ASTRO 310: Galactic & Extragalactic Astronomy
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Introduction to Stellar Dynamics: Potential Theory & Mass Distributions & Motions
Gravity & Stellar Systems

- Only force that affects stars & dark matter: Gravity

- Gas is more complicated since it is affected by other forces

- Newtonian gravity works well in most cases. Relativistic effects important only in tight binaries and near black holes in galaxy nuclei.
Behavior of stellar system and how to treat it mathematically depends on # of stars in system

- Binary stars: 2 stars
- Open clusters: $\sim 10^2$-$10^3$ stars
- Globular clusters: $\sim 10^5$-$10^6$ stars
- Large galaxies: $\sim 10^{10}$-$10^{12}$ stars
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No exact solution for $n>2$ !
need to either solve numerically,
OR make simplifying assumptions and solve analytically
triple systems have non-repeating (chaotic) orbits
The three-body problem dates back to the 1680s. Isaac Newton had already shown that his new law of gravity could always predict the orbit of two bodies held together by gravity—such as a star and a planet—with complete accuracy. The orbit is basically always an ellipse. However, Newton couldn't come up with a similar solution for the case of three bodies orbiting one another. For 2 centuries, scientists tried different tacks until the German mathematician Heinrich Bruns pointed out that the search for a general solution for the three-body problem was futile, and that only specific solutions—one-offs that work under particular conditions—were possible. Generally, the motion of three bodies is now known to be nonrepeating.
Key Q: Does star’s motion depend much on individual star-star gravitational encounters, or does it depend more on the cumulative gravitational effect of all the more distant stars?

Gravitational potential $\Phi(x)$ of star cluster or galaxy

The total potential is the sum of a smoothly varying shallow potential well due to the average effect of many distant stars, plus a steep potential well near each star. In some (but not all!) cases the steep potential well near each star can be ignored.
describe the orbital motion of the Sun in the Galaxy ...
How does the orbital speed of the Sun relate to the mass of the galaxy?

A. instantaneous speed and radius gives total galaxy mass
B. instantaneous speed and radius gives interior galaxy mass
C. instantaneous speed and radius gives approximate interior galaxy mass
D. orbit-averaged speed and radius gives approximate interior galaxy mass
E. orbit-averaged speed and radius gives approximate total galaxy mass
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Newton’s Law of Gravity

\[ \frac{d}{dt}(mv) = -\frac{GmM}{r^3}r \]

this gives acceleration \( \frac{dv}{dt} \) on 1 star \( m \), due to gravitational force exerted on \( m \) by \( M \) at distance \( r \)

if there are \( N \) stars with masses \( m_\alpha \) \( (\alpha = 1,2,3,\ldots,N) \) at positions \( x_\alpha \); to get acceleration of star \( \alpha \), add forces on star \( \alpha \) from all the other stars:

\[ \frac{d}{dt}(m_\alpha v_\alpha) = -\sum_{\beta \neq \alpha} \frac{Gm_\alpha m_\beta}{|x_\alpha - x_\beta|^3}(x_\alpha - x_\beta) \]
it is often convenient to write the acceleration as the gradient in the gravitational potential

\[ \frac{d}{dt}(mv) = -m \nabla \Phi(x), \quad \text{with} \quad \Phi(x) = -\sum_{\alpha} \frac{G m_\alpha}{|x - x_\alpha|} \quad \text{for} \ x \neq x_\alpha \]

if we have (or can approximate by) a continuous distribution of matter:

\[ \Phi(x) = -\int \frac{G \rho(x')}{|x - x'|} \ d^3x', \]

in this case the acceleration, or \textbf{force per unit mass} is:

\[ F(x) = -\nabla \Phi(x) = -\int \frac{G \rho(x')(x - x')}{|x - x'|^3} \ d^3x'. \]

\[ \text{NOTE!! this F is really the acceleration!!} \]
Gravitational force and potential

Since force can be expressed as the gradient of a potential, the gravitational force is \textit{conservative}:
Gravitational force and potential

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• Work done by force independent of path taken between end points

• Energy is conserved
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Since force can be expressed as the gradient of a potential, the gravitational force is *conservative*:

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- Energy is conserved \([ \text{if } \phi \neq \phi(t) ]\)

Advantages of potential:

- Potential is a scalar field, simpler than vector field
- In many cases, best way to calculate force is to first calculate potential, then takes its gradient
preceding equation for $\phi(x)$ is integral equation. this can be expressed in another form which is sometime more convenient, a differential equation:

$$\nabla^2 \Phi(x) = 4\pi G \rho(x)$$

Poisson’s Equation
derive $\phi(r)$ in case of spherical symmetry...
spherical coordinate system \((r, \varphi, \theta)\)
gravitational potential in case of spherical symmetry

mass blob with density distribution $\rho(r')$

integrate over all space (all $r'$) to learn gravitational potential at some position $r$
Newton’s theorem #2 for spherically symmetric gravitational field

The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell’s matter were concentrated into a point at its center.
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The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell’s matter were concentrated into a point at its center.

$I_{IN}$: by Newton’s 2nd theorem, this integral $I_{IN}$ must be same as if all matter were at the center, i.e., $r' = 0$

So $|r' - r| \rightarrow r$
Newton’s theorem #1 for *spherically symmetric* gravitational field

A body that is inside a uniformly dense spherical shell of matter experiences no net gravitational force from that shell.

force from $A_1$ balances force from $A_2$

Geometry of spherical shells precisely compensates for $1/r^2$ dependence of gravitational force (more mass in $A_1$ but it is further away)
A body that is inside a uniformly dense spherical shell of matter experiences no net gravitational force from that shell for \( M_{\text{out}} \), force from \( S_1 \) cancels force from \( S_2 \) (equal and opposite).

Geometry of spherical shells precisely compensates for \( 1/r^2 \) dependence of gravitational force (more mass in \( S_2 \) but it is further away).

Net force on \( m \) is all from \( M_{\text{in}} \).
$I_{\text{OUT}}$ : Since contribution to force from matter outside $r$ must be ZERO, the contribution to potential from matter outside $r$ must be constant within radius $r$, and we can choose to evaluate integral at any radius smaller than $r$.

$\rightarrow$ so choose $r \rightarrow 0$

so $|r' - r| \rightarrow r'$
gravitational potential in case of spherical symmetry

in the case of a spherically symmetric mass distribution:

\[
\Phi(r) = - \left[\frac{G M(<r)}{r} + 4\pi G \int_r^{\infty} \rho(r')r' \, dr' \right]
\]
circular speed: speed of test particle in circular orbit at radius $r$

acceleration = force per unit mass = 
\[ a = \frac{F}{m} = \frac{d\phi}{dr} = \frac{v_c^2}{r} \]

\[ v_c = \sqrt{ra} = \sqrt{r \frac{d\phi}{dr}} = \sqrt{\frac{GM_{\text{in}}}{r}} \]

depends only on \textbf{interior} mass $M(r)$ in the special case of spherical symmetry

\textit{if not spherical symmetry, also depends on \textbf{exterior} mass!}
to first order, orbital speed depends on \textit{interior mass} not \textit{exterior mass}
What if mass distr. not spherically symmetric? *(galaxies are NOT spherically symmetric!)*

How much does this change orbital speeds?
Rotation curve of exponential disk

B+C have similar velocities at R>4R_{disk}
→ little additional mass and Keplerian behavior (V~R^{-1/2}) beyond R>4R_{disk} for A, B, C

A peaks at R=2R_{disk}
A ~15% higher than B (at peak)

A slower than B at R<1R_{disk}
→ Slowed by exterior mass

A faster than B at R>1R_{disk}
→ sped by interior mass concentrated in disk

Figure 2-17. The circular-speed curves of: an exponential disk (full curve); a point with the same total mass (dotted curve); the spherical body for which M(r) is given by equation (2-170) (dashed curve).

Binney & Tremaine 1987

A: flattened exponential disk
B: spherical distribution with same interior mass as exponential disk
C: point mass with same total mass as exponential disk
Orbital speeds depend mostly on *interior mass only*. How can we measure the *total mass* of the galaxy?
Orbital speeds depend mostly on interior mass only. How can we measure the total mass of the galaxy?

2 ways:
1. measure orbital speeds at large $r$, beyond all the mass (so that $M_{in} = M_{tot}$)
   (need test particles at large $r$ – not always available)

2. measure escape speed
   (can do this from inside mass distribution!)
escape speed: escaping from a continuous distribution of matter

\[ E_{\text{tot}} = K + P \]
\[ E = 0 \quad \text{escape condition} \]
\[ E_1 = E_2 \quad \text{conservation of energy} \]
\[ K_1 + P_1 = K_2 + P_2 \]
\[ K_1 - K_2 = P_2 - P_1 \]
\[ K(r) - K(\infty) = P(\infty) - P(r) \]
\[ \frac{1}{2}mv_{\text{esc}}^2 - 0 = 0 - m\phi \]

\[ v_{\text{esc}} = \sqrt{2|\phi|} \]

\[ \phi(r) \text{ and therefore } v_{\text{esc}} \text{ depends on both } M_{\text{in}} \text{ and } M_{\text{out}} \]
potential energy

defined to be work done against gravity in assembling distribution of matter from infinity

for single star of mass m: \[ PE = m \phi(x) \]
gravitational potential in case of spherical symmetry

mass blob with density distribution \( \rho(r') \)

integrate over all space (all \( r' \)) to learn gravitational potential at some position \( r \)

in the case of a spherically symmetric mass distribution:

\[
\Phi(r) = - \left[ \frac{G \mathcal{M}(< r)}{r} + 4\pi G \int_0^\infty \rho(r')r' \, dr' \right]
\]

\( \Phi(r) \) \quad \text{interior mass}

\( \text{exterior mass} \)
to first order, orbital speed depends on interior mass not exterior mass but escape speed depends on both!
circular & escape speeds

\[ v_c = \sqrt{r \frac{d\phi}{dr}} = \sqrt{GM_{\text{in}}/r} \]

\[ v_{\text{esc}} = \sqrt{2\phi(r)} \]

if all mass is interior then \( v_{\text{esc}} = \sqrt{2GM_{\text{in}}/r} \)

but in general \( v_{\text{esc}} \geq \sqrt{2GM_{\text{in}}/r} \)

exterior mass beyond \( r \): \( M_{\text{out}} = M(>r) \)

important for escape speed but not circular speed!

(strictly true in case of spherical symmetry; if not spherically symmetric, circular speed depends a little on exterior mass)
Suppose we can only observe stars in the solar neighborhood....

...how can we estimate the escape speed & therefore also the total mass of the galaxy?
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...how can we estimate the escape speed & therefore also the total mass of the galaxy?

Observe speeds of high velocity stars. The fastest stars that are bound to the galaxy have speeds just less than the escape speed.
hypervelocity stars in Milky Way

orbital speed of Sun = 220 km/s

speed of hypervelocity stars ~400-1000 km/s

velocity vectors of 4 hypervelocity stars near the Sun
artists’ impressions of hypervelocity stars escaping Milky Way
How do we know if fast star is escaping or not?
How do we know if fast star is escaping or not?

2 ways:
1. examine distribution of stellar velocities, search for “break” in distribution at escape speed

2. examine direction of motion – stars moving outward may be escaping, stars moving inward may be bound
Mira – variable Red Giant star $1.2 \, M_{\text{sun}}$ moving at 130 km/s through galaxy (not fast enough to escape) making gaseous tail 13 light-years long

UV image from GALEX telescope
UV emission is from molecular Hydrogen
UV image from GALEX telescope
UV emission is from molecular Hydrogen
high-velocity neutron star (& pulsar) plowing through the ISM: Guitar Nebula

moving at 1600 km/s through the ISM (greater than escape speed from Galaxy)
high-velocity neutron star (& pulsar) plowing through the ISM: Guitar Nebula
Guitar Nebula: HST image of head of Nebula, showing most recent part of trail created by fast-moving neutron star.