM/r = v_{\rm ff}^2$$

with $v_{\rm ff}^2$ the free-fall velocity. The continuity equation (2.23) now gives

$$\rho \cong \rho(r_{\rm s}) \left(\frac{r_{\rm s}}{r}\right)^{3/2} \text{ for } r \lesssim r_{\rm s}.$$

Finally, we can, in principle, get the gas temperature, using the perfect gas law and the polytropic relation

$$T \cong T(r_{\rm s}) \left(\frac{r_{\rm s}}{r}\right)^{[3/2](\gamma-1)} \text{ for } r \lesssim r_{\rm s}.$$

However, the steady increase in T for decreasing r predicted by this equation is probably unrealistic: radiative losses must begin to cool the gas, so a better energy equation than (2.25) is needed at this point.

The radius $r_{\rm acc}$ defined by (2.38) has a simple interpretation: at a radius r the ratio of internal (thermal) energy to gravitational binding energy of a gas element of mass m is

$$\frac{\text{thermal energy}}{\text{binding energy}} \sim \frac{mc_{\rm s}^2(r)}{2} \frac{r}{GMm} \sim \frac{r}{r_{\rm acc}} \text{ for } r \gtrsim r_{\rm acc}$$

since $c_{\rm s}(r) \sim c_{\rm s}(\infty)$ for $r > r_{\rm acc}$. Hence, for $r \gg r_{\rm acc}$ the gravitational pull of the star has little effect on the gas. We call $r_{\rm acc}$ the accretion radius: it gives the range of influence of the star on the gas cloud which we sought at the outset. Note that in terms of r the relation (2.36) giving the steady accretion rate can be rewritten as

$$\dot{M} \sim \pi r_{\rm acc}^2 c_{\rm s}(\infty)\rho(\infty). \tag{2.39}$$

Dimensionally $r_{\rm acc}$ must have a form like (2.33); however, since the proper specification of the accretion flow involves a 'surface' condition like (2.30) the numerical factor in the formula for $r_{\rm acc}$ is in general undetermined, and the concept of an 'accretion radius' is not well defined. A Type 3 solution for the same $c_{\rm s}(\infty), \rho(\infty)$ would give a smaller accretion rate $\dot{M}$ than (2.36). If an $\dot{M}$ greater than the value (2.36) is externally imposed (e.g. by mass exchange in a binary system) the flow must become supersonic near the star and must involve discontinuities (i.e. shocks).

We have treated the problem of steady spherical accretion at some length. The main conclusions we can draw from this study and apply generally are:

(i) The steady accretion rate $\dot{M}$ is determined by ambient conditions at infinity (equation (2.36)) and a 'surface' condition (e.g. equation (2.30)). For accretion by isolated stars from the interstellar medium, the resulting value of $\dot{M}$ is too low to be of much observational importance. Clearly, we must look to close binaries to find more powerful accreting systems.

(ii) The star's gravitational pull seriously influences the gas's behaviour only inside the accretion radius $r_{\rm acc}$.

(iii) A steady accretion flow with $\dot{M}$ greater than or equal to the value (2.36) must possess a sonic point; i.e. the inflow velocity must become supersonic near the stellar surface.

The immediate consequence of point (iii) is that, since for a star (although not for a black hole) the accreting material must eventually join the star with a very small velocity, some way of stopping the highly supersonic accretion flow must be found. Consideration of how this stopping process can work leads us naturally into the area of plasma physics, which we touched on briefly at the beginning of this chapter. In the next chapter we shall develop in more detail the plasma concepts we shall need.

3 Plasma

3.1 Introductic

Whenever we need t
the mean free path l
this chapter we shall
our study of accretio
A *plasma* differs fr
two gases of electrica
different particle mas
The electrons and
attractions and repul
distance and do not
acts with many other
more complicated tha
very short-range. A
masses m_e and m_i.
transfer only a small
ble for electrons and
timescales. These tw
the disparity in elect
character. A further
ated by a large-scale
accreting on to highl

3.2 Charge neu

Let us begin by exam
of the Coulomb force
Note first that the
approximately equal,
even a small charge in
to move the plasma p
suppose that there is